Transport-collapse scheme for heterogeneous scalar conservation laws

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Abstract

We extend Brenier's transport collapse scheme on the Cauchy problem for heterogeneous scalar conservation laws. It is based on averaging out the solution to the corresponding kinetic equation, and it necessarily converges toward the entropy admissible solution. We also provide numerical examples.

Introduction

We deal with is the initial value problem for heterogeneous scalar conservation laws. In order to introduce it, let $\Omega \subset \mathbf{R}^d$ be a bounded smooth domain and $\mathbf{R}^+ = [0, \infty)$. We consider for $f \in C^2(\mathbf{R}; \mathbf{R}^d)$

$$\partial_t u + \operatorname{div}_x f(t, x, u) = 0, \quad (t, \mathbf{x}) \in \mathbf{R}^+ \times \Omega, \tag{1}$$

$$u|_{t=0} = u_0(\mathbf{x}), \tag{2}$$

If not stated otherwise, we assume that $u_0 \in L^1(\mathbf{R}^d) \cap BV(\mathbf{R}^d)$. We also assume that $a \leq u_0 \leq b$ for some constant $a, b > 0$.

A typical problem described by (1), (2), arises e.g. in traffic flow models or fluid dynamics.

Definition 1. A bounded function u is called an entropy admissible solution to with the initial conditions (2) if for every convex function $V \in C^2(\mathbf{R})$, every $\lambda \in \mathbf{R}$ and every $\varphi \in C^1_c([0, \infty) \times \mathbf{R}^d)$, it holds

$$\iint\limits_{\mathbf{R}^+ \times \mathbf{R}^d} \Big[V(u)\partial_t \varphi + \int_a^u f'_\lambda(t, \mathbf{x}, v)\, V'(v) dv \cdot \nabla \varphi \tag{3}$$

$$+ \int_a^u \operatorname{div}_{\mathbf{x}} f(t, \mathbf{x}, v) V''(v) dv\, \varphi \Big] d\mathbf{x} dt + \int_{\mathbf{R}^d} V(u_0(\mathbf{x}))\varphi(0, \mathbf{x}) d\mathbf{x} \leq 0.$$

Equivalent and more usual definition of admissible solution is given by the Kruzhkov entropies $V(u) = |u - \lambda|$, $\lambda \in \mathbf{R}$, and it states that a bounded function u is called an entropy admissible solution to (1), (2) if for every $\lambda \in \mathbf{R}$ it holds

$$\partial_t |u - \lambda| + \operatorname{div}_{\mathbf{x}}[\operatorname{sgn}(u - \lambda)(f(t, \mathbf{x}, u) - f(t, \mathbf{x}, \lambda))] \tag{4}$$

$$+ \operatorname{sgn}(u - \lambda) \operatorname{div}_{\mathbf{x}} f(t, \mathbf{x}, \lambda) \leq 0$$

in the sense of distributions on $\mathcal{D}'(\mathbf{R}^d_+)$, and it holds $esslim_{t \to 0} \int_\Omega |u(t, \mathbf{x}) - u_0(\mathbf{x})| d\mathbf{x} = 0$. By finding derivative with respect to λ in (4) one reaches to the kinetic formulation (see e.g. [10, 8] for different variants).

Kinetic formulation

Theorem. [5] The function $u \in C([0, \infty); L^1(\mathbf{R}^d)) \cap L^\infty_{loc}((0, \infty); L^\infty(\mathbf{R}^d))$ is the entropy admissible solution to (1), (2) if and only if there exists a non-negative Radon measure $m(t, \mathbf{x}, \lambda)$ such that $m((0, T) \times \mathbf{R}^{d+1}) < \infty$ for all $T > 0$ and such that the function $\chi(\lambda, u) = \begin{cases} 1, & 0 \leq \lambda \leq u \\ -1, & u \leq \lambda \leq 0 \\ 0, & else \end{cases}$, represents the distributional solution to

$$\partial_t \chi + \operatorname{div}_{(\mathbf{x}, \lambda)}[F(t, \mathbf{x}, \lambda)\chi] = \partial_\lambda m(t, \mathbf{x}, \lambda), \quad (t, \mathbf{x}) \in \mathbf{R}^+ \times \mathbf{R}^d, \tag{5}$$

$$\chi(\lambda, u(t = 0, \mathbf{x})) = \chi(\lambda, u_0(\mathbf{x})), \tag{6}$$

where $F = (f'_\lambda, -\sum_{j=1}^d \partial_{x_j} f_j)$.

Remark that through the kinetic concept, one reduces the nonlinear equation (1) on the linear (so called kinetic) equation. However, derivative of a measure figures in the equation (see the right-hand side of (5)) and it has one more variable (so called kinetic or velocity variable). Due to the former reason, the kinetic equation is not convenient for numerical implementation.

Never the less, if we neglect the derivative of the measure, and then average out the solution to the obtained linear equation with respect to the kinetic variable, we can obtain entropy solution to the considered problem. Such a procedure is proposed in [2] for equation homogeneous case, interestingly more than ten years before the kinetic concept was formalized in [10]. We aim to extend the transport-collapse scheme [2] for the initial value problem for heterogeneous scalar conservation laws.

Let us now state properties of the function χ.

Proposition [2, page 1018] It holds

a) $\forall u, v \in L^1(\mathbf{R}^d)$ such that $u \geq v \implies \chi(\lambda, u) \geq \chi(\lambda, v)$;

b) $\forall u \in L^1(\mathbf{R}^d)$, $\forall g \in L^\infty(\mathbf{R}^d \times \mathbf{R})$, it holds $\iint \chi(\lambda, u) g(\mathbf{x}, \lambda) d\mathbf{x} d\lambda = \int (\int_a^u g(\mathbf{x}, \lambda) d\lambda)\, d\mathbf{x}$;
In particular, if $g = G'_\lambda$ and $G(a) = 0$, then $\iint \chi(\lambda, u) g(\mathbf{x}, \lambda) d\mathbf{x} d\lambda = \int G(\mathbf{x}, u) d\mathbf{x}$

c) $TV(u) = \int TV(\chi(\lambda, \cdot)) d\lambda$.

Transport collapse operator

The idea of the transport collapse scheme for the initial value problem (1), (2) is to solve problem (5), (6) when we omit the right-hand side in (5):

$$\partial_t h + \operatorname{div}_{\mathbf{x}, \lambda}[F(t, \mathbf{x}, \lambda) h] = 0, \quad h|_{t=0} = \chi(\lambda, u_0(\mathbf{x})). \tag{7}$$

The solution of this equation is obtained via the method of characteristics. They are given by

$$\begin{cases} \dot{\mathbf{x}} = f'_\lambda, \quad \mathbf{x}|_{t=0} = \mathbf{x}_0, \\ \dot{\lambda} = -\sum_{j=1}^d \partial_{x_j} f_j(t, \mathbf{x}, \lambda), \quad \lambda|_{t=0} = \lambda_0. \end{cases} \tag{8}$$

The solution to (7) has the form

$$h(t, \mathbf{x}, \lambda) = \chi(\lambda_0(t, \mathbf{x}, \lambda), u_0(\mathbf{x}_0(t, \mathbf{x}, \lambda))). \tag{9}$$

Definition. The transport collapse operator $T(t)$ is defined for every $u \in L^1(\mathbf{R}^d)$ by

$$T(t)u(\mathbf{x}) = \int \chi(\lambda_0(t, \mathbf{x}, \lambda), u(\mathbf{x}_0(t, \mathbf{x}, \lambda))) d\lambda. \tag{10}$$

It satisfies the following properties which are the same as the ones from [2, Proposition 1].

Proposition. It holds for every $u, v \in L^1(\mathbf{R}^d)$

a) $u \leq v$ a.e. implies $T(t)u \leq T(t)v$ a.e;

b) $\int T(t)u(\mathbf{x}) d\mathbf{x} = \int u(\mathbf{x}) d\mathbf{x}$;

c) the operator $T(t)$ is non-expansive
$$\|T(t)u - T(t)v\|_{L^1(\mathbf{R}^d)} \leq \|u - v\|_{L^1(\mathbf{R}^d)},$$
and, in particular, $\|T(t)u\|_{L^1(\mathbf{R}^d)} \leq \|u\|_{L^1(\mathbf{R}^d)}$;

d) $TV(T(t)u) \leq (1 + C_1 t) TV(u) + tC_2$, where TV is the total variation and C_1 and C_2 are appropriate constants depending on the C^2-bounds of the flux f;

e) $\|T(t)u - u\|_{L^1(\mathbf{R}^d)} \leq C_2 TV(u) t + tC_1$ for the constants C_1 and C_2 from the previous item;

Proposition. For any smooth positive test function φ, any $u \in L^1(\mathbf{R})$ such that $a \leq u \leq b$, and convex Lipschitz function $V : \mathbf{R} \to \mathbf{R}$, we have

$$\int (V(T(t)u) - V(u))(\mathbf{x})\varphi(\mathbf{x}) d\mathbf{x} \tag{11}$$

$$\leq \int_0^t \int B_V(t', \mathbf{x}, u(\mathbf{x})) \nabla \varphi d\mathbf{x} dt' \tag{12}$$

$$+ \int_0^t \int \int_a^u \operatorname{div}_{\mathbf{x}} f(t, \mathbf{x}, \lambda) V''(\lambda) d\lambda dt' + o(t), \quad t \to 0$$

where $B_V(t, \mathbf{x}, u) = \int_a^u f'_\lambda(t, \mathbf{x}, \lambda) V'(\lambda) d\lambda$, and $o(t)$ depends only on the L^∞-bound of u.

Main theorem

Theorem. Denote

$$S_n(t)u = (1 - \alpha) T(\frac{t}{n})^k u + \alpha T(\frac{t}{n})^{k+1} u, \tag{13}$$

where

$$t = \frac{(k + \alpha)}{n}, \quad k \in \mathbf{N}, \quad \alpha \in [0, 1). \tag{14}$$

For each initial value $u_0 \in L^1(\mathbf{R}^d)$ such that $a \leq u_0 \leq b$, the unique entropy solution of (1), (2) at time t is given by the formula

$$u(t, \cdot) = L^1 - \lim_{n \to \infty} S_n(t) u.$$

Numerical simulations

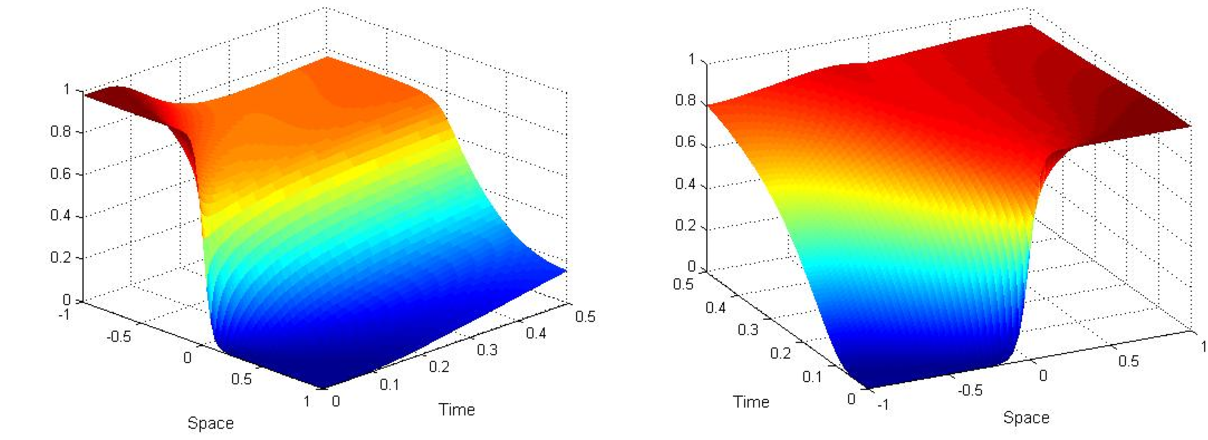

Figure 1: Cauchy problem with initial condition $u_0(x) = H_\epsilon(-x)$ (left) and $u_0(x) = H_\epsilon(x)$ (right).

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