Optimal control of parabolic equations by spectral decomposition

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Outline

- Problem formulation
- Characterisation of the solution
- Numerical recovery
- Numerical examples

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The problem framework

An optimal control problem for an abstract heat equation:

$$\begin{cases} -y'(t) &= \mathcal{A}y(t) \qquad \text{for } t > 0 \\ y(0) &= u \in H. \end{cases} \tag{1}$$

H - Hilbert space
$\mathcal{A}$ – positive semidefinite, self-adjoint unbounded operator on H,
– with dense domain $D(\mathcal{A})$,
– with compact resolvent.
Keynote example: $\mathcal{A} = -\Delta$, the Dirichlet Laplacian in $L^2(\Omega)$.

$\{\mathcal{S}_t\}_{t\geq 0}$ – strongly continuous semigroup of non-expansive linear operators generated by $-\mathcal{A}$

Control u – initial datum aiming to:

(1) steer the solution (arbitrarily closed) to a desired target in a given time horizon,
(2) minimise a given energy functional.

The system (1) is controllable to a *target state* $y^T \in H$ in time $T > 0$ if there is $u \in H$ such that

$$\mathcal{S}_T u = y^T.$$

In general, system (1) is NOT controllable to an arbitrary target.
E.g. $\mathcal{A} = -\Delta$, the Dirichlet Laplacian in $L^2(\Omega)$

$$(\forall\, t > 0) \quad S_t H \subseteq D(\mathcal{A}) = \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$$

– no target state $y^T \in H \setminus D(\mathcal{A})$ can be attained in any time.

System (1) is approximately controllable:
for every target time $T > 0$, target state y^T, tolerance $\varepsilon > 0$,
there exists an initial datum $u \in H$ such that

$$\|\mathcal{S}_T u - y^T\|_H \leq \varepsilon.$$

The problem

Given a tolerance $\varepsilon > 0$, a control time $T > 0$, and a target state y^T, find

$$(\mathcal{P}) \qquad \hat{u} = \underset{u \in H}{\operatorname{argmin}} \left\{ J(u) : \|\mathcal{S}_T u - y^T\|_H \leq \varepsilon \right\},$$

where

$$J(u) = \frac{\alpha}{2} \|u\|_H^2 + \frac{1}{2} \int_0^T \beta(t) \|\mathcal{S}_t u - y^d(t)\|_H^2 \, dt,$$

with

- $y_d \in L^2(0, T; H)$, target trajectory;
- $\alpha > 0$, weight of the control cost;
- $0 \leq \beta \in L^2(0, T)$, weight of the control on the trajectory.

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The solution is unique!

Indeed, by means of the *indicator function*:

$$I_C(y) = \begin{cases} 0 & \text{if } y \in C \\ +\infty & \text{else.} \end{cases}$$

the problem $(\mathcal{P})$ is restated as

$$\min_{u \in H} \left\{ J(u) + I_{\bar{B}}(\mathcal{S}_T u) \right\},$$

where $\bar{B} = \overline{B(y^T; \varepsilon)}$.

$J + I_{\bar{B}} \circ \mathcal{S}_{\mathcal{T}}$ is proper, strongly convex and lower-semicontinuous

$\implies$ problem $(\mathcal{P})$ has a unique solution $\hat{u}$.

Let $\tilde{u}$ denote the unique solution of the unconstrained problem

$$\tilde{u} = \operatorname*{argmin}_{u \in H} \; J(u),$$

and $\tilde{y} = \mathcal{S}_T \tilde{u}$ the corresponding final state.

Proposition

If $\|\tilde{y} - y^T\|_H \leq \varepsilon$, then $\hat{u} = \tilde{u}$.
Otherwise, the optimal final state verifies $\hat{y} \in \partial \bar{B}$.

Standard approaches to the problem

$\beta = 0$ (no desired trajectory)
HUM (Hilbert Uniqueness Method), some penalised version:

- based on the dual problem,
- discretisation of the system,
- approximation by a finite dimensional problem,
- iterative algorithm for getting the control.

Glowinski, R., Lions, J. L. Exact and approximate controllability for distributed parameter systems, Acta Numer. (1994), 269-378.

Boyer, F. On the penalized HUM approach and its application to the numerical approximation of null-controls for parabolic problems, ESAIM: Proceedings (2013), no. 41, 15-58.

$\beta \neq 0$
– even a more complex numerical treatment (convex optimisation techniques)

We present a different approach based on spectral decomposition of the solution by eigenfunctions of $\mathcal{A}$,

Geometrical interpretation

Introduce sublevel sets of $\psi = (J \circ S_{-T})$:

$$
\begin{aligned}
W_c &= \{y \in H : \psi(y) \le c\} \\
&= \{y \in H : \ y = \mathcal{S}_T u \text{ for some } u \in H \text{ with } J\,(u) \le c\}\,.
\end{aligned}
$$

W_c is empty for $c < \tilde{c} = J\,(\tilde{u})$.

$(W_c)_{c \ge \tilde{c}}$ – a nested family of nonempty closed convex sets centred at $\tilde{y}$, that increases with c.

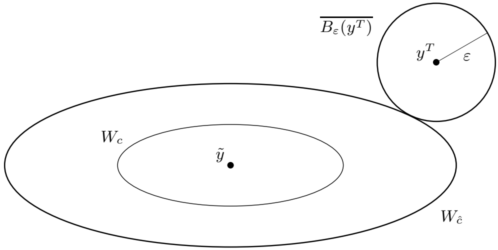

Figure: Sublevel sets W_c and the target ball.

The target ball is hit for the first time by $W_{\hat{c}}$, where $\hat{c} = J(\hat{u})$.
The intersection $\hat{y}$ is the optimal final state.

$$\hat{y} - y^T = -\hat{\gamma}\nabla\psi(\hat{y}),$$

for some $\hat{\gamma} > 0$.
Together with

$$\|\hat{y} - y^T\|_H = \varepsilon$$

we get a fully determined system for $\hat{\gamma}, \hat{y}$.

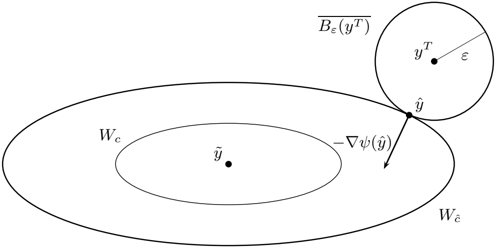

Figure: Optimal final state $\hat{y}$ as intersection of $W_{\hat{c}}$ and $\overline{B_\varepsilon(y^T)}$

Spectral decomposition

Denote:

$(\varphi_n)_{n\in\mathbf{N}}$ – an orthonormal basis of H, consisting of eigenfunctions of $\mathcal{A}$
$(\lambda_n)_{n\in\mathbf{N}}$ – a sequence of corresponding (nonnegative) eigenvalues λ_n,

$$\lim_n \lambda_n = +\infty$$

y_n – the n-th Fourier coefficient of $y \in H$.

The ellipsoids W_c can be now characterised as

$$W_c = \left\{ \sum_n y_n \varphi_n : \underbrace{\sum_n \left(a_n y_n^2 + b_n y_n + c_n\right)}_{\psi(y)=J(u)} \leq c \right\}, \tag{2}$$

where

$$a_n = \left(\frac{\alpha}{2} + \frac{1}{2}\int_0^T \beta\left(t\right) e^{-2\lambda_n t} dt\right) e^{2\lambda_n T};$$

$$b_n = -e^{\lambda_n T} \int_0^T \beta\left(t\right) e^{-\lambda_n t} y_n^d\left(t\right) dt;$$

$$c_n = \frac{1}{2}\int_0^T \beta\left(t\right)\left(y_n^d(t)\right)^2 dt.$$

The geometrical interpretation

$$\hat{y} - y^T = -\hat{\gamma}\nabla\psi\left(\hat{y}\right),$$

together with

$$\left(\nabla\psi(y)\right)_n = 2a_n y_n + b_n, \quad n \in \mathbf{N},$$

we get an explicit formula for the Fourier coefficients of the optimal final state $\hat{y}$:

$$\hat{y}_n = \frac{y_n^T - \hat{\gamma} b_n}{1 + 2\hat{\gamma} a_n}. \tag{3}$$

It remains to determine the constant $\hat{\gamma} > 0$.

Condition $\|\hat{y} - y^T\| = \varepsilon$ together with (3) implies

$$G(\hat{\gamma}) := \sum_n \left(\frac{\hat{\gamma}\ \left(2a_n y_n^T + b_n\right)}{1 + 2\gamma a_n}\right)^2 = \varepsilon^2. \tag{4}$$

G – strictly increasing,
 – G(0)=0,
 – $\lim_{\gamma\to\infty} = \|\tilde{y} - y^T\| =: \varepsilon_c$.

The equation

$$G(\hat{\gamma}) = \varepsilon^2$$

has the unique solution for every $\varepsilon \in \big(0, \|\tilde{y} - y^T\|_H\big)$.

Theorem [Generalised HUM]*

The solution of the optimal control problem $(\mathcal{P})$ is given by

$$\hat{u} = \sum_n \hat{u}_n \varphi_n = \sum_n e^{\lambda_n T} \hat{y}_n \varphi_n,$$

where the Fourier coefficients $(\hat{y}_n)_{n\in\mathbf{N}}$ of the final state $\hat{y}$ are given by (3) and (4).

* L., M., Molinari, C., Peypouquet, J.: Optimal control of parabolic equations by spectral decomposition. Optimization 66 (8), 1359-1381 (2017)

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Obtained explicit formulas incorporate infinite series.
Truncation required.
Denote by G_N the truncation of series G to first N terms.
Solve

$$G_N(\gamma) = \varepsilon^2.$$

For every $\varepsilon < \lim_{\gamma\to\infty} G_N(\gamma) < \varepsilon_c = \|\tilde{y} - y^T\|_H$ it has a unique solution, denoted by $\hat{\gamma}_N$.
Introduce the truncated approximation of the optimal final state, $\hat{y}^N$:

$$\hat{y}^N = \sum_{n=1}^{N} \hat{y}_n^N \varphi_n, \quad \text{with} \quad \hat{y}_n^N = \frac{y_n^T - \hat{\gamma}_N b_n}{1 + 2\hat{\gamma}_N a_n}. \tag{5}$$

Truncated Fourier series with approximate coefficients.

Theorem

The following estimate holds

$$\|\hat{y}^N - \hat{y}\|_H^2 \leq 4\|y^T - y^{T,N}\|_H^2 + \frac{4\|\beta\|_{L^2(0,T)}^2}{\alpha^2 e^{2\lambda_N T}}\|y^d - y^{d,N}\|_{L^2(0,T;H)}^2,$$

where $y^{T,N} = \sum_{n=0}^{N} y_n^T \varphi_n$ and $y^{d,N}(t) = \sum_{n=0}^{N} y_n^d(t)\, \varphi_n$ the truncated series representation of the target final state and the reference trajectory in the distributed cost, respectively.

Numerical algorithm

– produces the approximate optimal final state $\hat{y}^N$, with precision ρ.

Step 1. Determine N such that

$$\max\left\{\|y^T - y^{T,N}\|_H^2, \frac{\|\beta\|^2_{L^2(0,T)}}{\alpha^2 e^{2\lambda_N T}}\|y^d - y^{d,N}\|^2_{L^2(0,T;H)}\right\} \le \frac{\rho^2}{8}.$$

Step 2. Compute $\varepsilon_N := \lim_{\gamma\to\infty} G_N(\gamma)$.
Step 3. For $\varepsilon \ge \varepsilon_N$ the approximative solution is $\tilde{u}^N = \sum_{n=1}^N -\frac{b_n}{2a_n} e^{\lambda_n T}\varphi_n$.
Otherwise, proceed to Step 4.
Step 4. Solve equation $G_N(\gamma) = \varepsilon^2$ numerically to find $\hat{\gamma}_N$ (bisection method or other).
Step 5. Compute the approximate optimal final state $\hat{y}^N$ using (5), and the approximate optimal control $\hat{u}^N$ by

$$\hat{u}^N = \sum_{n=1}^N \hat{u}_n^N \varphi_n = \sum_{n=1}^N e^{\lambda_n T}\hat{y}_n^N \varphi_n.$$

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Example 1 - Energy minimisation in 2D

The heat equation on $\Omega = (0,1) \times (0,1)$

$$\begin{cases} \frac{d}{dt}y - \Delta y = 0 & \Omega \times (0,T) \\ y = 0 & \partial\Omega \times (0,T) \\ y(0) = u & (0,\pi). \end{cases} \tag{6}$$

Target time $T = 0.001$
We use the eigenfunctions of the Dirichlet Laplacian on the rectangle

$$\varphi_{j,k}(x_1, x_2) = \sin(j\pi x_1)\sin(k\pi x_2), \qquad j,k = 1,2,\ldots,$$

with corresponding eigenvalues

$$\lambda_{j,k} = (j\pi)^2 + (k\pi)^2.$$

Example 1 - Energy minimisation in 2D

$\alpha = 1$
$\beta = 0$ - no prescribed reference trajectory.

$$(\mathcal{P}) \qquad \hat{u} \in \arg \min_{u \in L^2(0,\pi)} \left\{ \frac{1}{2} \|u\|_{L^2}^2 \,:\; \mathcal{S}_T u \in \overline{B_\varepsilon \left(y^T\right)} \right\}.$$

We choose a reachable final target y^T.
We introduce

$$u(x_1, x_2) = \exp\left(-\left(x_1^2 + x_2^2\right)\right) \cdot \sin\left(5\pi x_1^3\right) \cdot \sin\left(5\pi x_2^7\right),$$

and we set $u^T = u^N$, the Fourier representation of u using the first 15×15 coefficients.

The final target, given as $y^T = S_T u^T$, has a finite series representation.

The aim: Explore the differences between the initial datum u^T that generates the target and the solution $\hat{u}$ for various values of ε.

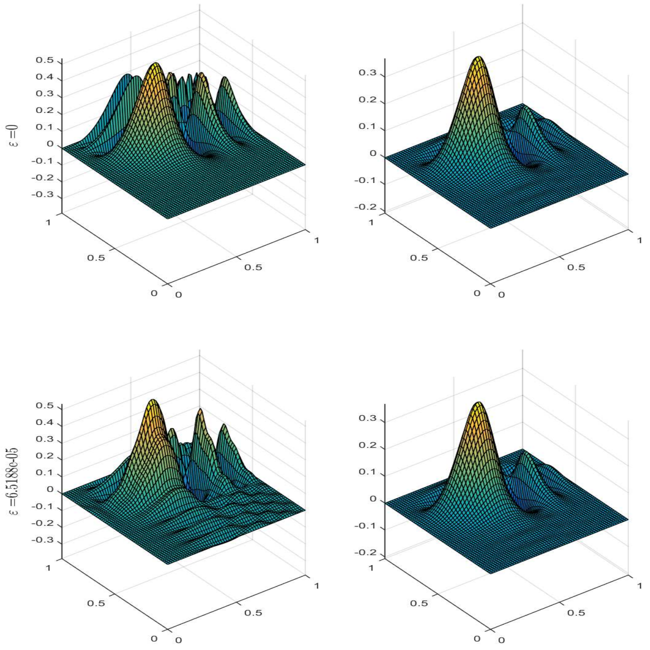

Figure: Initial data (left) and final state (right) for different values of ε.

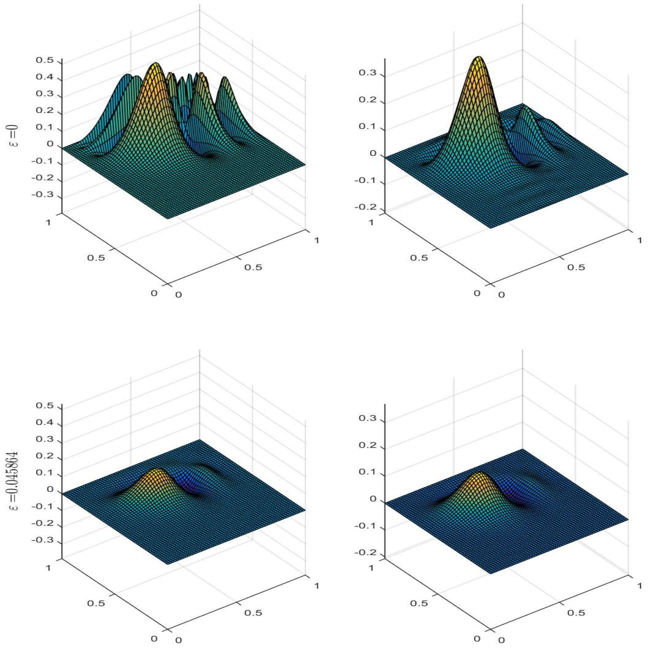

Figure: Initial data (left) and final state (right) for different values of ε.

ε^2	$J\left(\hat{u}^N\right)$
0	0.2235
$10^{-6} \cdot \|y^T\|_{L^2}^2$	0.1772
$0.5 \cdot \|y^T\|_{L^2}^2$	0.0057
$0.99 \cdot \|y^T\|_{L^2}^2$	$1.486e^{-6}$

Table: Values of the energy functional J for the solution corresponding to different tolerances ε.

Example 2 - Energy minimisation and trajectory regulation, 1D

The heat equation on $\Omega = (0, \pi)$, $T = 0.01$

- $\alpha = 10^{-4}$;
- $\beta(t) = \mathbb{1}_{[t_1, t_2]}(t)$, with $t_1 = T/3$ and $t_2 = 2T/3$;
- $y^d(x,t)$ as a smoothing regularisation (through classical mollifier) of the function $x \longmapsto \mathbb{1}_{[x_1,x_2]}(x)$, with $x_1 = \pi/5$ and $x_2 = 2\pi/5$;
- $y^T(x)$ as a smoothing regularisation (again, through mollifier) of the function $x \longmapsto \mathbb{1}_{[x_3,x_4]}(x)$, with $x_3 = 3\pi/5$ and $x_4 = 4\pi/5$.

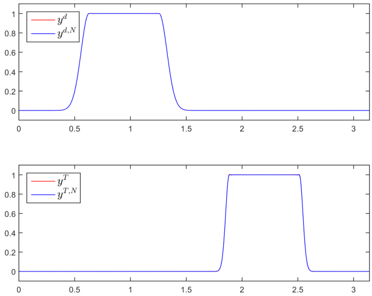

Figure: TOP: reference trajectory y^d for the distributed cost. BOTTOM: target final state y^T, in comparison with their reconstructions after Fourier decomposition with N=185 coefficients (indistinguishable).

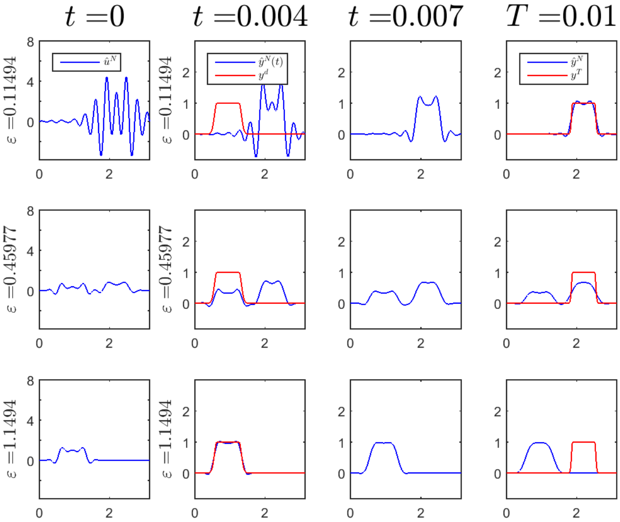

Figure: For the three values of ε: evolution of the solution in time and comparison with the reference trajectory $y^d(t)$ $(t = 0.004)$, and with the target y^T $(T = 0.01)$.

Conclusion

The new approach:

- exploring spectral representation of the solution by eigenfunctions of $\mathcal{A}$,
- an explicit expression of the optimal final state $\hat{y} = S_T \hat{u}$ in terms of the given problem data.
- the numerical issues are reduced to finding the unique root of a scalar function.

Price to pay:

- knowledge of eigenfunctions.

If the problem has to be considered many times for different data, but the same operator, this can be done offline.

The method applicable to distributed control problem (recent result):

* L, M., Molinari, C.: Optimal distributed control of the heat-type equations by spectral decomposition, submitted.

- dual problem involved
- more complex relations.

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Thanks for your attention!