Explicit solutions of multiple state optimal design problems

Krešimir Burazin

UNIVERSITY J. J. STROSSMAYER OF OSIJEK
DEPARTMENT OF MATHEMATICS
Trg Ljudevita Gaja 6
31000 Osijek, Hrvatska
http://www.mathos.unios.hr

kburazin@mathos.hr

Joint work with **Marko Vrdoljak**

WeConMApp

[SEMINAR ZA OPTIMIZACIJU I PRIMJENE] 27.5.2015

Stationary diffusion equation

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f \in \mathrm{L}^2(\Omega)$, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R}))$ given;
stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Physical interpretations:

- Thermal conductivity; $\mathbf{A}$ – (thermal) conductivity of material, f – external heat density, u – temperature
- Electrical conductivity; $\mathbf{A}$ – (electrical) conductivity of material, f – electric charge density, u – electrical potential

Energy functional (total amount of heat/electrical energy dissipated in Ω):

$$J = \int_\Omega f(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{A}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x})\,d\mathbf{x}$$

Stationary diffusion equation

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f \in \mathrm{L}^2(\Omega)$, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R}))$ given; stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Physical interpretations:

- Thermal conductivity; $\mathbf{A}$ – (thermal) conductivity of material, f – external heat density, u – temperature
- Electrical conductivity; $\mathbf{A}$ – (electrical) conductivity of material, f – electric charge density, u – electrical potential

Energy functional (total amount of heat/electrical energy dissipated in Ω):

$$J = \int_\Omega f(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{A}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x})\,d\mathbf{x}$$

Stationary diffusion equation

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f \in \mathrm{L}^2(\Omega)$, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R}))$ given; stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Physical interpretations:

- Thermal conductivity; $\mathbf{A}$ – (thermal) conductivity of material, f – external heat density, u – temperature
- Electrical conductivity; $\mathbf{A}$ – (electrical) conductivity of material, f – electric charge density, u – electrical potential

Energy functional (total amount of heat/electrical energy dissipated in Ω):

$$J = \int_\Omega f(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{A}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x})\,d\mathbf{x}$$

Stationary diffusion equation

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f \in \mathrm{L}^2(\Omega)$, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R}))$ given; stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Physical interpretations:

- Thermal conductivity; $\mathbf{A}$ – (thermal) conductivity of material, f – external heat density, u – temperature
- Electrical conductivity; $\mathbf{A}$ – (electrical) conductivity of material, f – electric charge density, u – electrical potential

Energy functional (total amount of heat/electrical energy dissipated in Ω):

$$J = \int_\Omega f(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{A}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x})\,d\mathbf{x}$$

Stationary diffusion equation

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f \in \mathrm{L}^2(\Omega)$, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R}))$ given; stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

Physical interpretations:

- Thermal conductivity; $\mathbf{A}$ – (thermal) conductivity of material, f – external heat density, u – temperature
- Electrical conductivity; $\mathbf{A}$ – (electrical) conductivity of material, f – electric charge density, u – electrical potential

Energy functional (total amount of heat/electrical energy dissipated in Ω):

$$J = \int_\Omega f(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{A}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x})\,d\mathbf{x}$$

Optimal design problem (single state)

Composite of two isotropic materials with conductivities $0 < \alpha < \beta$:
$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi \, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.
For given Ω, α, β, q_α and f we want to find such material $\mathbf{A}$ which maximizes or minimizes the cost functional:

$$J(\chi) = \int_\Omega f(\mathbf{x})u(\mathbf{x})d\mathbf{x} \longrightarrow \max / \min \,,$$

where u is the solution of the state equation

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{cases}$$

Interpretations:

- Maximize the amount of heat kept inside body
- Maximize the torsional rigidity of a rod made of two materials
- Maximize the flow rate of two viscous immiscible fluids through pipe

Optimal design problem (single state)

Composite of two isotropic materials with conductivities $0 < \alpha < \beta$: $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi \, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.

For given Ω, α, β, q_α and f we want to find such material $\mathbf{A}$ which maximizes or minimizes the cost functional:

$$J(\chi) = \int_\Omega f(\mathbf{x})u(\mathbf{x})d\mathbf{x} \longrightarrow \max / \min \,,$$

where u is the solution of the state equation

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{cases}$$

Interpretations:

- Maximize the amount of heat kept inside body
- Maximize the torsional rigidity of a rod made of two materials
- Maximize the flow rate of two viscous immiscible fluids through pipe

Optimal design problem (single state)

Composite of two isotropic materials with conductivities $0 < \alpha < \beta$:
$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.
For given Ω, α, β, q_α and f we want to find such material $\mathbf{A}$ which maximizes or minimizes the cost functional:

$$J(\chi) = \int_\Omega f(\mathbf{x})u(\mathbf{x})d\mathbf{x} \longrightarrow \max / \min\,,$$

where u is the solution of the state equation

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{cases}$$

Interpretations:

- Maximize the amount of heat kept inside body
- Maximize the torsional rigidity of a rod made of two materials
- Maximize the flow rate of two viscous immiscible fluids through pipe

Optimal design problem (single state)

Composite of two isotropic materials with conductivities $0 < \alpha < \beta$:
$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi \, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.
For given Ω, α, β, q_α and f we want to find such material $\mathbf{A}$ which maximizes or minimizes the cost functional:

$$J(\chi) = \int_\Omega f(\mathbf{x})u(\mathbf{x})d\mathbf{x} \longrightarrow \max / \min \,,$$

where u is the solution of the state equation

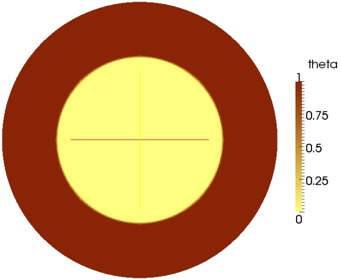
$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{cases}$$

Interpretations:

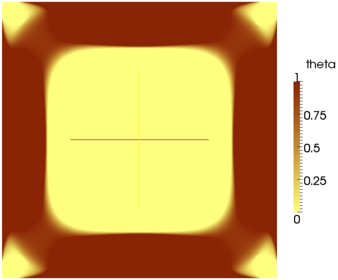
- Maximize the amount of heat kept inside body
- Maximize the torsional rigidity of a rod made of two materials
- Maximize the flow rate of two viscous immiscible fluids through pipe

Maximization, Ω circle / square, $f \equiv 1$

Murat and Tartar, 1985



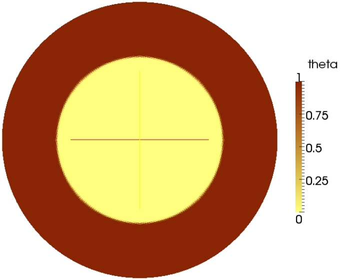
Goodman, R.V. Kohn, L. Reyna, 1986



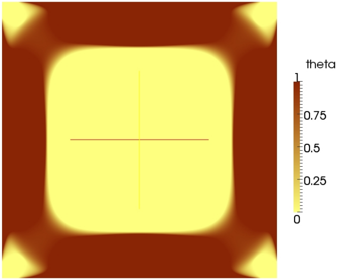
In general, there might exist no classical optimal design. The relaxation is needed, by introducing composite materials,...

Maximization, Ω circle / square, $f \equiv 1$

Murat and Tartar, 1985



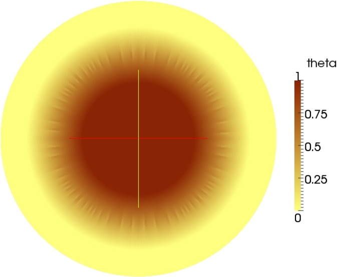
Goodman, R.V. Kohn, L. Reyna, 1986



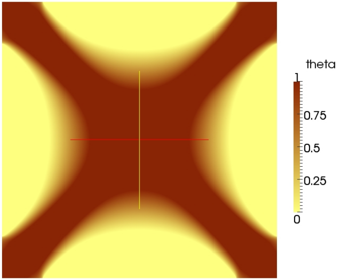
In general, there might exist no classical optimal design. The relaxation is needed, by introducing composite materials,...

Energy minimization

Murat and Tartar, 1985



K. Lurie, A. Cherkaev, 1984



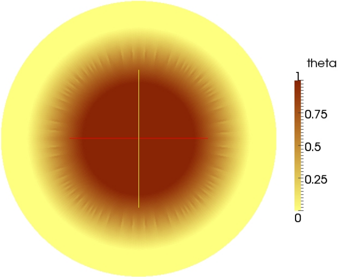
$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \quad \cdots \quad \theta \in \mathrm{L}^\infty(\Omega; [0,1])$

$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \qquad \mathbf{A} \in \mathcal{K}(\theta)$ a.e. on Ω

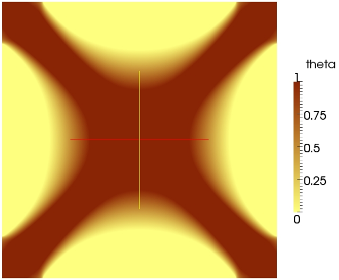
classical material — composite mateiral - relaxation

Energy minimization

Murat and Tartar, 1985



K. Lurie, A. Cherkaev, 1984



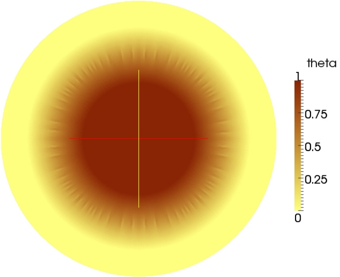
$$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \quad \cdots \quad \theta \in \mathrm{L}^\infty(\Omega; [0,1])$$

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \qquad \mathbf{A} \in \mathcal{K}(\theta) \quad \text{a.e. on } \Omega$$

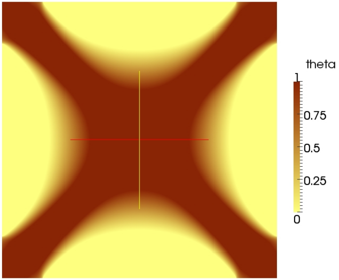
classical material composite mateiral - relaxation

Energy minimization

Murat and Tartar, 1985



K. Lurie, A. Cherkaev, 1984



$$\chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \quad \cdots \quad \theta \in \mathrm{L}^\infty(\Omega; [0,1])$$

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \qquad \mathbf{A} \in \mathcal{K}(\theta) \quad \text{a.e. on } \Omega$$

classical material — composite mateiral - relaxation

Homogenised material

Definition

A sequence of matrix functions $\mathbf{A}^\varepsilon$ is said to H-converge to $\mathbf{A}^*$ if for every f the sequence of solutions of

$$\left\{ \begin{array}{l} -\mathsf{div}\,(\mathbf{A}^\varepsilon \nabla u_\varepsilon) = f \\ u_\varepsilon \in \mathrm{H}^1_0(\Omega) \end{array} \right.$$

satisfies $u_\varepsilon \rightharpoonup u$ in $\mathrm{H}^1_0(\Omega)$, $\mathbf{A}^\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbf{A}^* \nabla u$ in $\mathrm{L}^2(\Omega; \mathbf{R}^d)$, where u is the solution of the homogenised equation

$$\left\{ \begin{array}{l} -\mathsf{div}\,(\mathbf{A}^* \nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{array} \right.$$

Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *and conductivities* $\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$ *satisfy* $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε)*.*

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.}\}$$

Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *and conductivities* $\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$ *satisfy* $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε).

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.}\}$$

Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *and conductivities* $\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$ *satisfy* $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε)*.*

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times\mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.}\}$$

Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

2D:

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

3D:

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{\mathrm{L}^\infty(\Omega;\{0,1\})} I$

Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

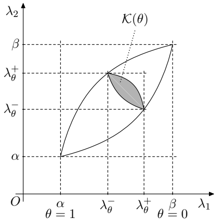
$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

2D:



3D:

$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{\mathrm{L}^\infty(\Omega;\{0,1\})} I$

Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

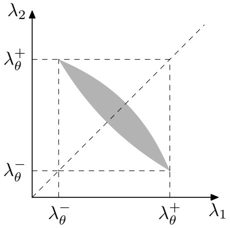
$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \dots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{L^\infty(\Omega;\{0,1\})} I$

2D:



3D:

Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

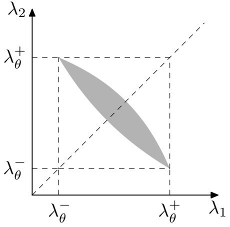
$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

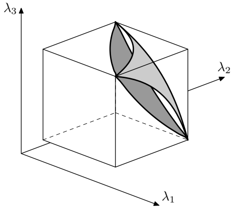
$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{L^\infty(\Omega;\{0,1\})} I$

2D:



3D:



Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

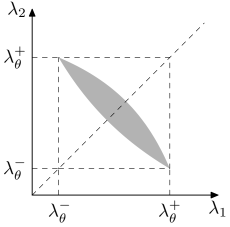
$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^d \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

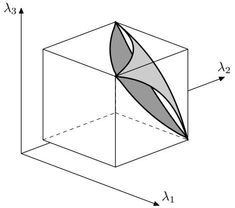
$$\sum_{j=1}^d \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{\mathrm{L}^\infty(\Omega;\{0,1\})} I$

2D:



3D:



Multiple state optimal design problem

State equations

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1, \dots, m$$

State function $\mathrm{u} = (u_1, \dots, u_m)$

$$\begin{cases} I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min \\ \quad \mathrm{u} = (u_1, \dots, u_m) \text{ state function for } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \quad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \,, \; \int_\Omega \chi \, d\mathbf{x} = q_\alpha \,, \end{cases}$$

for some given weights $\mu_i > 0$. Relaxed problem:

$$J(\theta, \mathbf{A}) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min \qquad \text{on}$$

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \; \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

Multiple state optimal design problem

State equations

$$\begin{cases} -\text{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1, \dots, m$$

State function $\mathrm{u} = (u_1, \dots, u_m)$

$$\begin{cases} I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min \\ \qquad \mathrm{u} = (u_1, \dots, u_m) \text{ state function for } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \,, \; \int_\Omega \chi \, d\mathbf{x} = q_\alpha \,, \end{cases}$$

for some given weights $\mu_i > 0$. Relaxed problem:

$$J(\theta, \mathbf{A}) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min \qquad \text{on}$$

$$\mathcal{A} := \left\{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \; \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \right\}$$

Multiple state optimal design problem

State equations

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m$$

State function $\mathrm{u} = (u_1, \ldots, u_m)$

$$\begin{cases} I(\chi) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min \\ \quad \mathrm{u} = (u_1, \ldots, u_m) \text{ state function for } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \quad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \,, \; \int_\Omega \chi \, d\mathbf{x} = q_\alpha \,, \end{cases}$$

for some given weights $\mu_i > 0$. Relaxed problem:

$$J(\theta, \mathbf{A}) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min \qquad \text{on}$$

$$\mathcal{A} := \{ (\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \; \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha \right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{T}} I$$

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha \right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \dots, m$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{T}} I$$

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \dots, m$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \quad \Longleftrightarrow \quad \min_{\mathcal{T}} I$$

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \ldots, m$$

$$\min_{\mathcal{A}} J \Longleftrightarrow \min_{\mathcal{B}} J \Longleftrightarrow \min_{\mathcal{T}} I$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega;[0,1]) : \textstyle\int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\text{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$\min_{\mathcal{A}} J \iff \min_{\mathcal{T}} I$

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega;[0,1]) : \textstyle\int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\text{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m$$

$\min_{\mathcal{A}} J \iff \min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{T}} I$$

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \ldots, m$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\text{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_\mathcal{A} J \quad \Longleftrightarrow \quad \min_\mathcal{T} I$$

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\text{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \dots, m$$

$$\min_\mathcal{A} J \Longleftrightarrow \min_\mathcal{B} J \Longleftrightarrow \min_\mathcal{T} I$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{T}} I$$

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \dots, m$$

$$\min_{\mathcal{A}} J \iff \min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha \right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\text{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_\mathcal{A} J \quad \Longleftrightarrow \quad \min_\mathcal{T} I$$

B. Multiple state equations: Simpler relaxation fails; in spherically symmetric case or when $m < d$, it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha \right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

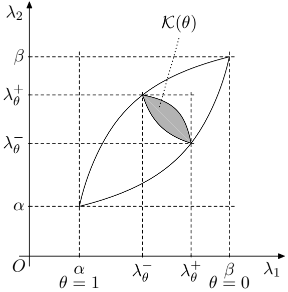
$$\begin{cases} -\text{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \dots, m$$

$$\min_\mathcal{A} J \Longleftrightarrow \min_\mathcal{B} J \Longleftrightarrow \min_\mathcal{T} I$$

$\min_{\mathcal{B}} J$

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \; \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

2D:



Further relaxation:

$\mathcal{B}$ ... $\int_\Omega \theta \, d\mathbf{x} = q_\alpha$

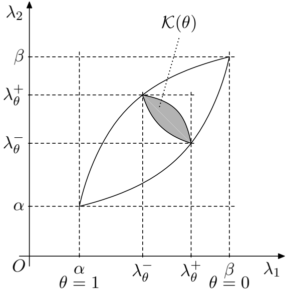
$\lambda_\theta^- \leq \lambda_{\min}(\mathbf{A}) \,, \; \lambda_{\max}(\mathbf{A}) \leq \lambda_\theta^+$

$\mathcal{B}$ is convex and compact and J is continuous on $\mathcal{B}$, so there is a solution of $\min_{\mathcal{B}} J$.

$\min_{\mathcal{B}} J$

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \, , \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

2D:



Further relaxation:

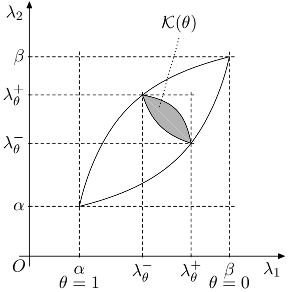
$$\mathcal{B} \qquad \ldots \quad \int_\Omega \theta \, d\mathbf{x} = q_\alpha$$
$$\lambda_\theta^- \le \lambda_{\min}(\mathbf{A}) \, , \ \lambda_{\max}(\mathbf{A}) \le \lambda_\theta^+$$

$\mathcal{B}$ is convex and compact and J is continuous on $\mathcal{B}$, so there is a solution of $\min_{\mathcal{B}} J$.

$\min_{\mathcal{B}} J$

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

2D:



Further relaxation:

$\mathcal{B}$ $\quad \dots \quad \int_\Omega \theta \, d\mathbf{x} = q_\alpha$

$$\lambda_\theta^- \leq \lambda_{\min}(\mathbf{A}) \,, \ \lambda_{\max}(\mathbf{A}) \leq \lambda_\theta^+$$

$\mathcal{B}$ is convex and compact and J is continuous on $\mathcal{B}$, so there is a solution of $\min_{\mathcal{B}} J$.

Simpler problem $\min_{\mathcal{T}} I$

$$I(\theta) = \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\theta \in \mathcal{T} = \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \right\}$$

and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m\,,$$

$\mathcal{T}$ is compact and convex and I is continuous on $\mathcal{T}$

$\cdots$ $\min_{\mathcal{T}} I$ has solution

Simpler problem $\min_{\mathcal{T}} I$

$$I(\theta) = \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\theta \in \mathcal{T} = \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \right\}$$

and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m\,,$$

$\mathcal{T}$ is compact and convex and I is continuous on $\mathcal{T}$

$\cdots$ $\min_{\mathcal{T}} I$ has solution

$\min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I \iff \min_{\mathcal{A}} J$ **if** $m < d$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}_0^1(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$.
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$, *then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$.
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$, *then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda_{\theta^*}^+ \nabla u_i^*$ *almost everywhere on* Ω *(e.g.* $\mathbf{A}^* = \lambda_{\theta^*}^+ \mathbf{I}$*) is an optimal design for the problem* $\min_{\mathcal{B}} J$. *If aditionally* $m < d$, *then* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$, *and thus it is also minimizer for* J *on* $\mathcal{A}$.

$$\min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I \iff \min_{\mathcal{A}} J \text{ if } m < d$$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$.
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$, *then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$.
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$, *then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda^+_{\theta^*} \nabla u_i^*$ *almost everywhere on* Ω *(e.g.* $\mathbf{A}^* = \lambda^+_{\theta^*} \mathbf{I}$*) is an optimal design for the problem* $\min_{\mathcal{B}} J$. *If aditionally* $m < d$, *then* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$, *and thus it is also minimizer for* J *on* $\mathcal{A}$.

$$\min_{\mathcal{B}} J \Longleftrightarrow \min_{\mathcal{T}} I \Longleftrightarrow \min_{\mathcal{A}} J \text{ if } m < d$$

Theorem

- *There is unique* $\mathsf{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$*.*
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$*.*
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$*, then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda^+_{\theta^*} \nabla u_i^*$ *almost everywhere on* Ω *(e.g.* $\mathbf{A}^* = \lambda^+_{\theta^*} \mathbf{I}$*) is an optimal design for the problem* $\min_{\mathcal{B}} J$*. If aditionally* $m < d$*, then* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$*, and thus it is also minimizer for* J *on* $\mathcal{A}$*.*

$\min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I \iff \min_{\mathcal{A}} J$ **if** $m < d$

Theorem

- *There is unique* $\mathsf{u}^* \in \mathrm{H}_0^1(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$*.*
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$*.*
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$*, then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda_{\theta^*}^+ \nabla u_i^*$ *almost everywhere on* Ω *(e.g.* $\mathbf{A}^* = \lambda_{\theta^*}^+ \mathbf{I}$*) is an optimal design for the problem* $\min_{\mathcal{B}} J$*. If aditionally* $m < d$*, then* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$*, and thus it is also minimizer for* J *on* $\mathcal{A}$*.*

Spherical symmetry: $\min_{\mathcal{A}} J \iff \min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric, and let the right-hand sides $f_i = f_i(r), r \in \omega, i = 1, \ldots, m$ be radial functions. Then $\min_{\mathcal{A}} J = \min_{\mathcal{B}} J = \min_{\mathcal{T}} I$, and there exists a minimizer $(\theta^*, \mathbf{A}^*)$ of the optimal design problem $\min_{\mathcal{A}} J$ which is a radial function. More precisely,

a) *For any minimizer θ of functional I over $\mathcal{T}$, let us define a radial function $\theta^* : \Omega \longrightarrow \mathbf{R}$ as the average value over spheres of θ: for $r \in \omega$ we take*

$$\theta^*(r) := \fint_{\partial B(\mathbf{0},r)} \theta \, dS \,,$$

where S denotes the surface measure on a sphere. Then θ^* is also minimizer for I over $\mathcal{T}$.

Spherical symmetry: $\min_{\mathcal{A}} J \Longleftrightarrow \min_{\mathcal{B}} J \Longleftrightarrow \min_{\mathcal{T}} I$

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric, and let the right-hand sides $f_i = f_i(r), r \in \omega, i = 1, \dots, m$ be radial functions. Then $\min_{\mathcal{A}} J = \min_{\mathcal{B}} J = \min_{\mathcal{T}} I$, and there exists a minimizer $(\theta^*, \mathbf{A}^*)$ of the optimal design problem $\min_{\mathcal{A}} J$ which is a radial function.More precisely,

a) *For any minimizer θ of functional I over $\mathcal{T}$, let us define a radial function $\theta^* : \Omega \longrightarrow \mathbf{R}$ as the average value over spheres of θ: for $r \in \omega$ we take*

$$\theta^*(r) := \fint_{\partial B(\mathbf{0},r)} \theta \, dS \,,$$

where S denotes the surface measure on a sphere. Then θ^* is also minimizer for I over $\mathcal{T}$.

Spherical symmetry. . . cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ as a simple laminate with the lamination direction orthogonal to the radial vector $\mathbf{e}_r$, almost everywhere on Ω. To be specific, we define*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(|\mathbf{x}|), \lambda^-_{\theta^*}(|\mathbf{x}|), \lambda^+_{\theta^*}(|\mathbf{x}|), \dots, \lambda^+_{\theta^*}(|\mathbf{x}|)\right) .$$

in spherical basis $(\mathbf{e}_r(\mathbf{x}), \mathbf{e}_{\phi_1}(\mathbf{x}), \mathbf{e}_{\phi_2}(\mathbf{x}), \dots, \mathbf{e}_{\phi_{d-1}}(\mathbf{x}))$. Then $(\theta^*, \mathbf{A}^*)$ is a radial optimal design for $\min_{\mathcal{B}} J$. Moreover, $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$, and thus it is also a solution for $\min_{\mathcal{A}} J$.

c) *If $(\widetilde{\theta}, \widetilde{\mathbf{A}}) \in \mathcal{A}$ is a solution of the relaxed problem $\min_{\mathcal{A}} J$ with corresponding state function $\widetilde{\mathrm{u}}$, then $\widetilde{\theta}$ is optimal for $\min_{\mathcal{T}} I$, and $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ is also a minimizer of J on $\mathcal{B}$. Consequently, we have $\widetilde{\mathrm{u}} = \mathrm{u}^*$, and $\widetilde{\mathbf{A}}\mathbf{e}_r = \lambda_+(\widetilde{\theta})\mathbf{e}_r$, almost everywhere.*

Spherical symmetry. . . cont.

Theorem

b) *For any radial minimizer* θ^* *of* I *over* $\mathcal{T}$*, let us define* $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ *as a simple laminate with the lamination direction orthogonal to the radial vector* $\mathbf{e}_r$*, almost everywhere on* Ω*. To be specific, we define*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(|\mathbf{x}|), \lambda^-_{\theta^*}(|\mathbf{x}|), \lambda^+_{\theta^*}(|\mathbf{x}|), \ldots, \lambda^+_{\theta^*}(|\mathbf{x}|)\right) .$$

in spherical basis $(\mathbf{e}_r(\mathbf{x}), \mathbf{e}_{\phi_1}(\mathbf{x}), \mathbf{e}_{\phi_2}(\mathbf{x}), \ldots, \mathbf{e}_{\phi_{d-1}}(\mathbf{x}))$*.*
Then $(\theta^*, \mathbf{A}^*)$ *is a radial optimal design for* $\min_{\mathcal{B}} J$*. Moreover,* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$*, and thus it is also a solution for* $\min_{\mathcal{A}} J$*.*

c) *If* $(\widetilde{\theta}, \widetilde{\mathbf{A}}) \in \mathcal{A}$ *is a solution of the relaxed problem* $\min_{\mathcal{A}} J$ *with corresponding state function* $\widetilde{\mathrm{u}}$*, then* $\widetilde{\theta}$ *is optimal for* $\min_{\mathcal{T}} I$*, and* $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ *is also a minimizer of* J *on* $\mathcal{B}$*. Consequently, we have* $\widetilde{\mathrm{u}} = \mathrm{u}^*$*, and* $\widetilde{\mathbf{A}}\mathbf{e}_r = \lambda_+(\widetilde{\theta})\mathbf{e}_r$*, almost everywhere.*

Spherical symmetry… cont.

Theorem

b) *For any radial minimizer* θ^* *of* I *over* $\mathcal{T}$*, let us define* $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ *as a simple laminate with the lamination direction orthogonal to the radial vector* $\mathbf{e}_r$*, almost everywhere on* Ω*.To be specific, we define*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(|\mathbf{x}|), \lambda^-_{\theta^*}(|\mathbf{x}|), \lambda^+_{\theta^*}(|\mathbf{x}|), \dots, \lambda^+_{\theta^*}(|\mathbf{x}|)\right) .$$

in spherical basis $(\mathbf{e}_r(\mathbf{x}), \mathbf{e}_{\phi_1}(\mathbf{x}), \mathbf{e}_{\phi_2}(\mathbf{x}), \dots, \mathbf{e}_{\phi_{d-1}}(\mathbf{x}))$*.* *Then* $(\theta^*, \mathbf{A}^*)$ *is a radial optimal design for* $\min_{\mathcal{B}} J$*.Moreover,* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$*, and thus it is also a solution for* $\min_{\mathcal{A}} J$*.*

c) *If* $(\widetilde{\theta}, \widetilde{\mathbf{A}}) \in \mathcal{A}$ *is a solution of the relaxed problem* $\min_{\mathcal{A}} J$ *with corresponding state function* $\widetilde{\mathrm{u}}$*, then* $\widetilde{\theta}$ *is optimal for* $\min_{\mathcal{T}} I$*, and* $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ *is also a minimizer of* J *on* $\mathcal{B}$*. Consequently, we have* $\widetilde{\mathrm{u}} = \mathrm{u}^*$*, and* $\widetilde{\mathbf{A}}\mathbf{e}_r = \lambda_+(\widetilde{\theta})\mathbf{e}_r$*, almost everywhere.*

Spherical symmetry... cont.

Theorem

b) *For any radial minimizer* θ^* *of* I *over* $\mathcal{T}$*, let us define* $\mathbf{A}^* \in \mathcal{K}(\theta^*)$ *as a simple laminate with the lamination direction orthogonal to the radial vector* $\mathbf{e}_r$*, almost everywhere on* Ω*.To be specific, we define*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(|\mathbf{x}|), \lambda^-_{\theta^*}(|\mathbf{x}|), \lambda^+_{\theta^*}(|\mathbf{x}|), \ldots, \lambda^+_{\theta^*}(|\mathbf{x}|)\right) .$$

in spherical basis $(\mathbf{e}_r(\mathbf{x}), \mathbf{e}_{\phi_1}(\mathbf{x}), \mathbf{e}_{\phi_2}(\mathbf{x}), \ldots, \mathbf{e}_{\phi_{d-1}}(\mathbf{x}))$*.*
Then $(\theta^*, \mathbf{A}^*)$ *is a radial optimal design for* $\min_{\mathcal{B}} J$*.Moreover,* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$*, and thus it is also a solution for* $\min_{\mathcal{A}} J$*.*

c) *If* $(\widetilde{\theta}, \widetilde{\mathbf{A}}) \in \mathcal{A}$ *is a solution of the relaxed problem* $\min_{\mathcal{A}} J$ *with corresponding state function* $\widetilde{\mathrm{u}}$*, then* $\widetilde{\theta}$ *is optimal for* $\min_{\mathcal{T}} I$*, and* $(\widetilde{\theta}, \widetilde{\mathbf{A}})$ *is also a minimizer of* J *on* $\mathcal{B}$*. Consequently, we have* $\widetilde{\mathrm{u}} = \mathrm{u}^*$*, and* $\widetilde{\mathbf{A}}\mathbf{e}_r = \lambda_+(\widetilde{\theta})\mathbf{e}_r$*, almost everywhere.*

Optimality conditions for $\min_{\mathcal{T}} I$

Lemma

$\theta^* \in \mathcal{T}$ is a solution $\min_{\mathcal{T}} I$ if and only if there exists a Lagrange multiplier $c \geq 0$ such that

$$\theta^* \in \langle 0,1 \rangle \quad \Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 = c\,,$$

$$\theta^* = 0 \quad \Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 \geq c\,,$$

$$\theta^* = 1 \quad \Rightarrow \quad \sum_{i=1}^m \mu_i |\nabla u_i^*|^2 \leq c\,,$$

or equivalently

$$\sum_{i=1}^m \mu_i |\nabla u_i^*|^2 > c \quad \Rightarrow \quad \theta^* = 0\,,$$

$$\sum_{i=1}^m \mu_i |\nabla u_i^*|^2 < c \quad \Rightarrow \quad \theta^* = 1\,.$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^+_{\theta(r)}u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$
\begin{aligned}
|u'(r)| > \gamma \quad &\Rightarrow \quad \theta^*(r) = 0 \\
&\Rightarrow \quad g_\beta := \frac{\psi}{\beta} > \gamma \\
|u'(r)| < \gamma \quad &\Rightarrow \quad \theta^*(r) = 1 \\
&\Rightarrow \quad g_\alpha := \frac{\psi}{\alpha} < \gamma \\
\theta^* \in \langle 0, 1\rangle \quad &\Rightarrow \quad |u'(r)| = \gamma \\
&\Rightarrow \quad \theta^*(r) = \frac{\beta\gamma - \psi(r)}{\gamma(\beta-\alpha)}
\end{aligned}
$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^+_{\theta(r)}u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$
\begin{aligned}
|u'(r)| > \gamma \quad &\Rightarrow \quad \theta^*(r) = 0 \\
&\Rightarrow \quad g_\beta := \frac{\psi}{\beta} > \gamma \\
|u'(r)| < \gamma \quad &\Rightarrow \quad \theta^*(r) = 1 \\
&\Rightarrow \quad g_\alpha := \frac{\psi}{\alpha} < \gamma \\
\theta^* \in \langle 0, 1\rangle \quad &\Rightarrow \quad |u'(r)| = \gamma \\
&\Rightarrow \quad \theta^*(r) = \frac{\beta\gamma - \psi(r)}{\gamma(\beta-\alpha)}
\end{aligned}
$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^+_{\theta(r)} u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$|u'(r)| > \gamma \quad\Rightarrow\quad \theta^*(r) = 0$$

$$\Rightarrow\quad g_\beta := \frac{\psi}{\beta} > \gamma$$

$$|u'(r)| < \gamma \quad\Rightarrow\quad \theta^*(r) = 1$$

$$\Rightarrow\quad g_\alpha := \frac{\psi}{\alpha} < \gamma$$

$$\theta^* \in \langle 0, 1\rangle \quad\Rightarrow\quad |u'(r)| = \gamma$$

$$\Rightarrow\quad \theta^*(r) = \frac{\beta\gamma - \psi(r)}{\gamma(\beta-\alpha)}$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^{+}_{\theta(r)}u'\right)' = 1 - r \,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$|u'(r)| > \gamma \quad \Rightarrow \quad \theta^*(r) = 0$$
$$\Rightarrow \quad g_\beta := \frac{\psi}{\beta} > \gamma$$
$$|u'(r)| < \gamma \quad \Rightarrow \quad \theta^*(r) = 1$$
$$\Rightarrow \quad g_\alpha := \frac{\psi}{\alpha} < \gamma$$
$$\theta^* \in \langle 0, 1\rangle \quad \Rightarrow \quad |u'(r)| = \gamma$$
$$\Rightarrow \quad \theta^*(r) = \frac{\beta\gamma-\psi(r)}{\gamma(\beta-\alpha)}$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^+_{\theta(r)}u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$\begin{aligned}
|u'(r)| > \gamma \quad &\Rightarrow \quad \theta^*(r) = 0 \\
&\Rightarrow \quad g_\beta := \frac{\psi}{\beta} > \gamma \\
|u'(r)| < \gamma \quad &\Rightarrow \quad \theta^*(r) = 1 \\
&\Rightarrow \quad g_\alpha := \frac{\psi}{\alpha} < \gamma \\
\theta^* \in \langle 0, 1\rangle \quad &\Rightarrow \quad |u'(r)| = \gamma \\
&\Rightarrow \quad \theta^*(r) = \frac{\beta\gamma-\psi(r)}{\gamma(\beta-\alpha)}
\end{aligned}$$

Ball $\Omega = B(0,2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

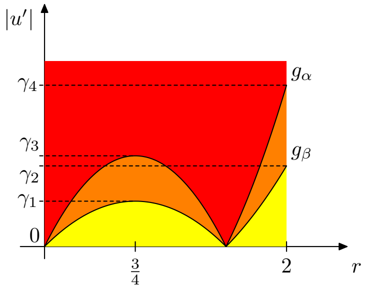
In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^+_{\theta(r)}u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$|u'(r)| > \gamma \quad \Rightarrow \quad \theta^*(r) = 0$$
$$\Rightarrow \quad g_\beta := \frac{\psi}{\beta} > \gamma$$
$$|u'(r)| < \gamma \quad \Rightarrow \quad \theta^*(r) = 1$$
$$\Rightarrow \quad g_\alpha := \frac{\psi}{\alpha} < \gamma$$
$$\theta^* \in \langle 0,1\rangle \quad \Rightarrow \quad |u'(r)| = \gamma$$
$$\Rightarrow \quad \theta^*(r) = \frac{\beta\gamma-\psi(r)}{\gamma(\beta-\alpha)}$$



Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

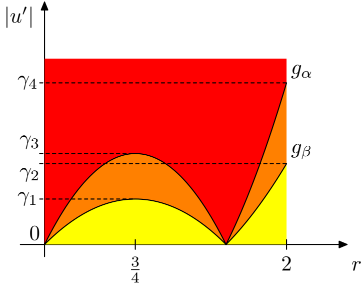
In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^+_{\theta(r)}u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$|u'(r)| > \gamma \quad\Rightarrow\quad \theta^*(r) = 0$$
$$\Rightarrow\quad g_\beta := \frac{\psi}{\beta} > \gamma$$
$$|u'(r)| < \gamma \quad\Rightarrow\quad \theta^*(r) = 1$$
$$\Rightarrow\quad g_\alpha := \frac{\psi}{\alpha} < \gamma$$
$$\theta^* \in \langle 0, 1\rangle \quad\Rightarrow\quad |u'(r)| = \gamma$$
$$\Rightarrow\quad \theta^*(r) = \frac{\beta\gamma - \psi(r)}{\gamma(\beta-\alpha)}$$



Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

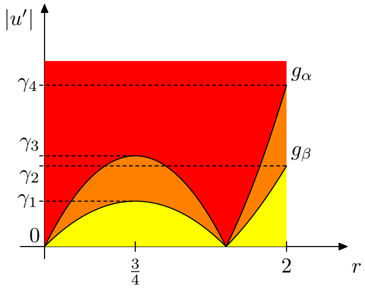
In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

State equation in polar coordinates $$-\frac{1}{r}\left(r\lambda^+_{\theta(r)}u'\right)' = 1 - r\,.$$

Integration gives $|u'(r)| = \frac{\psi(r)}{\alpha\theta(r)+\beta(1-\theta(r))}$, where $\psi(r) = \frac{|2r^2-3r|}{6}$.

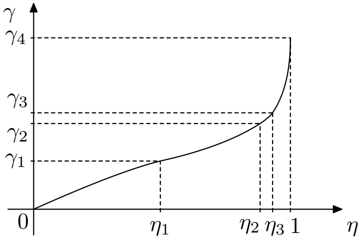
Conditions of optimality: there exists a constant $\gamma := \sqrt{c} > 0$ such that for optimal θ^* we have:

$$
\begin{aligned}
|u'(r)| > \gamma \quad &\Rightarrow \quad \theta^*(r) = 0\\
&\Rightarrow \quad g_\beta := \frac{\psi}{\beta} > \gamma\\
|u'(r)| < \gamma \quad &\Rightarrow \quad \theta^*(r) = 1\\
&\Rightarrow \quad g_\alpha := \frac{\psi}{\alpha} < \gamma\\
\theta^* \in \langle 0, 1\rangle \quad &\Rightarrow \quad |u'(r)| = \gamma\\
&\Rightarrow \quad \theta^*(r) = \frac{\beta\gamma - \psi(r)}{\gamma(\beta-\alpha)}
\end{aligned}
$$



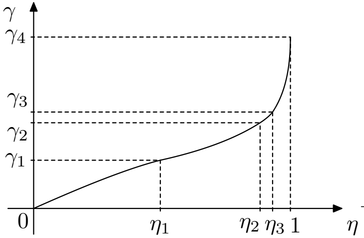
Ball with nonconstant right-hand side

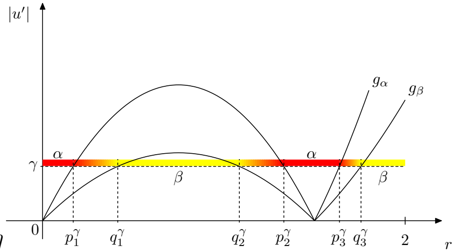
Lagrange multiplier γ is uniquely determined by the constraint $\fint_\Omega \theta^* \, d\mathbf{x} = \eta := \frac{q_\alpha}{|\Omega|} \in [0,1]$, which is alegraic equation for γ.



Ball with nonconstant right-hand side

Lagrange multiplier γ is uniquely determined by the constraint $\fint_\Omega \theta^* \, d\mathbf{x} = \eta := \frac{q_\alpha}{|\Omega|} \in [0,1]$, which is alegraic equation for γ.





Multiple (two) states on a ball $\Omega = B(\mathbf{0},2)$

- $f_1 = \chi_{B(\mathbf{0},1)}\,,\ f_2 \equiv 1\,,$
- $\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1,2$
- $\mu \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \min$

Solving state equation

$$u_i'(r) = \frac{\psi_i(r)}{\theta(r)\alpha + (1-\theta(r))\beta}\,,\ i = 1,2\,,$$

with

$$\psi_1(r) = \begin{cases} -\frac{r}{2}\,, & 0 \le r < 1\,, \\ -\frac{1}{2r}\,, & 1 \le r \le 2\,, \end{cases} \quad \text{and} \quad \psi_2(r) = -\frac{r}{2}\,.$$

Similarly as in the first example: $\psi := \mu\psi_1^2 + \psi_2^2$, $g_\alpha := \frac{\psi}{\alpha^2}$, $g_\beta := \frac{\psi}{\beta^2}$.

Multiple (two) states on a ball $\Omega = B(\mathbf{0}, 2)$

- $f_1 = \chi_{B(\mathbf{0},1)}\,,\ f_2 \equiv 1\,,$
- $\begin{cases} -\mathsf{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, 2$
- $\mu \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \min$

Solving state equation

$$u_i'(r) = \frac{\psi_i(r)}{\theta(r)\alpha + (1 - \theta(r))\beta}\,,\ i = 1, 2\,,$$

with

$$\psi_1(r) = \begin{cases} -\frac{r}{2}\,, & 0 \le r < 1\,, \\ -\frac{1}{2r}\,, & 1 \le r \le 2\,, \end{cases} \quad \text{and} \quad \psi_2(r) = -\frac{r}{2}\,.$$

Similarly as in the first example: $\psi := \mu\psi_1^2 + \psi_2^2$, $g_\alpha := \frac{\psi}{\alpha^2}$, $g_\beta := \frac{\psi}{\beta^2}$.

Multiple (two) states on a ball $\Omega = B(\mathbf{0}, 2)$

- $f_1 = \chi_{B(\mathbf{0},1)}\,,\ f_2 \equiv 1\,,$
- $\left\{ \begin{array}{l} -\mathsf{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. \qquad i = 1, 2$
- $\mu \int_\Omega f_1 u_1\, d\mathbf{x} + \int_\Omega f_2 u_2\, d\mathbf{x} \to \min$

Solving state equation

$$u_i'(r) = \frac{\psi_i(r)}{\theta(r)\alpha + (1-\theta(r))\beta}\,,\ i = 1, 2\,,$$

with

$$\psi_1(r) = \begin{cases} -\frac{r}{2}\,, & 0 \le r < 1\,, \\ -\frac{1}{2r}\,, & 1 \le r \le 2\,, \end{cases} \quad \text{and} \quad \psi_2(r) = -\frac{r}{2}\,.$$

Similarly as in the first example: $\psi := \mu\psi_1^2 + \psi_2^2$, $g_\alpha := \frac{\psi}{\alpha^2}$, $g_\beta := \frac{\psi}{\beta^2}$.

Multiple (two) states on a ball $\Omega = B(\mathbf{0}, 2)$

- $f_1 = \chi_{B(\mathbf{0},1)}\,,\ f_2 \equiv 1\,,$
- $\begin{cases} -\text{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, 2$
- $\mu \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \min$

Solving state equation

$$u_i'(r) = \frac{\psi_i(r)}{\theta(r)\alpha + (1 - \theta(r))\beta}\,,\ i = 1, 2\,,$$

with

$$\psi_1(r) = \begin{cases} -\frac{r}{2}\,, & 0 \le r < 1\,, \\ -\frac{1}{2r}\,, & 1 \le r \le 2\,, \end{cases} \quad \text{and} \quad \psi_2(r) = -\frac{r}{2}\,.$$

Similarly as in the first example: $\psi := \mu\psi_1^2 + \psi_2^2$, $g_\alpha := \frac{\psi}{\alpha^2}$, $g_\beta := \frac{\psi}{\beta^2}$.

Multiple (two) states on a ball $\Omega = B(\mathbf{0},2)$

- $f_1 = \chi_{B(\mathbf{0},1)}\,,\ f_2 \equiv 1\,,$
- $\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1,2$
- $\mu \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \min$

Solving state equation

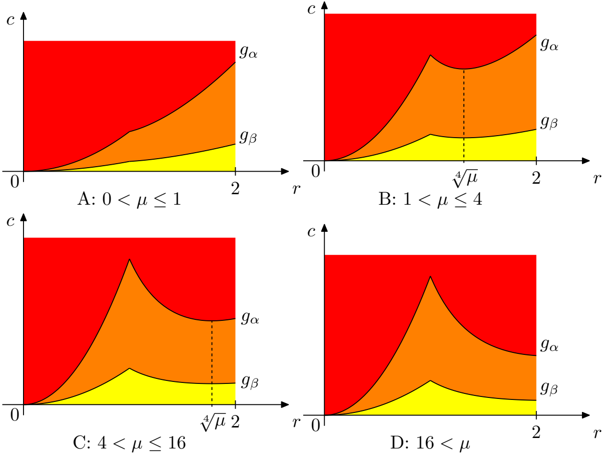
$$u_i'(r) = \frac{\psi_i(r)}{\theta(r)\alpha + (1-\theta(r))\beta}\,,\ i = 1,2\,,$$

with

$$\psi_1(r) = \begin{cases} -\frac{r}{2}\,, & 0 \le r < 1\,, \\ -\frac{1}{2r}\,, & 1 \le r \le 2\,, \end{cases} \quad \text{and} \quad \psi_2(r) = -\frac{r}{2}\,.$$

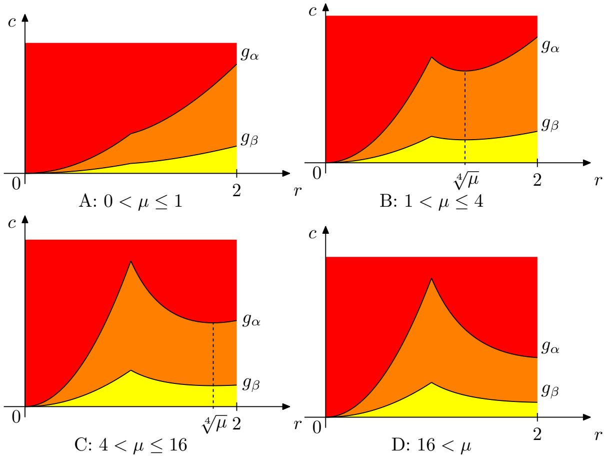
Similarly as in the first example: $\psi := \mu\psi_1^2 + \psi_2^2$, $g_\alpha := \frac{\psi}{\alpha^2}$, $g_\beta := \frac{\psi}{\beta^2}$.

Geometric interpretation of optimality conditions



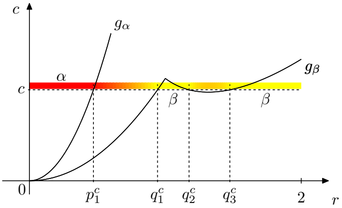
As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $\int_\Omega \theta^* \, d\mathbf{x} = \eta$.

Geometric interpretation of optimality conditions



As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $\fint_\Omega \theta^* \, d\mathbf{x} = \eta$.

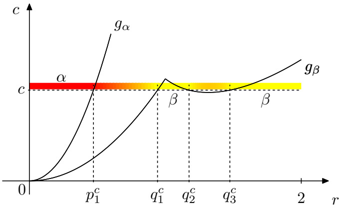
Optimal θ^* for case B



In orange region:

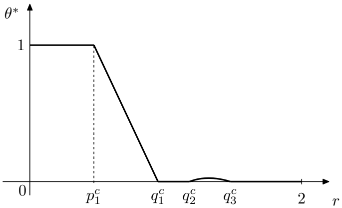
$$\theta^*(r) = \frac{1}{\beta - \alpha}\left(\beta - \sqrt{\frac{\psi(r)}{c}}\right)$$

Optimal θ^* for case B

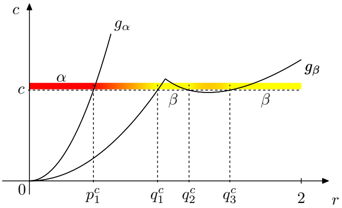


In orange region:

$$\theta^*(r) = \frac{1}{\beta - \alpha}\left(\beta - \sqrt{\frac{\psi(r)}{c}}\right)$$



Optimal θ^* for case B



In orange region:

$$\theta^*(r) = \frac{1}{\beta - \alpha}\left(\beta - \sqrt{\frac{\psi(r)}{c}}\right)$$