Optimality Criteria Method for Optimal Design Problems

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[INTERNATIONAL WORKSHOP ON PDEs:
ANALYSIS AND MODELLING]

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Stationary diffusion equation

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{Sym_d})$ satisfying

$$\mathbf{A}\xi \cdot \xi \geq \alpha|\xi|^2, \quad \mathbf{A}^{-1}\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2, \qquad \xi \in \mathbf{R}^d$$

and $f \in \mathrm{H}^{-1}(\Omega)$. Stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Ω - mixture of two isotropic materials with conductivities $0 < \alpha < \beta$:

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I},$$

where $\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$.

Stationary diffusion equation

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where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$.

Multiple state optimal design problem

$$\left\{ \begin{array}{l} J(\chi) = \int_\Omega \left[\chi(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u}(\mathbf{x}))\right] d\mathbf{x} \longrightarrow \min\,, \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})\,, \ \int_\Omega \chi\, d\mathbf{x} = q_\alpha\,, \end{array} \right.$$

$0 < q_\alpha < |\Omega|$, and g_α, g_β Caratheodory functions which satisfy growth condition

$$g_j(x, \mathsf{u}) \le a|\mathsf{u}|^s + b(x), \qquad j = \alpha, \beta,$$

for some $a > 0$, $b \in \mathrm{L}^1(\Omega)$ and $1 \le s < \frac{2d}{d-2}$,
and $\mathsf{u} = (u_1, \dots, u_m)$, $m \ge 2$, where u_i is the solution of

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{array} \right. , \qquad\qquad i = 1, \dots, m$$

where $f_i \in \mathrm{H}^{-1}(\Omega)$, $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, $0 < \alpha < \beta$.

Definition (Composite material)

If a sequence of characteristic functions $\chi_n \in \mathrm{L}^\infty(\Omega; \{0,1\})$ and conductivities $\mathbf{A}^n(x) = \chi_n(x)\alpha\mathbf{I} + (1-\chi_n(x))\beta\mathbf{I}$ satisfy

$$\begin{array}{ccc} \chi_n & \overset{*}{\rightharpoonup} & \theta \\ \mathbf{A}^n & \overset{H}{\rightharpoonup} & \mathbf{A}, \end{array}$$

then it is said that $\mathbf{A}$ is homogenised tensor of two-phase composite material with proportions θ of first material and microstructure defined by the sequence (χ_n).

Definition (H-convergence)

A sequence of matrix functions $\mathbf{A}^n$ is said to H-converge to $\mathbf{A}$ if for every f the sequence of solutions of

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}^n \nabla u_n) = f \\ u_n \in \mathrm{H}_0^1(\Omega) \end{cases}$$

satisfies $u_n \rightharpoonup u$ in $\mathrm{H}_0^1(\Omega)$, $\mathbf{A}^n \nabla u_n \rightharpoonup \mathbf{A}\nabla u$ in $\mathrm{L}^2(\Omega; \mathbf{R}^d)$, where u is the solution of the homogenised equation

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega)\,. \end{cases}$$

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A} = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where $$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

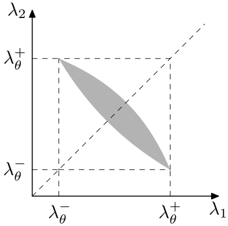
Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym_d}) : \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e. on } \Omega\}$$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}$

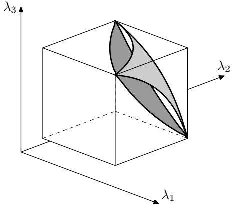
$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

2D:



$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

3D:



$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

Original problem:

$$\begin{cases} J(\chi) = \int_\Omega [\chi(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u})]\, d\mathbf{x} \longrightarrow \min\,, \\ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})\,,\ \int_\Omega \chi\, d\mathbf{x} = q_\alpha\,. \end{cases}$$

Generalized objective function is

$$J(\theta, \mathbf{A}) = \int_\Omega [\theta(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u}(\mathbf{x}))]\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x})\, d\mathbf{x}$$

where $\mathsf{u} = (u_1, \dots, u_m)$ and u_i is the solution of the state equation

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases}, \qquad i = 1, \dots, m.$$

Relaxed problem:

$$\begin{cases} J(\theta, \mathbf{A}) \longrightarrow \min\,, \\ (\theta, \mathbf{A}) \in \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym_d}) :\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e. on } \Omega\}\,. \end{cases} \tag{1}$$

Optimality criteria method

- G. Allaire, *Shape optimization by the homogenization method,* Springer-Verlag, 2002.
- F. Murat, L. Tartar, *Calcul des Variations et Homogénéisation, in Les Méthodes de l'Homogenisation Théorie et Applications en Physique (Bréau-sans-Nappe, 1983)*, Coll. Dir. Etudes et Recherches EDF, 57:319–369, Eyrolles, Paris 1985.
- M. Vrdoljak, *On Hashin-Shtrikman bounds for mixtures of two isotropic materials*, Nonlinear Anal. Real World Appl., 11:4597–4606, 2010.

Theorem

Let $(\theta^*, \mathbf{A}^*)$ be a local minimizer for relaxation problem (1) with corresponding states u_i^* and adjoint states p_i^*, solutions of

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}^*\nabla p_i^*) = \theta \dfrac{\partial g_\alpha}{\partial u_i}(\cdot, \mathsf{u}) + (1-\theta)\dfrac{\partial g_\beta}{\partial u_i}(\cdot, \mathsf{u}) \\ p_i^* \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m\,.$$

We introduce symmetric matrix $\mathbf{N}^* = \mathit{Sym} \sum_{i=1}^{m} \sigma_i^* \otimes \tau_i^*$, for $\sigma_i^* = \mathbf{A}^*\nabla u_i^*$, $\tau_i^* = \mathbf{A}^*\nabla p_i^*$ and define function $g(\theta, \mathbf{N}^*) = \min_{\mathbf{A}\in\mathcal{K}(\theta)} (\mathbf{A}^{-1} : \mathbf{N}^*)$. Then

$$(\mathbf{A}^*)^{-1}(\mathbf{x}) : \mathbf{N}^*(\mathbf{x}) = g(\theta^*(\mathbf{x}), \mathbf{N}^*(\mathbf{x})), \qquad \textit{a.e } x \in \Omega.$$

Moreover, if we define function

$$R^*(\mathbf{x}) = g_\alpha(\mathbf{x}, \mathsf{u}^*) - g_\beta(\mathbf{x}, \mathsf{u}^*) + l + \tfrac{\partial g}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{N}^*(\mathbf{x}))$$

the optimal θ^* satisfies (a. e. on Ω)

$$\begin{array}{rl} \theta^*(\mathbf{x}) = 0 & \textit{if } R^*(\mathbf{x}) > 0 \\ \theta^*(\mathbf{x}) = 1 & \textit{if } R^*(\mathbf{x}) < 0 \\ 0 \le \theta^*(\mathbf{x}) \le 1 & \textit{if } R^*(\mathbf{x}) = 0\,. \end{array}$$

Algorithm

Take some initial θ^0 and $\mathbf{A}^0$. For k from 0 to N:

1. Calculate u_i^k, $i = 1, \dots, m$, the solution of

$$\begin{cases} -\text{div}\,(\mathbf{A}^k \nabla u_i^k) = f_i \\ u_i^k \in \mathrm{H}_0^1(\Omega)\,. \end{cases}$$

2. Calculate p_i^k, $i = 1, \dots, m$, the solution of

$$\begin{cases} -\text{div}\,(\mathbf{A}^k \nabla p_i^k) = \theta^k \dfrac{\partial g_\alpha}{\partial u_i}(\cdot, \mathsf{u}^k) + (1-\theta^k)\dfrac{\partial g_\beta}{\partial u_i}(\cdot, \mathsf{u}^k) \\ p_i^k \in \mathrm{H}_0^1(\Omega),\ \mathsf{u}^k = (u_1^k, \dots, u_m^k) \end{cases}$$

   and define $\sigma_i^k := \mathbf{A}^k \nabla u_i^k$, $\tau_i^k := \mathbf{A}^k \nabla p_i^k$ and $\mathbf{N}^k := \mathrm{Sym} \displaystyle\sum_{i=1}^m (\sigma_i^k \otimes \tau_i^k)$.

3. For $\mathbf{x} \in \Omega$, let $\theta^{k+1}(\mathbf{x})$ be the zero of function

$$\theta \mapsto g_\alpha(\mathbf{x}, \mathsf{u}^k(\mathbf{x})) - g_\alpha(\mathbf{x}, \mathsf{u}^k(\mathbf{x})) + l + \frac{\partial g}{\partial \theta}(\theta, \mathbf{N}^k(\mathbf{x}))\,,$$

   and if a zero doesn't exist, take 0 (or 1) in case when this function is positive (or negative) on $\langle 0, 1\rangle$.

4. Let $(\mathbf{A}^{k+1})^{-1}(\mathbf{x})$ be the minimizer in the definition of $g(\theta^{k+1}(\mathbf{x}), \mathbf{N}^k(\mathbf{x}))$.

Theorem (d=2)

If dimension $d = 2$, then for given $\theta \in [0, 1]$ and matrix $\mathbf{N}$ with eigenvalues $\eta_1 \geq \eta_2$ we have

A. *If $\eta_2 > 0$ and $\theta^A := \left(\alpha \frac{\sqrt{\eta_1}}{\sqrt{\eta_2}} - \beta \right) \frac{1}{\alpha - \beta}$, then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} \frac{1}{\beta} \left(\beta^2 - \alpha^2\right) \left(\frac{\sqrt{\eta_1} + \sqrt{\eta_2}}{(\theta(\alpha - \beta) + \beta + \alpha} \right)^2, & \theta < \theta^A \\ \frac{(\beta - \alpha)\, \eta_1}{(\theta(\alpha - \beta) + \beta)^2} + \eta_2 \left(\frac{1}{\alpha} - \frac{1}{\beta} \right), & \theta \geq \theta^A \end{cases}.$$

B. *If $\eta_1 < 0$ and $\theta^B := \left(\frac{\sqrt{-\eta_1}}{\sqrt{-\eta_2}} - 1 \right) \frac{\beta}{\alpha - \beta}$, then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} -\frac{1}{\alpha} \left(\beta^2 - \alpha^2\right) \left(\frac{\sqrt{-\eta_1} + \sqrt{-\eta_2}}{\theta(\alpha - \beta) + 2\beta} \right)^2, & \theta > \theta^B \\ \frac{(\beta - \alpha)\, \eta_1}{(\theta(\alpha - \beta) + \beta)^2} + \eta_2 \left(\frac{1}{\alpha} - \frac{1}{\beta} \right), & \theta \leq \theta^B \end{cases}.$$

C. *If $\eta_1 \geq 0$ and $\eta_2 \leq 0$, then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \frac{(\beta - \alpha)\, \eta_1}{(\theta(\alpha - \beta) + \beta)^2} + \eta_2 \left(\frac{1}{\alpha} - \frac{1}{\beta} \right).$$

Theorem (d=3)

For $d=3$, given $\theta \in [0,1]$ and matrix $\mathbf{N}$ with eigenvalues $\eta_1 \geq \eta_2 \geq \eta_3$ we have

A. *If $\eta_3 = 0$ then $\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \frac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2)$.*

B. *If $\eta_3 > 0$ and additionally $\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1} > 0$, it holds*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} \beta^{-1}(\beta-\alpha)(\alpha+2\beta)\left(\dfrac{\sqrt{\eta_1}+\sqrt{\eta_2}+\sqrt{\eta_3}}{2\theta(\alpha-\beta)+\alpha+2\beta}\right)^2, & \theta < \theta_1^B, \\ \beta^{-1}(\beta^2-\alpha^2)\left(\dfrac{\sqrt{\eta_2}+\sqrt{\eta_3}}{\theta(\alpha-\beta)+\alpha+\beta}\right)^2 + (\beta-\alpha)\dfrac{\eta_1}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_1^B \leq \theta < \theta_2^B, \\ \eta_3(\alpha^{-1}-\beta^{-1}) + \dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \geq \theta_2^B, \end{cases}$$

where $\theta_1^B = 1-\frac{\alpha(\sqrt{\eta_2}+\sqrt{\eta_3}-2\sqrt{\eta_1})}{(\alpha-\beta)(\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1})}$ and $\theta_2^B = 1-\frac{\alpha(\sqrt{\eta_3}-\sqrt{\eta_2})}{(\alpha-\beta)\sqrt{\eta_3}}$.
If $\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1} \leq 0$ then we omit the first case in the above formula.

C. *If $\eta_3 < 0$ then, if η_2 and η_1 are negative as well then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} -\alpha^{-1}(\beta-\alpha)(2\alpha+\beta)\left(\dfrac{\sqrt{-\eta_1}+\sqrt{-\eta_2}+\sqrt{-\eta_3}}{2\theta(\alpha-\beta)+3\beta}\right)^2, & \theta > \theta_1^C, \\ -\alpha^{-1}(\beta^2-\alpha^2)\left(\dfrac{\sqrt{-\eta_2}+\sqrt{-\eta_3}}{\theta(\alpha-\beta)+2\beta}\right)^2 + \eta_1\dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_2^C < \theta \leq \theta_1^C, \\ (\alpha^{-1}-\beta^{-1})\eta_3 + \dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \leq \theta_2^C, \end{cases}$$

where $\theta_1^C = \frac{\beta(\sqrt{-\eta_2}+\sqrt{-\eta_3}-2\sqrt{-\eta_1})}{(\beta-\alpha)(\sqrt{-\eta_2}+\sqrt{-\eta_3}-\sqrt{-\eta_1})}$ and $\theta_2^C = \frac{\beta(\sqrt{-\eta_3}-\sqrt{-\eta_2})}{(\beta-\alpha)\sqrt{-\eta_3}}$.
If $\eta_2 < 0$ and $\eta_1 \geq 0$ then θ_1^C is not defined and we can express $\frac{\partial g}{\partial \theta}(\theta, \mathbf{N})$ by the second and third term in the formula above, omitting the assumption $\theta \leq \theta_1^C$ in the second case.
If $\eta_2 \geq 0$ then both θ_1^C and θ_2^C are not defined and $\frac{\partial g}{\partial \theta}$ is given by formula in the third case above, for any $\theta \in [0,1]$.

Example 1.

We consider two dimensional problem of weighted energy minimization

$$J(\theta, \mathbf{A}) = 2\int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \longrightarrow \min,$$

where $\Omega \subseteq \mathbf{R}^2$ is a ball $B(\mathbf{0}, 2)$, $\alpha = 1$, $\beta = 2$, while u_1 and u_2 are state functions for

$$\left\{ \begin{array}{l} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. , \quad i = 1, 2,$$

where we take $f_1 = \chi_{B(\mathbf{0},1)}$ and $f_2 \equiv 1$ for right-hand sides.

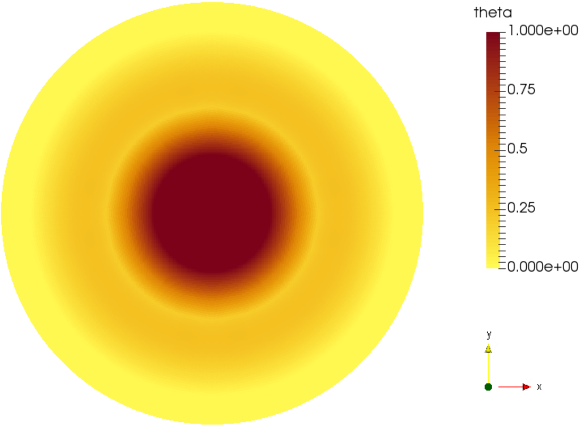
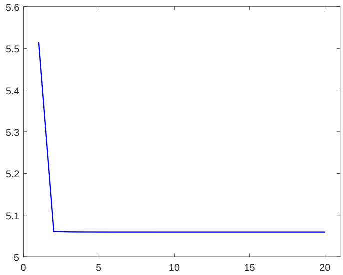
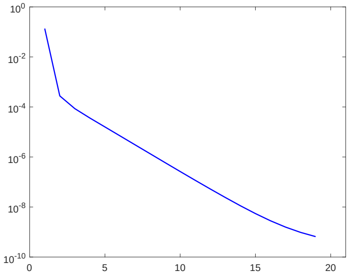


Figure: Optimal distribution of materials in circle with volume constraint 25% of the first material.



(a) Cost functional J.



(b) $\|\theta^k - \theta^{k+1}\|_{L^2}$ in terms of the iteration number k.

Figure: Convergence history for energy minimization in circle.

Comparison between numerical and exact solution

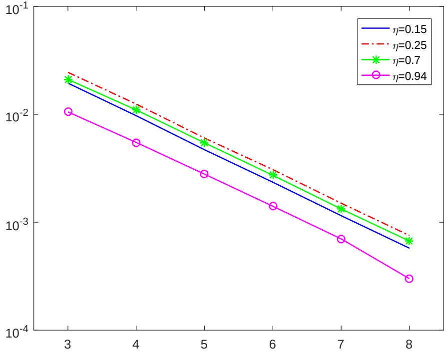


Figure: L^1 norm of difference between numerical and exact solution with respect to mesh refinement (each refinement introduces four times finer mesh) for various choices of volume fractions η of the first phase.

Example 2.

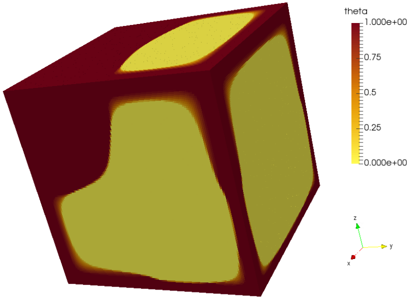
Second example is three dimensional energy minimization problem

$$J(\theta, \mathbf{A}) = \int_\Omega (f_1 u_1 + f_2 u_2)\, d\mathbf{x} \longrightarrow \min,$$

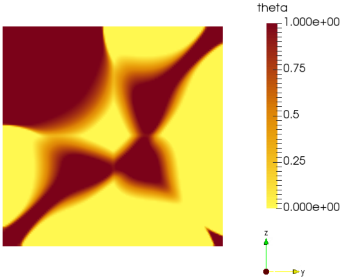
with $\alpha = 1$, $\beta = 2$ and two state equations

$$\left\{ \begin{array}{l} -\text{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. , \quad i = 1, 2.$$

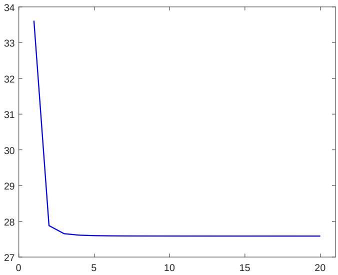
We take cube $\Omega = [-1, 1]^3$ as domain and set function f_1 to be zero on the upper half $(z > 0)$ and 10 on the lower half of the cube, while function f_2 to be zero on the left half $(y < 0)$ and 10 on the right half of the cube.



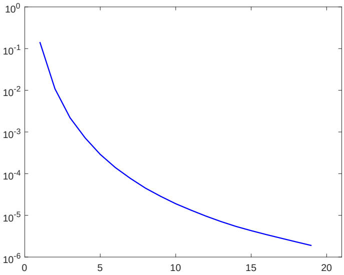
(a) Optimal distribution of materials in a cube.



(b) Intersection of the cube with x=0 plane.



(c) Cost functional J.



(d) $\|\theta^k - \theta^{k+1}\|_{L^2}$ in terms of the iteration number k.

Figure: Convergence history for energy minimization.

Convergence of the optimality criteria method

- single-state problems, case of a self-adjoint optimization problem

$$J(\theta, \mathbf{A}) = \pm \int_\Omega f u \, d\mathbf{x} + l \int_\Omega \theta \, d\mathbf{x}$$

- G. Allaire, *Shape optimization by the homogenization method,* Springer-Verlag, 2002.

- multiple-state problems, case of minimization of the conic sum of energies

$$J(\theta, \mathbf{A}) = \sum_{i=1}^{m} \mu_i \int_\Omega f_i u_i \, d\mathbf{x} + l \int_\Omega \theta \, d\mathbf{x}, \quad \mu_i > 0, \; i = 1, \dots m.$$

Theorem

If $m < d$, then, in every step of the algorithm, $\mathbf{A}^{k+1}$ is a simple laminate with the lamination direction orthogonal to σ_i^k, $i = 1, \ldots, m$.

Theorem

If Ω is spherically symmetric, θ^k and f_i are radial functions for $i = 1, \ldots, m$, and $\mathbf{A}^k$ is a simple laminate with the lamination direction orthogonal to the radial vector, then θ^{k+1} is also a radial function, while $\mathbf{A}^{k+1}$ is a simple laminate with the lamination direction orthogonal to σ_i^k, $i = 1, \ldots, m$.

Linearized elasticity system

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{Sym}^4_{\mathrm{d}})$ defined as

$$\mathbf{A}(\mathbf{x}) = \chi(\mathbf{x})\mathbf{A} + (1 - \chi(\mathbf{x}))\mathbf{B},$$

where $\mathbf{A}, \mathbf{B} = 2\mu_{\mathbf{A},\mathbf{B}}\mathbf{I}_4 + \left(\kappa_{\mathbf{A},\mathbf{B}} - \frac{2\mu_{\mathbf{A},\mathbf{B}}}{d}\right)\mathbf{I}_2 \otimes \mathbf{I}_2$, for $0 < \mu_{\mathbf{A}} \leq \mu_{\mathbf{B}}$ and $0 < \kappa_{\mathbf{A}} \leq \kappa_{\mathbf{B}}$. The state equation reads

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}e(\mathsf{u})) = \mathsf{f} \\ \mathsf{u} \in \mathrm{H}^1_0(\Omega; \mathbf{R}^d) \end{cases}$$

where $e(\mathsf{u}) = \frac{1}{2}(\nabla\mathsf{u} + \nabla\mathsf{u}^\top)$ and $\mathsf{f} \in \mathrm{H}^{-1}(\Omega; \mathbf{R}^{\mathrm{d}})$.

Classical optimal design problem:

$$\begin{cases} J(\chi) = \displaystyle\int_\Omega \left[\chi(\mathbf{x}) g_{\mathbf{A}}(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\chi(\mathbf{x})) g_{\mathbf{B}}(\mathbf{x}, \mathsf{u}(\mathbf{x}))\right] d\mathbf{x} \longrightarrow \min\,, \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})\,, \ \displaystyle\int_\Omega \chi \, d\mathbf{x} = q_\alpha\,. \end{cases}$$

Relaxed formulation:

$$\begin{cases} J(\theta, \mathbf{A}) = \displaystyle\int_\Omega \left[\theta(\mathbf{x}) g_{\mathbf{A}}(\mathbf{x}, \mathsf{u}(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_{\mathbf{B}}(\mathbf{x}, \mathsf{u}(\mathbf{x}))\right] d\mathbf{x} + \\ \qquad\qquad + \, l \displaystyle\int_\Omega \theta(\mathbf{x}) d\mathbf{x} \longrightarrow \min, \\ (\theta, \mathbf{A}) \in \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{Sym}_{\mathrm{d}}^4) : \ \mathbf{A} \in \mathrm{G}_\theta \text{ a.e. on } \Omega\}\,. \end{cases}$$

Compliance optimization

Theorem

If $(\theta^*, \mathbf{A}^*)$ *is a minimizer of the objective function* $J(\theta, \mathbf{A}) = \int_\Omega \mathsf{f}(\mathbf{x}) \cdot \mathsf{u}(\mathbf{x})\, d\mathbf{x} + l \int_\Omega \theta(\mathbf{x}) d\mathbf{x}$ *and if* σ *is the unique corresponding minimizer of*

$$\min_{\substack{\tau \in \mathrm{L}^2(\Omega;\mathrm{Sym_d}) \\ -\mathrm{div}\tau = \mathsf{f} \text{ in } \Omega}} \int_\Omega \mathbf{A}^{-1}\tau : \tau \, d\mathbf{x} + l \int_\Omega \theta \, d\mathbf{x},$$

then σ *satisfies*

$$\begin{cases} \sigma = \mathbf{A}^* e(\mathsf{u}) & in\, \Omega \\ -\mathrm{div}\sigma = \mathsf{f} & in\, \Omega \\ \mathsf{u} = 0 & on\, \partial\Omega, \end{cases}$$

with $\mathsf{u} \in \mathrm{H}^1_0(\Omega, \mathbf{R}^{\mathrm{d}})$*, and, almost everywhere in* Ω

$$\mathbf{A}^{*-1}\sigma : \sigma = g(\theta^*, \sigma),$$

while θ^* *is the unique minimizer of the convex optimization* $\min_{0 \le \theta \le 1} (g(\theta, \sigma) + l\theta)$*, where* $g(\theta, \sigma)$ *is the lower Hashin-Shtrinkman bound on complementary energy defined by*

$$\mathbf{A}^{*-1}\sigma : \sigma \ge \mathbf{B}^{-1}\sigma : \sigma + \theta \max_{\eta \in \mathrm{Sym_d}} \left(2\sigma : \eta - \left(\mathbf{A}^{-1} - \mathbf{B}^{-1}\right)^{-1} \eta : \eta - (1-\theta)(\mathbf{B}\eta : \eta - h(\mathbf{B}\eta)) \right),$$

where $h(\eta) = \min_{\mathsf{e} \in S^{d-1}} \frac{1}{\mu_{\mathbf{B}}} \left(|\eta \mathsf{e}|^2 - (\eta \mathsf{e} \cdot \mathsf{e})^2 \right) + \frac{1}{2\mu_{\mathbf{B}} + \lambda_{\mathbf{B}}} (\eta \mathsf{e} \cdot \mathsf{e})^2.$

Thank you for your attention!