Anisotropic distributions and compactness by compensation

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What are H-measures?

Mathematical objects introduced (1989/90) by:

- Luc Tartar, who was motivated by possible applications in homogenisation, and independently by
- Patrick Gérard, whose motivation were problems in kinetic theory.

**Theorem 1.** If $u_n \rightharpoonup 0$ and $v_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$, then there exist their subsequences and a complex valued Radon measure $\mu$ on $\mathbb{R}^d \times S^{d-1}$, such that for any $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)$ and $\psi \in C(S^{d-1})$ one has

$$
\lim_{n'} \int_{\mathbb{R}^d} \varphi_1 u_{n'} \varphi_2 v_{n'} (\psi \circ \pi) d\xi = \langle \mu, \varphi_1 \varphi_2 \boxtimes \psi \rangle,
$$

where $\pi : \mathbb{R}^d \setminus \{0\} \longrightarrow S^{d-1}$ is the projection along rays.
Question: How to replace $L^2$ with $L^p$?

Notice: if we denote by $A_\psi$ the Fourier multiplier operator with symbol $\psi \in L^\infty(\mathbb{R}^d)$:

$$A_\psi(u) = (\psi \hat{u})^\vee,$$

we can rewrite the equality from the theorem as

$$\langle \mu, \varphi_1 \varphi_2 \square \psi \rangle = \lim_{n'} \int_{\mathbb{R}^d} \varphi_1 u_{n'} \varphi_2 v_{n'} (\psi \circ \pi) d\xi$$

$$= \lim_{n'} \int_{\mathbb{R}^d} \varphi_1 u_{n'}(x) A_{\psi \circ \pi}(\varphi_2 u_{n'})(x) dx .$$
Theorem 2. Let $\psi \in L^\infty(\mathbb{R}^d)$ have partial derivatives of order less than or equal to $\kappa = \lceil d/2 \rceil + 1$. If for some $k > 0$

\[(\forall r > 0)(\forall \alpha \in \mathbb{N}_0^d) \quad |\alpha| \leq \kappa \implies \int_{r/2 \leq |\xi| \leq r} |\partial^\alpha \psi(\xi)|^2 d\xi \leq k^2 r^{d-2|\alpha|},\]

then for any $p \in (1, \infty)$ and the associated multiplier operator $A_\psi$ there exists a constant $C_d$ such that

\[\|A_\psi\|_{L^p \to L^p} \leq C_d \max\{p, 1/(p - 1)\}(k + \|\psi\|_{L^\infty(\mathbb{R}^d)}).
\]

For $\psi \in C^\kappa(S^{d-1})$, extended by homogeneity to $\mathbb{R}^d \setminus \{0\}$, we can take $k = \|\psi\|_{C^\kappa(S^{d-1})}$. 

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What is the First commutation lemma?

- \( A_\psi u := (\psi \hat{u})^\vee \)
- \( M_b u := bu \)

\[
[A_\psi, M_b] := A_\psi M_b - M_b A_\psi
\]

**Question:** Why do we need such a result?

\[
\langle \mu, \varphi_1 \varphi_2 \otimes \psi \rangle = \lim_{n'} \int_{\mathbb{R}^d} \varphi_1 u_{n'}(x) \overline{A_{\psi \circ \pi} (\varphi_2 u_{n'})(x)} dx
\]

\[
= \lim_{n'} \int_{\mathbb{R}^d} M_{\varphi_1} u_{n'}(x) \overline{A_{\psi \circ \pi} (M_{\varphi_2} u_{n'})(x)} dx.
\]
Compactness on $L^2$ - Cordes’ result$^1$

**Theorem**

If bounded continuous functions $b$ and $\psi$ satisfy

$$\lim_{|\xi| \to \infty} \sup_{|h| \leq 1} \{ |\psi(\xi+h) - \psi(\xi)| \} = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \sup_{|h| \leq 1} \{ |b(x+h) - b(x)| \} = 0 ,$$

then the commutator $[A_\psi, M_b]$ is a compact operator on $L^2(\mathbb{R}^d)$.

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Compactness on $L^2$ - Tartar’s version

For given $M, \varrho \in \mathbb{R}^+$ we denote the set

$$Y(M, \varrho) = \{(\xi, \eta) \in \mathbb{R}^{2d} : |\xi|, |\eta| \geq M \& |\xi - \eta| \leq \varrho\}.$$
Compactness on $L^2$ - Tartar's version

Lemma (general form of the First commutation lemma)

If $b \in C_0(\mathbb{R}^d)$, while $\psi \in L^\infty(\mathbb{R}^d)$ satisfies the condition
\[
(\forall \varrho, \varepsilon \in \mathbb{R}^+ ) ( \exists M \in \mathbb{R}^+) \ |\psi(\xi) - \psi(\eta)| \leq \varepsilon \ (s.s. \ (\xi, \eta) \in Y(M, \varrho) ), \quad (1)
\]
then $[A_\psi, M_b]$ is a compact operator on $L^2(\mathbb{R}^d)$.

Lemma

Let $\pi : \mathbb{R}^d_* \to \Sigma$ be a smooth projection to a smooth compact hypersurface $\Sigma$, such that $\|\nabla \pi(\xi)\| \to 0$ for $|\xi| \to \infty$, and let $\psi \in C(\Sigma)$. Then $\psi \circ \pi$ (\psi extended by homogeneity of order 0) satisfies (1).

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Where is it used?


What about the $L^p$ variant of the First commutation lemma?

One variant can be found in the article by Cordes - complicated proof and higher regularity assumptions. Namely, the symbol is required to satisfy:

- $\psi \in C^{2\kappa}(\mathbb{R}^d)$,
- for every $\alpha \in \mathbb{N}_0^d, |\alpha| \leq 2\kappa$:
  $$ (1 + |\xi|)^{|\alpha|} D^\alpha \psi(\xi) \text{ is bounded.} $$

A different variant was given by Antonić and Mitrović$^4$:

**Lemma**

Assume $\psi \in C^\kappa(S^{d-1})$ and $b \in C_0(\mathbb{R}^d)$. Let $(v_n)$ be a bounded sequence, both in $L^2(\mathbb{R}^d)$ and in $L^r(\mathbb{R}^d)$, for some $r \in (2, \infty]$, and such that $v_n \rightharpoonup 0$ in the sense of distributions.

Then $[A_\psi, M_b]v_n \rightarrow 0$ strongly in $L^q(\mathbb{R}^d)$, for any $q \in [2, r)$.

The proof was based on a simple interpolation inequality of $L^p$ spaces:

$$ \|f\|_{L^q} \leq \|f\|_{L^2}^{\theta} \|f\|_{L^r}^{1-\theta}, \text{ where } 1/q = \theta/2 + (1-\theta)/r. $$

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A variant of Krasnoselskij’s type of result\textsuperscript{5}

Lemma
Assume that linear operator $A$ is compact on $L^2(\mathbb{R}^d)$ and bounded on $L^r(\mathbb{R}^d)$, for some $r \in \langle 1, \infty \rangle \setminus \{2\}$. Then $A$ is also compact on $L^p(\mathbb{R}^d)$, for any $p$ between 2 and $r$ (i.e. such that $1/p = \theta/2 + (1 - \theta)/r$, for some $\theta \in \langle 0, 1 \rangle$).

Corollary
If $b \in C_0(\mathbb{R}^d)$, while $\psi \in C^\infty(\mathbb{R}^d)$ satisfies the conditions of the Hörmander-Mihlin theorem, then the commutator $[A\psi, M_b]$ is a compact operator on $L^p(\mathbb{R}^d)$, for any $p \in \langle 1, \infty \rangle$.

Theorem
Let $\psi \in \mathcal{C}^{\kappa}(\mathbb{R}^d \setminus \{0\})$ be bounded and satisfy Hörmander’s condition, while $b \in \mathcal{C}_c(\mathbb{R}^d)$. Then for any $u_n \rightharpoonup 0$ in $L^\infty(\mathbb{R}^d)$ and $p \in \langle 1, \infty \rangle$ one has:

$$\left( \forall \varphi, \phi \in \mathcal{C}_c^\infty(\mathbb{R}^d) \right) \phi C(\varphi u_n) \longrightarrow 0 \quad \text{in} \quad L^p(\mathbb{R}^d).$$

Corollary
Let $(u_n)$ be a bounded, uniformly compactly supported sequence in $L^\infty(\mathbb{R}^d)$, converging to 0 in the sense of distributions. Assume that $\psi \in \mathcal{C}^{\kappa}(\mathbb{R}^d \setminus \{0\})$ satisfies Hörmander’s condition and condition from the general form of the First commutation lemma.
Then for any $b \in L^s(\mathbb{R}^d)$, $s > 1$ arbitrary, it holds

$$\lim_{n \to \infty} \| b A_\psi(u_n) - A_\psi(b u_n) \|_r(\mathbb{R}^d) = 0, \quad r \in \langle 1, s \rangle.$$
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H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović as an extension of H-measures to the $L^p - L^q$ context.

Existing applications are related to the velocity averaging$^6$ and $L^p - L^q$ compactness by compensation$^7$.

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**Theorem 3.** If \( u_n \rightarrow 0 \) in \( L^p_{\text{loc}}(\mathbb{R}^d) \) and \( v_n \rightharpoonup^* v \) in \( L^q_{\text{loc}}(\mathbb{R}^d) \) for some \( p \in (1, \infty) \) and \( q \geq p' \), then there exist subsequences \((u_{n'})\), \((v_{n'})\) and a complex valued distribution \( \mu \in D'(\mathbb{R}^d \times S^{d-1}) \), such that, for every \( \varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^d) \) and \( \psi \in C^\kappa(S^{d-1}) \), for \( \kappa = [d/2] + 1 \), one has:

\[
\lim_{n' \to \infty} \int_{\mathbb{R}^d} A_\psi(\varphi_1 u_{n'})(x)(\varphi_2 v_{n'})(x)dx = \lim_{n' \to \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'})(x)\overline{A_\psi(\varphi_2 v_{n'})(x)}dx
= \langle \mu, \varphi_1 \overline{\varphi_2} \boxtimes \psi \rangle,
\]

where \( A_\psi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) is the Fourier multiplier operator with symbol \( \psi \in C^\kappa(S^{d-1}) \).
Distributions of anisotropic order

Let $X$ and $Y$ be open sets in $\mathbb{R}^d$ and $\mathbb{R}^r$ (or $C^\infty$ manifolds of dimensions $d$ and $r$) and $\Omega \subseteq X \times Y$ an open set. By $C^{l,m}(\Omega)$ we denote the space of functions $f$ on $\Omega$, such that for any $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0^r$, if $|\alpha| \leq l$ and $|\beta| \leq m$, $\partial^{\alpha,\beta} f = \partial_x^\alpha \partial_y^\beta f \in C(\Omega)$.

$C^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$ p_{K_n}^{l,m}(f) := \max_{|\alpha| \leq l, |\beta| \leq m} \| \partial^{\alpha,\beta} f \|_{L^\infty(K_n)} , $$

where $K_n \subseteq \Omega$ are compacts, such that $\Omega = \bigcup_{n \in \mathbb{N}} K_n$ and $K_n \subseteq \text{Int} K_{n+1}$.

Consider the space

$$ C^{l,m}_c(\Omega) := \bigcup_{n \in \mathbb{N}} C^{l,m}_{K_n}(\Omega) , $$

and equip it by the topology of strict inductive limit.
**Definition.** A *distribution of order* $l$ *in* $x$ *and order* $m$ *in* $y$ is any linear functional on* $C^{l,m}_c(\Omega)$, *continuous in the strict inductive limit topology. We denote the space of such functionals by* $D'_{l,m}(\Omega)$.

**Conjecture.** Let $X,Y$ be $C^\infty$ manifolds and let $u$ be a linear functional on $C^{l,m}_c(X \times Y)$. If $u \in D'(X \times Y)$ and satisfies
\[
(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y))(\exists C > 0)(\forall \varphi \in C^\infty_K(X))(\forall \psi \in C^\infty_L(Y))
\]
\[
|\langle u, \varphi \boxtimes \psi \rangle| \leq C p^l_K(\varphi) p^m_L(\psi),
\]
then $u$ can be uniquely extended to a continuous functional on $C^{l,m}_c(X \times Y)$ (i.e. it can be considered as an element of $D'_{l,m}(X \times Y)$). □
From the proof of the existence theorem, we already have $\mu \in D'(\mathbb{R}^d \times S^{d-1})$ and the following bound with $\varphi := \varphi_1 \varphi_2$:

$$|\langle \mu, \varphi \Box \psi \rangle| \leq C \|\psi\|_{C^\kappa(S^{d-1})} \|\varphi\|_{C_{\text{Kl}}}(\mathbb{R}^d),$$

where $C$ does not depend on $\varphi$ and $\psi$.

If the conjecture were true, then the H-distribution $\mu$ from the preceding theorem belongs to the space $D'_{0,\kappa}(\mathbb{R}^d \times S^{d-1})$, i.e. it is a distribution of order 0 in $x$ and of order not more than $\kappa$ in $\xi$.

But the conjecture is not true. Indeed, take a distribution $u = \frac{-1}{\pi} \partial_y \ln |x - y|$ on $\mathbb{R}^2$. It is an element of $D'_{0,1}(\mathbb{R} \times \mathbb{R})$. It holds:

$$\langle u, \varphi(x)\psi(y) \rangle = \frac{1}{\pi} \int \varphi(x) \int \ln |x - y|\psi'(y) \, dy \, dx = \int \varphi(x)H\psi(x) \, dx,$$

$$|\langle u, \varphi(x)\psi(y) \rangle| \leq C_{\text{supp } \varphi, \text{supp } \psi} \|\varphi\|_{L^\infty} \|\psi\|_{L^\infty}.$$

If $u$ were locally finite measure on $\mathbb{R}^2$, in case $\text{supp } g$ does not intersect the diagonal we would get $\langle u, g \rangle = \frac{1}{\pi} \int \frac{g(x,y)}{x-y} \, dx \, dy$. 
Let $X$ and $Y$ be two $C^\infty$ manifolds. Then the following statements hold:

a) Let $K \in \mathcal{D}'(X \times Y)$. Then for every $\varphi \in \mathcal{D}(X)$, the linear form $K_\varphi$ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution on $Y$. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathcal{D}(X)$ to $\mathcal{D}'(Y)$ is linear and continuous.

b) Let $A : \mathcal{D}(X) \to \mathcal{D}'(Y)$ be a continuous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$
Schwartz kernel theorem for anisotropic distributions

Let $X$ and $Y$ be two $C^\infty$ manifolds of dimensions $d$ and $r$, respectively. Then the following statements hold:

a) Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for every $\varphi \in C^l_c(X)$, the linear form $K_\varphi$ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution of order not more than $m$ on $Y$. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $C^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.

b) Let $A : C^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A \varphi, \psi \rangle.$$  

Furthermore, $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$. 

How to prove it?

Use the structure theorem of distributions (Dieudonné).

Two steps:

**Step I:** assume the range of $A$ is $C(Y)$
**Step II:** use structure theorem and go back to Step I

Consequence: H-distributions are of order 0 in $x$ and of finite order not greater than $d(\kappa + 2)$ with respect to $\xi$. 
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Motivation - Maxwell’s equations

Let $\Omega \subseteq \mathbb{R}^3$. Denote by $E$ and $H$ the electric and magnetic field, and by $D$ and $B$ the electric and magnetic induction. Let $\rho$ denote the charge, and $j$ the current density. Maxwell’s system of equations reads:

$$\partial_t B + \text{rot } E = G,$$
$$\text{div } B = 0,$$
$$\partial_t D + j - \text{rot } H = F,$$
$$\text{div } D = \rho.$$

Assume that properties of the material can be expressed by following linear constitutive equations:

$$D = \varepsilon E, \quad B = \mu H.$$ 

The energy of electromagnetic field at time $t$ is given by:

$$T(t) = \frac{1}{2} \int_\Omega (D \cdot E + B \cdot H) dx.$$

It’s natural to consider

$$D, B \in L^\infty([0, T]; L^2_{\text{div}}(\Omega; \mathbb{R}^3)),$$
$$E, H \in L^\infty([0, T]; L^2_{\text{rot}}(\Omega; \mathbb{R}^3)),$$
$$J \in L^\infty([0, T]; L^2(\Omega; \mathbb{R}^3)), \quad F, G \in L^2([0, T]; L^2(\Omega; \mathbb{R}^3)).$$
Let us consider a family of problems:

\[
\begin{align*}
\partial_t B^n + \text{rot} E^n &= G^n, \\
\partial_t D^n + J^n - \text{rot} H^n &= F^n,
\end{align*}
\]

with constitution equations:

\[
D^n = \epsilon^n E^n, \quad B^n = \mu^n H^n, \quad J^n = \sigma^n E^n.
\]

What can we say about energy \( T(t) \) if we know

\[
T^n(t) = \frac{1}{2} \int_{\Omega} (D^n \cdot E^n + B^n \cdot H^n) dx.
\]
**Theorem 4.** Assume that $\Omega$ is open and bounded subset of $\mathbb{R}^3$, and that it holds:

$$ u_n \rightharpoonup u \text{ in } L^2(\Omega; \mathbb{R}^3), $$
$$ v_n \rightharpoonup v \text{ in } L^2(\Omega; \mathbb{R}^3), $$
$$ \text{rot } u_n \text{ bounded in } L^2(\Omega; \mathbb{R}^3), \text{ div } v_n \text{ bounded in } L^2(\Omega). $$

Then

$$ u_n \cdot v_n \rightharpoonup u \cdot v $$

in the sense of distributions.
Theorem 5. (Quadratic theorem) Assume that $\Omega \subseteq \mathbb{R}^d$ is open and that $
abla \subseteq \mathbb{R}^r$ is defined by

$$\Lambda := \left\{ \lambda \in \mathbb{R}^r : (\exists \xi \in \mathbb{R}^d \setminus \{0\}) \sum_{k=1}^{d} \xi_k A^k \lambda = 0 \right\},$$

where $Q$ is a real quadratic form on $\mathbb{R}^r$, which is nonnegative on $\Lambda$, i.e.

$$(\forall \lambda \in \Lambda) \quad Q(\lambda) \geq 0.$$

Furthermore, assume that the sequence of functions $(u_n)$ satisfies

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^2_{\text{loc}}(\Omega; \mathbb{R}^r),$$

$$\left( \sum_k A^k \partial_k u_n \right) \quad \text{relatively compact in} \quad H^{-1}_{\text{loc}}(\Omega; \mathbb{R}^q).$$

Then every subsequence of $(Q \circ u_n)$ which converges in distributions to its limit $L$, satisfies

$$L \geq Q \circ u$$

in the sense of distributions.
The most general version of the classical $L^2$ results has recently been proved by E. Yu. Panov\textsuperscript{9}:

Assume that the sequence $(u_n)$ is bounded in $L^p(\mathbb{R}^d;\mathbb{R}^r)$, $2 \leq p < \infty$, and converges weakly in $\mathcal{D}'(\mathbb{R}^d)$ to a vector function $u$. Let $q = p'$ if $p < \infty$, and $q > 1$ if $p = \infty$. Assume that the sequence

$$\sum_{k=1}^{\nu} \partial_k(A^k u_n) + \sum_{k,l=\nu+1}^{d} \partial_{kl}(B^{kl} u_n)$$

is precompact in the anisotropic Sobolev space $W^{-1,-2;\tilde{q}}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^m)$, where $m \times r$ matrices $A^k$ and $B^{kl}$ have variable coefficients belonging to $L^{2\tilde{q}}(\mathbb{R}^d)$, $\tilde{q} = \frac{p}{p-2}$ if $p > 2$, and to the space $C(\mathbb{R}^d)$ if $p = 2$.

We introduce the set $\Lambda(x)$

$$\Lambda(x) = \left\{ \lambda \in C^r \mid (\exists \xi \in \mathbb{R}^d \setminus \{0\}) : \lambda \right.$$ 

$$\left( i \sum_{k=1}^{\nu} \xi_k A^k(x) - 2\pi \sum_{k,l=\nu+1}^{d} \xi_k \xi_l B^{kl}(x) \right) \lambda = 0 \right\},$$

(2)

and consider the bilinear form on $C^r$

$$q(x, \lambda, \eta) = Q(x) \lambda \cdot \eta,$$

(3)

where $Q \in L^{\bar{q}}_{loc}(\mathbb{R}^d; \text{Sym}_r)$ if $p > 2$ and $Q \in C(\mathbb{R}^d; \text{Sym}_r)$ if $p = 2$.

Finally, let $q(x, u_n, u_n) \rightharpoonup \omega$ weakly in the space of distributions.
The following theorem holds

**Theorem 6.** Assume that $(\forall \lambda \in \Lambda(x)) \ q(x, \lambda, \lambda) \geq 0 \ (a.e. \ x \in \mathbb{R}^d)$ and $u_n \rightharpoonup u$, then $q(x, u(x), u(x)) \leq \omega$.

The connection between $q$ and $\Lambda$ given in the previous theorem, we shall call the consistency condition.
We need Fourier multiplier operators with symbols defined on a manifold $P$ determined by $d$-tuple $\alpha \in (\mathbb{R}^+)^d$:

$$P = \left\{ \xi \in \mathbb{R}^d : \sum_{k=1}^{d} |\xi_k|^{2\alpha_k} = 1 \right\},$$

where $\alpha_k \in \mathbb{N}$ or $\alpha_k \geq d$. In order to associate an $L^p$ Fourier multiplier to a function defined on $P$, we extend it to $\mathbb{R}^d \setminus \{0\}$ by means of the projection

$$(\pi_P(\xi))_j = \xi_j \left( |\xi_1|^{2\alpha_1} + \cdots + |\xi_d|^{2\alpha_d} \right)^{-1/2\alpha_j}, \quad j = 1, \ldots, d.$$
We need the following extension of the results given above.

**Theorem 7.** Let \((u_n)\) be a bounded sequence in \(L^p(\mathbb{R}^d)\), \(p > 1\), and let \((v_n)\) be a bounded sequence of uniformly compactly supported functions in \(L^q(\mathbb{R}^d)\), \(1/q + 1/p < 1\). Then, after passing to a subsequence (not relabelled), for any \(\bar{s} \in (1, \frac{pq}{p+q})\) there exists a continuous bilinear functional \(B\) on \(L^{\bar{s}'}(\mathbb{R}^d) \otimes C^d(P)\) such that for every \(\varphi \in L^{\bar{s}'}(\mathbb{R}^d)\) and \(\psi \in C^d(P)\), it holds

\[
B(\varphi, \psi) = \lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(x) u_n(x) (A_{\psi_P} v_n)(x) dx,
\]

where \(A_{\psi_P}\) is the Fourier multiplier operator on \(\mathbb{R}^d\) associated to \(\psi \circ \pi_P\). The bilinear functional \(B\) can be continuously extended\(^{10}\) as a linear functional on \(L^{\bar{s}'}(\mathbb{R}^d; C^d(P))\).

For separable Banach space $E$, the dual of $L^p(\mathbb{R}^d; E)$ consists of all weakly-\* measurable functions $B : \mathbb{R}^d \to E'$ such that
\[
\int_{\mathbb{R}^d} \|B(x)\|_{E'}^{p'} \, dx
\]
is finite\textsuperscript{11}.

Sometimes the dual is denoted by $L_{w^*}^{p'}(\mathbb{R}^d; E')$.

Lemma

Assume that sequences \((u_n)\) and \((v_n)\) are bounded in \(L^p(\mathbb{R}^d; \mathbb{R}^r)\) and \(L^q(\mathbb{R}^d; \mathbb{R}^r)\), respectively, and converge toward \(0\) and \(v\) in the sense of distributions.

Furthermore, assume that sequence \((u_n)\) satisfies:

\[
G_n := \sum_{k=1}^{d} \partial_{\alpha_k}^k (A^k u_n) \to 0 \text{ in } W^{-\alpha_1, \ldots, -\alpha_d; p}(\Omega; \mathbb{R}^m),
\]

where either \(\alpha_k \in \mathbb{N}, k = 1, \ldots, d\) or \(\alpha_k > d, k = 1, \ldots, d\), and elements of matrices \(A^k\) belong to \(L^{\bar{s}'}(\mathbb{R}^d)\), \(\bar{s} \in (1, \frac{pq}{p+q})\).

Finally, by \(\mu\) denote a matrix \(H\)-distribution corresponding to subsequences of \((u_n)\) and \((v_n - v)\). Then the following relation holds

\[
\left( \sum_{k=1}^{d} (2\pi i \xi_k)^{\alpha_k} A^k \right) \mu = 0.
\]
Strong consistency condition

Introduce the set

\[ \Lambda_D = \left\{ \mu \in L^\bar{s}(\mathbb{R}^d; (C^d(P))')^r : \left( \sum_{k=1}^{n} (2\pi i \xi_k)^{\alpha_k} A^k \right) \mu = 0_m \right\}, \]

where the given equality is understood in the sense of \( L^\bar{s}(\mathbb{R}^d; (C^d(P))')^m \).

Let us assume that coefficients of the bilinear form \( q \) from (3) belong to space \( L^t_{loc}(\mathbb{R}^d) \), where \( 1/t + 1/p + 1/q < 1 \).

**Definition**

We say that set \( \Lambda_D \), bilinear form \( q \) from (3) and matrix \( \mu = [\mu_1, \ldots, \mu_r] \), \( \mu_j \in L^\bar{s}(\mathbb{R}^d; (C^d(P))')^r \) satisfy the strong consistency condition if (\( \forall j \in \{1, \ldots, r\} \)) \( \mu_j \in \Lambda_D \), and it holds

\[ \langle \phi Q \otimes 1, \mu \rangle \geq 0, \quad \phi \in L^\bar{s}(\mathbb{R}^d; \mathbb{R}_0^+). \]
Theorem 8. Assume that sequences \((u_n)\) and \((v_n)\) are bounded in \(L^p(\mathbb{R}^d; \mathbb{R}^r)\) and \(L^q(\mathbb{R}^d; \mathbb{R}^r)\), respectively, and converge toward \(u\) and \(v\) in the sense of distributions. Assume that (4) holds and that
\[ q(x; u_n, v_n) \rightharpoonup \omega \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d). \]

If the set \(\Lambda_D\), the bilinear form (3), and matrix \(H\)-distribution \(\mu\), corresponding to subsequences of \((u_n - u)\) and \((v_n - v)\), satisfy the strong consistency condition, then
\[ q(x; u, v) \leq \omega \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d). \]
Application to the parabolic type equation

Now, let us consider the non-linear parabolic type equation

\[ L(u) = \partial_t u - \text{div} \text{ div} \left( g(t, x, u)A(t, x) \right), \]

on \((0, \infty) \times \Omega\), where \(\Omega\) is an open subset of \(\mathbb{R}^d\). We assume that

\[ u \in L^p((0, \infty) \times \Omega), \quad g(t, x, u(t, x)) \in L^q((0, \infty) \times \Omega), \quad 1 < p, q, \]

\[ A \in L^s_{loc}((0, \infty) \times \Omega)^{d \times d}, \quad \text{where} \quad 1/p + 1/q + 1/s < 1, \]

and that the matrix \(A\) is strictly positive definite, i.e.

\[ A\xi \cdot \xi > 0, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \quad (a.e. (t, x) \in (0, \infty) \times \Omega). \]

Furthermore, assume that \(g\) is a Carathéodory function and non-decreasing with respect to the third variable.
Then we have the following theorem.

**Theorem 9.** Assume that sequences \((u_r)\) and \(g(\cdot, u_r)\) are such that \(u_r, g(u_r) \in L^2(\mathbb{R}^+ \times \mathbb{R}^d)\) for every \(r \in \mathbb{N}\); assume that they are bounded in \(L^p(\mathbb{R}^+ \times \mathbb{R}^d)\), \(p \in (1, 2]\), and \(L^q(\mathbb{R}^+ \times \mathbb{R}^d)\), \(q > 2\), respectively, where \(1/p + 1/q < 1\); furthermore, assume \(u_r \rightharpoonup u\) and, for some, \(f \in W^{-1, -2;p}(\mathbb{R}^+ \times \mathbb{R}^d)\), the sequence

\[
L(u_r) = f_r \to f \quad \text{strongly in} \quad W^{-1, -2;p}(\mathbb{R}^+ \times \mathbb{R}^d).
\]

Under the assumptions given above, it holds

\[
L(u) = f \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d).
\]
References


- N. Antonić, M. Erceg, M. Mišur, *Distributions of anisotropic order and applications*, in preparation, 24 pages