One-scale variants of H-measures

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Joint work with Marko Erceg and Martin Lazar
H-measures and variants without a characteristic scale
   Classical H-measures
   Parabolic H-measures and similar variants
   H-distributions and variants

One-scale H-measures
   Semiclassical measures
   One-scale H-measures
   Other variants

Localisation principle
   Motivation
   One-scale H-measures
What are H-measures?

Before 1990: Tools to describe passage from one scale to another in the models of continuum mechanics included compactness by compensation, Young measures, and defect measures. The same tools were successful in passing to weak limits for sequences of solutions to PDEs.
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Start from a sequence $u_n \rightharpoonup 0$ in $L^2_{loc}(\mathbb{R}^d)$, and $\varphi \in C_c(\mathbb{R}^d)$, and take the Fourier transform:

$$\widehat{\varphi u_n}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} (\varphi u_n)(x) dx .$$

As $\varphi u_n$ is supported on a fixed compact set $K$, so $|\widehat{\varphi u_n}(\xi)| \leq C$. 
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As $\varphi u_n$ is supported on a fixed compact set $K$, so $|\widehat{\varphi u_n}(\xi)| \leq C$.
Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\xi) \rightharpoonup 0$ pointwise.
By the Lebesgue dominated convergence theorem on bounded sets, we get $\widehat{\varphi u_n} \rightarrow 0$ strong, i.e. strongly in $L^2_{loc}(\mathbb{R}^d)$. 

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By the Lebesgue dominated convergence theorem on bounded sets, we get $\hat{\varphi u_n} \rightarrow 0$ strong, i.e. strongly in $L^2_{loc}(\mathbb{R}^d)$.

On the other hand, by the Plancherel theorem: $\|\hat{\varphi u_n}\|_{L^2(\mathbb{R}^d)} = \|\varphi u_n\|_{L^2(\mathbb{R}^d)}$. 

The limit is a measure

If $\varphi u_n$ does not converge to zero in $L^2(\mathbb{R}^d)$, then neither does $\varphi u_n$; therefore some information must go to infinity.
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He took $\psi \in C(S^{d-1})$, and considered the limits of the integrals:

$$\lim_{n} \int_{\mathbb{R}^d} \psi(\xi/|\xi|)|\widehat{\varphi u_n}|^2 d\xi = \int_{S^{d-1}} \psi(\xi) d\nu_{\varphi}(\xi).$$

Limit is a linear functional in $\psi$, thus an integral over the sphere of some nonegative Radon measure, which depends on $\varphi$.

(a bounded sequence of Radon measures has an accumulation point)
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The crucial question was how does this limit depend on $\varphi$. 
Existence of H-measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2(\Omega; \mathbb{C}^r)$, then there exist a subsequence $(u_{n'}')$ and $\mu_H \in \mathcal{M}_b(\Omega \times S^{d-1}; M_r(\mathbb{C}))$ such that for every $\varphi_1, \varphi_2 \in C_0(\Omega)$ and $\psi \in C(S^{d-1})$

$$
\lim_{n' \to \infty} \int_{\mathbb{R}^d} \widehat{\varphi_1 u_{n'}(\xi)} \otimes \widehat{\varphi_2 u_{n'}(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi = \langle \mu_H, \varphi_1 \varphi_2 \boxtimes \psi \rangle.
$$

Measure $\mu_H$ we call the **H-measure** corresponding to the (sub)sequence $(u_n)$.
Existence of H-measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2_{\text{loc}}(\Omega; C^r)$, then there exist a subsequence $(u_{n'})$ and $\mu_H \in \mathcal{M}(\Omega \times S^{d-1}; \mathcal{M}_r(C))$ such that for every $\varphi_1, \varphi_2 \in C_c(\Omega)$ and $\psi \in C(S^{d-1})$

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The distribution of order zero $\mu_H$ we call the H-measure corresponding to the (sub)sequence $(u_n)$. 


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The distribution of order zero \( \mu_H \) we call the H-measure corresponding to the (sub)sequence \((u_n)\).

Above we use the notation

\[
v \cdot u := \sum v_i \bar{u}_i, \quad (v \otimes u)a := (a \cdot u)v , \text{ while } (f \boxtimes g)(x, \xi) := f(x)g(\xi) .
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**Theorem.**

\[
u_n \rightharpoonup 0 \Leftrightarrow \mu_H = 0.
\]
Example 1: Oscillation

Take a periodic function $v \in L^2(\mathbb{R}^d/\mathbb{Z}^d)$, extend it to $\mathbb{R}^d$, and write

$$v(x) = \sum_{k \in \mathbb{Z}^d} \hat{v}_k e^{2\pi i k \cdot x}.$$
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Associated H-measure

\[
\mu_H = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{v}_k|^2 \delta_{\frac{k}{|k|}}(\xi) \lambda(x).
\]
Example 2: Concentration

For $U \in L^2(\mathbb{R}^d)$ define

$$u_n(x) = n^{\frac{d}{2}} U(nx).$$
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Associated H-measure

$$\mu_H = \int_{\mathbb{R}^d} |\hat{U}(y)|^2 \delta_{\frac{y}{|y|}} (\xi) \delta_0(x) dy.$$
Parabolic H-measures — rough idea in comparison

Take a sequence $u_n \rightarrow 0$ in $L^2(\mathbb{R}^2)$, and integrate $|\widehat{\phi u_n}|^2$ along rays and project onto $S^1$. 

![Diagram of a circle and a line segment](image-url)
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Take a sequence \( u_n \to 0 \) in \( L^2(\mathbb{R}^2) \), and integrate \( |\varphi u_n|^2 \) along rays and project onto \( S^1 \) parabolas and project onto \( P^1 \)

\[
\begin{align*}
S^1 \quad & \tau = \sqrt{\xi^2 + \tau^2} \\
& \{ (\tau, \xi) : \tau^2 + \xi^2 = 1 \}
\end{align*}
\]

\[
\begin{align*}
P^1 \quad & \rho = \sqrt{\frac{\xi^2}{4} + \frac{\tau^2}{4}} \\
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parabolas and project onto $\mathbb{P}^1$

In $\mathbb{R}^2$ we have a compact curve (a surface in higher dimensions):

$S^1 \ldots r^2(\tau, \xi) := \tau^2 + \xi^2 = 1$
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and projection $\mathbb{R}^2_\ast = \mathbb{R}^2 \setminus \{0\}$ onto the curve (surface):

$$p(\tau, \xi) := \left( \frac{\tau}{r(\tau, \xi)}, \frac{\xi}{r(\tau, \xi)} \right)$$
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\]
Analytic picture

**Multiplication** by $b \in L^\infty(\mathbb{R}^2)$, a bounded operator $M_b$ on $L^2(\mathbb{R}^2)$:

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norm equal to $\|b\|_{L^\infty(\mathbb{R}^2)}$. 

Delicate part: $a$ is given only on $S^1$ or $P^1$. We extend it by the projections, $p$ or $\pi$:

if $\alpha$ is a function defined on a compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e. $a(\tau, \xi) := \alpha(\tau)\rho(\tau, \xi)$, $\xi \rho(\tau, \xi)$.

The precise scaling is contained in the projections, not the surface.
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Fourier multiplier $P_a$, for $a \in L^\infty(\mathbb{R}^2)$:

$$\widehat{P_a u} = a\hat{u}.$$
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Existence of parabolic H-measures

**Theorem.** If \( u_n \rightharpoonup 0 \) in \( L^2(\mathbb{R}^d; \mathbb{R}^r) \), then there exists its subsequence and a complex matrix Radon measure \( \mu_H \) on

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\mathbb{R}^d \times S^{d-1}
\]

such that for any \( \varphi_1, \varphi_2 \in C_0(\mathbb{R}^d) \) and \( \psi \in C(S^{d-1}) \)

one has

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\lim_{n'} \int_{\mathbb{R}^d} \widehat{\varphi_1 u_{n'}} \otimes \widehat{\varphi_2 u_{n'}} (\psi \circ p) \, d\xi = \langle \mu, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle
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\[
= \int_{\mathbb{R}^d \times S^{d-1}} \varphi_1(x) \bar{\varphi}_2(x) \psi(\xi) \, d\mu_H(x, \xi) = \int_{\mathbb{R}^d \times P^{d-1}} \varphi_1(x) \bar{\varphi}_2(x) \psi(\xi) \, d\mu_P(x, \xi).
\]

**Theorem.**

\[
u_n \xrightarrow{L^2_{\text{loc}}} 0 \iff \mu_H = 0.
\]
Example 1: Oscillation

Periodic function (take $\hat{v}_{0,0} = 0$, as before):

$$v(t, x) = \sum_{(\omega, k) \in \mathbb{Z}^{1+d}} \hat{v}_{\omega, k} e^{2\pi i (\omega t + k \cdot x)}.$$
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$$v(t, x) = \sum_{(\omega, k) \in \mathbb{Z}^{1+d}} \hat{v}_{\omega, k} e^{2\pi i (\omega t + k \cdot x)}.$$ 

For $\alpha, \beta \in \mathbb{R}^+$, a sequence of periodic functions with periods approaching zero:

$$u_n(t, x) := v(n^\alpha t, n^\beta x) = \sum_{(\omega, k) \in \mathbb{Z}^{1+d}} \hat{v}_{\omega, k} e^{2\pi i (n^\alpha \omega t + n^\beta k \cdot x)}.$$
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Periodic function (take \( \hat{v}_{0,0} = 0 \), as before):

\[
v(t, x) = \sum_{(\omega, k) \in \mathbb{Z}^{1+d}} \hat{v}_{\omega, k} e^{2\pi i (\omega t + k \cdot x)}.
\]

For \( \alpha, \beta \in \mathbb{R}^+ \), a sequence of periodic functions with periods approaching zero:

\[
\hat{u}_n(t, x) := v(n^\alpha t, n^\beta x) = \sum_{(\omega, k) \in \mathbb{Z}^{1+d}} \hat{v}_{\omega, k} e^{2\pi i (n^\alpha \omega t + n^\beta k \cdot x)}.
\]

Their Fourier transforms are:

\[
\hat{u}_n(\tau, \xi) = \sum_{(\omega, k) \in \mathbb{Z}^{1+d}} \hat{v}_{\omega, k} \delta_{n^\alpha \omega}(\tau) \delta_{n^\beta k}(\xi).
\]
Example 1: Oscillation (cont.)

\[ u_n(t, x) := v(n^\alpha t, n^\beta x) = \sum_{(\omega, k) \in \mathbb{Z}^{1+d}} \hat{v}_{\omega, k} e^{2\pi i (n^\alpha \omega t + n^\beta k \cdot x)}. \]

\((u_n)\) is a pure sequence, and its variant H-measure \(\mu_P(t, x, \tau, \xi)\) is

\[ \lambda(t, x) \begin{cases} 
\sum_{(\omega, k) \in \mathbb{Z}^{1+d}} |\hat{v}_{\omega, k}|^2 \delta_{(\omega/|\omega|, 0)}(\tau, \xi) + \sum_{k \in \mathbb{Z}^d} |\hat{v}_{0, k}|^2 \delta_{(0, k/|k|)}(\tau, \xi), & \alpha > 2\beta \\
\sum_{(\omega, k) \in \mathbb{Z}^{1+d}} |\hat{v}_{\omega, k}|^2 \delta_{(0, k/|k|)}(\tau, \xi) + \sum_{\omega \in \mathbb{Z}} |\hat{v}_{\omega, 0}|^2 \delta_{(\omega/|\omega|, 0)}(\tau, \xi), & \alpha < 2\beta \\
\sum_{(\omega, k) \in \mathbb{Z}^{1+d}} |\hat{v}_{\omega, k}|^2 \delta_{\left(\frac{\omega}{\rho^2(\omega, k)}, \frac{k}{\rho(\omega, k)}\right)}(\tau, \xi), & \alpha = 2\beta, 
\end{cases} \]
Example 2: Concentration

For \( v \in L^2(\mathbb{R}^{1+d}) \) and \( \alpha, \beta \in \mathbb{R}^+ \)

\[
u n(t, x) := n^{\alpha+\beta d} v(n^{2\alpha} t, n^{2\beta} x),\]

bounded in \( L^2(\mathbb{R}^{1+d}) \) with constant norm \( \|u_n\|_{L^2(\mathbb{R}^{1+d})} = \|v\|_{L^2(\mathbb{R}^{1+d})} \), and weakly converges to zero.
Example 2: Concentration

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\((u_n)\) is pure, with variant H-measure \( \langle \mu_P, \phi \boxtimes \psi \rangle = \)

\[
\phi(0, 0) \begin{cases} 
\int_{\mathbb{R}^{1+d}} |\hat{v}(\sigma, \eta)|^2 \psi(\frac{\sigma}{|\sigma|}, 0) d\sigma d\eta + \int_{\mathbb{R}^d} |\hat{v}(0, \eta)|^2 \psi(0, \frac{\eta}{|\eta|}) d\eta, & \alpha > 2\beta \\
\int_{\mathbb{R}^{1+d}} |\hat{v}(\sigma, \eta)|^2 \psi(0, \frac{\eta}{|\eta|}) d\sigma d\eta + \int_{\mathbb{R}} |\hat{v}(\sigma, 0)|^2 \psi(\frac{\sigma}{|\sigma|}, 0) d\sigma, & \alpha < 2\beta \\
\int_{\mathbb{R}^{1+d}} |\hat{v}(\sigma, \eta)|^2 \psi \left(\frac{\sigma}{\rho^2(\sigma, \eta)}, \frac{\eta}{\rho(\sigma, \eta)}\right) d\sigma d\eta, & \alpha = 2\beta.
\end{cases}
\]
Other variants

I. Ivec, D. Mitrović (2011): for fractional scalar conservation laws
M. Lazar, D. Mitrović (2012): velocity averaging
Other variants

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H-measures can be tailored to the equations, following the above ideas (and surmounting technical details, which do appear). The objects are quadratic in nature, and are suited essentially to linear problems.
H-distributions

Introduced by D. Mitrović and N.A. (2011)
The objects are no longer measures, but distributions (of finite order in $\xi$).
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There is also independent work of F. Rindler on microlocal defect forms (preprint on arXiv).
Existence of H-distributions

ψ : R^d → C is a Fourier multiplier on L^p(R^d) if

\[ \bar{F}(\psi F(\theta)) \in L^p(R^d), \quad \text{for } \theta \in S(R^d), \]

and

\[ S(R^d) \ni \theta \mapsto \bar{F}(\psi F(\theta)) \in L^p(R^d) \]

can be extended to a continuous mapping \( A_\psi : L^p(R^d) \to L^p(R^d) \).
Existence of H-distributions

$\psi : \mathbb{R}^d \to \mathbb{C}$ is a \textit{Fourier multiplier} on $L^p(\mathbb{R}^d)$ if

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and

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can be extended to a continuous mapping $A_\psi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$.

\textbf{Theorem.} If $u_n \rightharpoonup 0$ in $L^p_{\text{loc}}(\mathbb{R}^d)$ and $v_n \rightharpoonup^* v$ in $L^q_{\text{loc}}(\mathbb{R}^d)$ for some $q \geq \max\{p', 2\}$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and $\mu_D \in \mathcal{D}'(\mathbb{R}^d \times S^{d-1})$ of order not more than $\kappa = \lceil d/2 \rceil + 1$ in $\xi$, such that for every $\varphi_1, \varphi_2 \in C^\infty_c(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$ we have:

$$\lim_{n'} \int_{\mathbb{R}^d} A_\psi(\varphi_{1u_{n'}})(x)(\varphi_{2v_{n'}})(x) dx = \lim_{n'} \int_{\mathbb{R}^d} (\varphi_{1u_{n'}})(x)A_\psi(\varphi_{2v_{n'}})(x) dx$$

$$= \langle \mu_D, \varphi_1 \varphi_2 \psi \rangle,$$

where $A_\psi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is the multiplier with symbol $\psi \in C^\kappa(S^{d-1})$. \blacksquare
Existence of H-distributions

ψ : \( \mathbb{R}^d \to \mathbb{C} \) is a Fourier multiplier on \( L^p(\mathbb{R}^d) \) if

\[
\tilde{F}(\psi F(\theta)) \in L^p(\mathbb{R}^d), \quad \text{for } \theta \in S(\mathbb{R}^d),
\]

and

\[
S(\mathbb{R}^d) \ni \theta \mapsto \tilde{F}(\psi F(\theta)) \in L^p(\mathbb{R}^d)
\]
can be extended to a continuous mapping \( A_\psi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \).

**Theorem.** If \( u_n \rightharpoonup 0 \text{ in } L^p_{\text{loc}}(\mathbb{R}^d) \) and \( v_n \rightharpoonup^* v \text{ in } L^q_{\text{loc}}(\mathbb{R}^d) \) for some \( q \geq \max\{p', 2\} \), then there exist subsequences \((u_{n'}),(v_{n'})\) and \( \mu_D \in \mathcal{D}'(\mathbb{R}^d \times S^{d-1}) \) of order not more than \( \kappa = [d/2] + 1 \) in \( \xi \), such that for every \( \varphi_1, \varphi_2 \in C^\infty_c(\mathbb{R}^d) \) and \( \psi \in C^\kappa(S^{d-1}) \) we have:

\[
\lim_{n'} \int_{\mathbb{R}^d} A_\psi(\varphi_1 u_{n'})(x)\overline{(\varphi_2 v_{n'})}(x)dx = \lim_{n'} \int_{\mathbb{R}^d} (\varphi_1 u_{n'})(x)\overline{A_\psi(\varphi_2 v_{n'})(x)}dx
\]
\[
= \langle \mu_D, \varphi_1 \overline{\varphi_2} \psi \rangle,
\]

where \( A_\psi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) is the multiplier with symbol \( \psi \in C^\kappa(S^{d-1}) \).

\( \mu_D \) is the H-distribution corresponding to (a subsequence of) \((u_n)\) and \((v_n)\).
Existance of H-distributions

\( \psi : \mathbb{R}^d \rightarrow \mathbb{C} \) is a Fourier multiplier on \( L^p(\mathbb{R}^d) \) if

\[
\bar{F}(\psi F(\theta)) \in L^p(\mathbb{R}^d), \quad \text{for } \theta \in S(\mathbb{R}^d),
\]

and

\[
S(\mathbb{R}^d) \ni \theta \mapsto \bar{F}(\psi F(\theta)) \in L^p(\mathbb{R}^d)
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can be extended to a continuous mapping \( A_\psi : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d) \).

**Theorem.** If \( u_n \rightharpoonup 0 \) in \( L^p_{\text{loc}}(\mathbb{R}^d) \) and \( v_n \rightharpoonup^* v \) in \( L^q_{\text{loc}}(\mathbb{R}^d) \) for some \( q \geq \max\{p', 2\} \), then there exist subsequences \( (u_{n'}) \), \( (v_{n'}) \) and \( \mu_D \in D'(\mathbb{R}^d \times S^{d-1}) \) of order not more than \( \kappa = [d/2] + 1 \) in \( \xi \), such that for every \( \varphi_1, \varphi_2 \in C_\infty^c(\mathbb{R}^d) \) and \( \psi \in C^\kappa(S^{d-1}) \) we have:

\[
\lim_{n'} \int_{\mathbb{R}^d} A_\psi(\varphi_1 u_{n'}) (x) (\overline{\varphi_2 v_{n'}}) (x) dx = \lim_{n'} \int_{\mathbb{R}^d} (\varphi_1 u_{n'}) (x) \overline{A_\psi(\varphi_2 v_{n'})} (x) dx = \langle \mu_D, \varphi_1 \overline{\varphi_2} \psi \rangle,
\]

where \( A_\psi : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d) \) is the multiplier with symbol \( \psi \in C^\kappa(S^{d-1}) \).

\( \mu_D \) is the H-distribution corresponding to (a subsequence of) \( (u_n) \) and \( (v_n) \).

Of course, for \( q \in (1, \infty) \) the weak * convergence coincides with the weak convergence.
Some remarks

The question of replacing $L^2$ by $L^p$ was already raised by Gérard (1991), as it was important for nonlinear problems.
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If $(u_n), (v_n)$ are defined on $\Omega \subseteq \mathbb{R}^d$, extension by zero to $\mathbb{R}^d$ preserves the convergence, and we can apply the Theorem. $\mu_D$ is supported on $\text{Cl} \Omega \times S^{d-1}$. 

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In the Theorem we distinguish $u_n \in L^p(\mathbb{R}^d)$ and $v_n \in L^q(\mathbb{R}^d)$. If $p \geq 2$, $p' \leq 2$ so we can take $q \geq 2$; this covers the $L^2$ case (including $u_n = v_n$).

Thus we can take $u_n, v_n \rightharpoonup 0$ in $L^2_{\text{loc}}(\mathbb{R}^d)$, resulting in a distribution $\mu_D$ of order zero (a Radon measure, not necessary bounded), instead of a more general distribution.

The real improvement in Theorem is for $p < 2$. 

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The question of replacing $L^2$ by $L^p$ was already raised by Gérard (1991), as it was important for nonlinear problems.

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The real improvement in Theorem is for $p < 2$.

For applications, of interest is to extend the result to vector-valued functions. For $u_n \in L^p(\mathbb{R}^d; \mathbb{C}^k)$ and $v_n \in L^q(\mathbb{R}^d; \mathbb{C}^l)$, the result is a matrix valued distribution $\mu_D = [\mu^{ij}], i \in 1..k$ and $j \in 1..l$.

In contrast to H-measures, we cannot consider H-distributions corresponding to the same sequence, but only to a pair of sequences, and the H-distribution would correspond to a non-diagonal block for an H-measure.
H-measures and variants without a characteristic scale
  Classical H-measures
  Parabola H-measures and similar variants
  H-distributions and variants

One-scale H-measures
  Semiclassical measures
  One-scale H-measures
  Other variants

Localisation principle
  Motivation
  One-scale H-measures
One-scale H-measures

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One-scale H-measures (Tartar, 2009) are variant H-measures which have the advantages of both H-measures and semiclassical measures.
Further step would be to introduce multi-scale H-measures.
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Further step would be to introduce multi-scale H-measures.

A sample problem: consider $T > 0$, $\Omega \subseteq \mathbb{R}^d$, $U := \langle 0, T \rangle \times \Omega$, $(u_n)$ in $H^1_{\text{loc}}(U)$,

$u_n \xrightarrow{L^2_{\text{loc}}(U)} 0$, $A \in W^{1,\infty}(U)$, $f_n \xrightarrow{L^2_{\text{loc}}(U)} 0$, and $\varepsilon_n \downarrow 0$

$$\partial_t u_n - \varepsilon_n \text{div} (A \nabla u_n) = f_n.$$
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Further step would be to introduce multi-scale H-measures.
A sample problem: consider $T > 0$, $\Omega \subseteq \mathbb{R}^d$, $U := \langle 0, T \rangle \times \Omega$, $(u_n)$ in $H^1_{\text{loc}}(U)$, $u_n \overset{L^2_{\text{loc}}(U)}{\rightharpoonup} 0$, $A \in W^{1,\infty}(U)$, $f_n \overset{L^2_{\text{loc}}(U)}{\rightharpoonup} 0$, and $\varepsilon_n \searrow 0$

$$\partial_t u_n - \varepsilon_n \text{div} (A \nabla u_n) = f_n.$$

What can we say about solutions on the limit $n \to \infty$?
Semiclassical measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2(\Omega; \mathbb{C}^r)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(u_n')$ and $\mu_{sc} \in M_b(\Omega \times \mathbb{R}^d; M_r(\mathbb{C}))$ such that for every $\varphi_1, \varphi_2 \in C_0(\Omega)$ and $\psi \in S(\mathbb{R}^d)$

$$
\lim_{n'} \int_{\mathbb{R}^d} \overline{\varphi_1 u_{n'}(\xi)} \otimes \overline{\varphi_2 u_{n'}(\xi)} \psi(\varepsilon_n' \xi) \, d\xi = \langle \mu_{sc}, \varphi_1 \tilde{\varphi}_2 \boxtimes \psi \rangle.
$$

Measure $\mu_{sc}$ we call the **semiclassical measure with characteristic length $\varepsilon_n$** corresponding to the (sub)sequence $(u_n)$.
Semiclassical measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(u_n')$ and $\mu_{sc} \in \mathcal{M}(\Omega \times \mathbb{R}^d; \mathcal{M}_r(\mathbb{C}))$ such that for every $\varphi_1, \varphi_2 \in C_c(\Omega)$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$

$$\lim_{n'} \int_{\mathbb{R}^d} \varphi_1 u_n'(\xi) \otimes \varphi_2 u_n'(\xi) \psi(\varepsilon_n' \xi) \, d\xi = \langle \mu_{sc}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle.$$

The distribution of the zero order $\mu_{sc}$ we call the semiclassical measure with characteristic length $\varepsilon_n$ corresponding to the (sub)sequence $(u_n)$. 
Semiclassical measures

**Theorem.** If \( u_n \to 0 \) in \( L^2_{\text{loc}}(\Omega; \mathbb{C}^r) \), \( \varepsilon_n \downarrow 0 \), then there exist a subsequence \( (u_{n'}) \) and \( \mu_{sc} \in \mathcal{M}(\Omega \times \mathbb{R}^d; \mathcal{M}_r(\mathbb{C})) \) such that for every \( \varphi_1, \varphi_2 \in C_c(\Omega) \) and \( \psi \in \mathcal{S}(\mathbb{R}^d) \)

\[
\lim_{n'} \int_{\mathbb{R}^d} \widehat{\varphi_1 u_{n'}}(\xi) \otimes \widehat{\varphi_2 u_{n'}}(\xi) \psi(\varepsilon_n' \xi) \, d\xi = \langle \mu_{sc}, \varphi_1 \varphi_2 \Box \psi \rangle.
\]

The distribution of the zero order \( \mu_{sc} \) we call the semiclassical measure with characteristic length \( \varepsilon_n \) corresponding to the (sub)sequence \( (u_n) \).

**Theorem.**

\[
u_n \underbrace{\to 0}_{L^2_{\text{loc}}} \iff \mu_{sc} = 0 \quad \text{and} \quad (u_n) \text{ is } (\varepsilon_n) - \text{oscillatory}.
\]
Semiclassical measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(u_{n'})$ and $\mu_{sc} \in \mathcal{M}(\Omega \times \mathbb{R}^d; M_r(\mathbb{C}))$ such that for every $\varphi_1, \varphi_2 \in C_c(\Omega)$ and $\psi \in S(\mathbb{R}^d)$

$$
\lim_{n'} \int_{\mathbb{R}^d} \overline{\varphi_1 u_{n'}(\xi)} \otimes \overline{\varphi_2 u_{n'}(\xi)} \psi(\varepsilon_{n'} \xi) \, d\xi = \langle \mu_{sc}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle.
$$

The distribution of the zero order $\mu_{sc}$ we call the semiclassical measure with characteristic length $\varepsilon_n$ corresponding to the (sub)sequence $(u_n)$.

$(u_n)$ is $(\varepsilon_n)$-oscillatory if

$$(\forall \varphi \in C^\infty_c(\Omega)) \lim_{R \to \infty} \limsup_n \int_{|\xi| \geq \frac{R}{\varepsilon_n}} |\overline{\varphi u_n(\xi)}|^2 \, d\xi = 0.$$ 

**Theorem.**

$$u_n \overset{L^2_{\text{loc}}}{\rightharpoonup} 0 \iff \mu_{sc} = 0 \quad \& \quad (u_n) \text{ is } (\varepsilon_n) - \text{oscillatory}.$$
Example 1a: Oscillation — one characteristic length

\(\alpha > 0, \ k \in \mathbb{Z}^d \setminus \{0\}, \ \varepsilon_n \downarrow 0:\)

\[u_n(x) := e^{2\pi i n \alpha k \cdot x} \frac{L^2_{loc}}{\varepsilon_n} 0.\]
Example 1a: Oscillation — one characteristic length

\[\alpha > 0, \ k \in \mathbb{Z}^d \setminus \{0\}, \ \varepsilon_n \searrow 0:\]

\[u_n(x) := e^{2\pi i n \alpha_k \cdot x} \overset{L^2_{\text{loc}}}{\longrightarrow} 0.\]

\[\mu_H = \lambda(x) \otimes \delta_{\frac{k}{|k|}}(\xi)\]
Example 1a: Oscillation — one characteristic length

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\[ \mu_H = \lambda(x) \, \boxtimes \frac{\delta_k (\xi)}{|k|} \]

\[ \mu_{sc} = \lambda(x) \, \boxtimes \begin{cases} 
\delta_0 (\xi), & \lim_n n^{\alpha} \varepsilon_n = 0 \\
\delta_{ck} (\xi), & \lim_n n^{\alpha} \varepsilon_n = c \in (0, \infty) \\
0, & \lim_n n^{\alpha} \varepsilon_n = \infty \end{cases} \]
Example 1a: Oscillation — one characteristic length

\( \alpha > 0, \ k \in \mathbb{Z}^d \setminus \{0\}, \ \varepsilon_n \searrow 0: \)

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\]

\[
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\]

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        \delta_0(\xi), & \lim_n n^\alpha \varepsilon_n = 0 \\
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        0, & \lim_n n^\alpha \varepsilon_n = \infty
    \end{cases}
\]

\( n = 2 \)

- \( \sin(\sqrt{n}\pi x) \)
- \( \sin(n\pi x) \)
- \( \sin(n^2\pi x) \)
Example 1b: Oscillation — two characteristic lengths

\[ 0 < \alpha < \beta, \ k, s \in \mathbb{Z}^d \setminus \{0\}, \ \varepsilon_n \searrow 0: \]

\[ u_n(x) := e^{2\pi i n^\alpha k \cdot x} L^2_{loc} \xrightarrow{\ v_n(x) := e^{2\pi i n^\beta s \cdot x} L^2_{loc} \xrightarrow{\ v_n(x) := e^{2\pi i n^\beta s \cdot x} L^2_{loc} \xrightarrow{\} 0, \]
Example 1b: Oscillation — two characteristic lengths

$0 < \alpha < \beta$, $k, s \in \mathbb{Z}^d \setminus \{0\}$, $\varepsilon_n \searrow 0$:

$$u_n(x) := e^{2\pi i n^\alpha k \cdot x} L_{loc}^2 \downarrow 0,$$

$$v_n(x) := e^{2\pi i n^\beta s \cdot x} L_{loc}^2 \downarrow 0.$$ 

$\mu_H (\mu_{sc})$ is H-measure (semmiclassical measure with characteristic length $\varepsilon_n \searrow 0$) corresponding to $u_n + v_n$.

$$\mu_H = \lambda(x) \boxtimes \left( \delta_{\frac{k}{|k|}} + \delta_{\frac{s}{|s|}} \right)(\xi)$$
Example 1b: Oscillation — two characteristic lengths

\[ 0 < \alpha < \beta, \ k, s \in \mathbb{Z}^d \setminus \{0\}, \ \varepsilon_n \downarrow 0: \]

\[ u_n(x) := e^{2\pi i n^{\alpha} k \cdot x} \frac{L^2_{loc}}{\varepsilon_n} 0, \]

\[ v_n(x) := e^{2\pi i n^{\beta} s \cdot x} \frac{L^2_{loc}}{\varepsilon_n} 0. \]

\( \mu_H (\mu_{sc}) \) is H-measure (semiclassical measure with characteristic length \( \varepsilon_n \downarrow 0 \)) corresponding to \( u_n + v_n \).

\[ \mu_H = \lambda(x) \bigotimes \left( \delta_{\frac{k}{|k|}} + \delta_{\frac{s}{|s|}} \right)(\xi) \]

\[ \mu_{sc} = \lambda(x) \bigotimes \begin{cases} 
2\delta_0(\xi), & \text{lim}_n n^\beta \varepsilon_n = 0 \\
(\delta_{cs} + \delta_0)(\xi), & \text{lim}_n n^\beta \varepsilon_n = c \in \langle 0, \infty \rangle \\
\delta_0(\xi), & \text{lim}_n n^\beta \varepsilon_n = \infty \& \text{lim}_n n^\alpha \varepsilon_n = 0 \\
\delta_{ck}, & \text{lim}_n n^\alpha \varepsilon_n = c \in \langle 0, \infty \rangle \\
0, & \text{lim}_n n^\alpha \varepsilon_n = \infty \end{cases} \]
Compatification of $\mathbb{R}^d \setminus \{0\}$

We have:

a) $C_0(\mathbb{R}^d) \subseteq C(K_0,\infty(\mathbb{R}^d))$.

b) $\psi \in C(S^{d-1})$, $\psi \circ \pi \in C(K_0,\infty(\mathbb{R}^d))$, where $\pi(\xi) = \xi / |\xi|$.

$$
\Sigma_0 := \{0^{\xi_0} : \xi_0 \in S^{d-1}\}
$$

$$
\Sigma_\infty := \{\infty^{\xi_0} : \xi_0 \in S^{d-1}\}
$$

$$
K_{0,\infty}(\mathbb{R}^d) := (\mathbb{R}^d \setminus \{0\}) \cup \Sigma_0 \cup \Sigma_\infty
$$
Compatification of $\mathbb{R}^d \setminus \{0\}$

We have:

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$\Sigma_0 := \{0^{\xi_0} : \xi_0 \in S^{d-1}\}$

$\Sigma_\infty := \{\infty^{\xi_0} : \xi_0 \in S^{d-1}\}$

$K_{0,\infty}(\mathbb{R}^d) := (\mathbb{R}^d \setminus \{0\}) \cup \Sigma_0 \cup \Sigma_\infty$
Existence and definition of one-scale H-measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2(\Omega; \mathbb{C}^n)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(u_{n'})$ and $\mu_{sc} \in M_b(\Omega \times \mathbb{R}^d; M_r(\mathbb{C}))$ such that for every $\varphi_1, \varphi_2 \in C_0(\Omega)$ and $\psi \in S(\mathbb{R}^d)$

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \left( \hat{\varphi_1 u_{n'}}(\xi) \otimes \hat{\varphi_2 u_{n'}}(\xi) \right) \psi(\varepsilon_{n'} \xi) \, d\xi = \langle \mu_{sc}, \varphi_1 \varphi_2 \boxtimes \psi \rangle .$$

Measure $\mu_{sc}$ we call the semiclassical measure with characteristic length $\varepsilon_n$ corresponding to the (sub)sequence $(u_n)$.
Existence and definition of one-scale H-measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2(\Omega; C^r)$, $\varepsilon_n \downarrow 0$, then there exist a subsequence $(u_{n'})$ and $\mu_{K_0,\infty} \in \mathcal{M}_b(\Omega \times K_0,\infty(\mathbb{R}^d); M_r(C))$ such that for every $\varphi_1, \varphi_2 \in C_0(\Omega)$ and $\psi \in C(K_0,\infty(\mathbb{R}^d))$

$$\lim_{n'} \int_{\mathbb{R}^d} (\hat{\varphi_1 u_{n'}})(\xi) \otimes (\hat{\varphi_2 u_{n'}})(\xi) \psi(\varepsilon_{n'} \xi) d\xi = \langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle.$$ 

Measure $\mu_{K_0,\infty}$ we call 1-scale H-measure with characteristic length $\varepsilon_n$ corresponding to the (sub)sequence $(u_n)$. 

Some properties:

**Theorem.**

1. $\langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle = \langle \mu_{sc}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle$,
2. $\langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \tilde{\psi} \circ \pi \rangle = \langle \mu_{H}, \varphi_1 \bar{\varphi}_2 \boxtimes \tilde{\psi} \rangle$.

**Theorem.**

1. $\mu^*_{K_0,\infty} = \mu_{K_0,\infty}$,
2. $u_n L^2_{\text{loc}} \rightharpoonup 0 \iff \mu_{K_0,\infty} = 0$,
3. $\mu_{K_0,\infty}(\Omega \times \Sigma_{\infty}) = 0 = \Rightarrow (u_n)$ is $(\varepsilon_n)$-oscillatory.
Existence and definition of one-scale H-measures

**Theorem.** If \( u_n \to 0 \) in \( L^2(\Omega; C^r) \), \( \varepsilon_n \downarrow 0 \), then there exist a subsequence \((u_{n'})\) and \( \mu_{K_0,\infty} \in \mathcal{M}_b(\Omega \times K_0,\infty(\mathbb{R}^d); M_r(C)) \) such that for every \( \varphi_1, \varphi_2 \in C_0(\Omega) \) and \( \psi \in C(K_0,\infty(\mathbb{R}^d)) \)

\[
\lim_{n'} \int_{\mathbb{R}^d} \left( \widehat{(\varphi_1 u_{n'})}(\xi) \otimes \widehat{(\varphi_2 u_{n'})}(\xi) \psi(\varepsilon_{n'} \xi) \right) d\xi = \langle \mu_{K_0,\infty}, \varphi_1 \varphi_2 \boxtimes \psi \rangle.
\]

Measure \( \mu_{K_0,\infty} \) we call 1-scale H-measure with characteristic length \( \varepsilon_n \) corresponding to the (sub)sequence \((u_n)\).
Existence and definition of one-scale H-measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2_{\text{loc}}(\Omega; C')$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(u_{n'}')$ and $\mu_{K_0,\infty} \in \mathcal{M}(\Omega \times K_{0,\infty}(\mathbb{R}^d); M_r(C))$ such that for every $\varphi_1, \varphi_2 \in C_c(\Omega)$ and $\psi \in C(K_{0,\infty}(\mathbb{R}^d))$

$$\lim_{n'} \int_{\mathbb{R}^d} (\varphi_1 u_{n'}')(\xi) \otimes (\varphi_2 u_{n'}')(\xi) \psi(\varepsilon_n' \xi) \, d\xi = \langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle.$$ 

The distribution of the zero order $\mu_{K_0,\infty}$ we call 1-scale H-measure with characteristic length $\varepsilon_n$ corresponding to the (sub)sequence $(u_n)$.
Existence and definition of one-scale $H$-measures

**Theorem.** If $u_n \rightharpoonup 0$ in $L^2_{loc}(\Omega; \mathbb{C}^r)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(u_{n'})$ and $\mu_{K_0,\infty} \in \mathcal{M}(\Omega \times K_0,\infty(\mathbb{R}^d); M_r(\mathbb{C}))$ such that for every $\varphi_1, \varphi_2 \in C_c(\Omega)$ and $\psi \in C(K_0,\infty(\mathbb{R}^d))$

$$\lim_{n'} \int_{\mathbb{R}^d} (\widehat{\varphi_1 u_{n'}})(\xi) \otimes (\widehat{\varphi_2 u_{n'}})(\xi) \psi(\varepsilon_n' \xi) \, d\xi = \langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle.$$

The distribution of the zero order $\mu_{K_0,\infty}$ we call 1-scale $H$-measure with characteristic length $\varepsilon_n$ corresponding to the (sub)sequence $(u_n)$.

Some properties:

**Theorem.** $\varphi_1, \varphi_2 \in C_c(\Omega)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\tilde{\psi} \in C(S^{d-1})$.

a) $\langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle = \langle \mu_{sc}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle$,

b) $\langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \tilde{\psi} \circ \pi \rangle = \langle \mu_{H}, \varphi_1 \bar{\varphi}_2 \boxtimes \tilde{\psi} \rangle$. 
Existence and definition of one-scale H-measures

**Theorem.** If \( u_n \to 0 \) in \( L^2_{\text{loc}}(\Omega; C^r) \), \( \varepsilon_n \searrow 0 \), then there exist a subsequence \((u_{n'})\) and \( \mu_{K_0,\infty} \in \mathcal{M}(\Omega \times K_0,\infty(\mathbb{R}^d); M_r(C)) \) such that for every \( \varphi_1, \varphi_2 \in C_c(\Omega) \) and \( \psi \in C(K_0,\infty(\mathbb{R}^d)) \)

\[
\lim_{n'} \int_{\mathbb{R}^d} (\widehat{\varphi_1 u_{n'}})(\xi) \otimes (\widehat{\varphi_2 u_{n'}})(\xi) \psi(\varepsilon_{n'}\xi) \, d\xi = \langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle.
\]

The distribution of the zero order \( \mu_{K_0,\infty} \) we call 1-scale H-measure with characteristic length \( \varepsilon_n \) corresponding to the (sub)sequence \((u_n)\).

Some properties:

**Theorem.** \( \varphi_1, \varphi_2 \in C_c(\Omega), \psi \in \mathcal{S}(\mathbb{R}^d), \tilde{\psi} \in C(S^{d-1}). \)

\( a) \quad \langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle = \langle \mu_{sc}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle, \quad b) \quad \langle \mu_{K_0,\infty}, \varphi_1 \bar{\varphi}_2 \boxtimes \tilde{\psi} \circ \pi \rangle = \langle \mu_H, \varphi_1 \bar{\varphi}_2 \boxtimes \tilde{\psi} \rangle. \)

**Theorem.**

\( a) \quad \mu_{K_0,\infty}^* = \mu_{K_0,\infty} \quad b) \quad u_n \overset{L^2_{\text{loc}}}{\to} 0 \quad \Longleftrightarrow \quad \mu_{K_0,\infty} = 0 \quad c) \quad \mu_{K_0,\infty}(\Omega \times \Sigma_\infty) = 0 \quad \Longrightarrow \quad (u_n) \text{ is } (\varepsilon_n) \text{- oscillatory} \)
Example 1a revisited

\[ u_n(x) = e^{2\pi in^\alpha k \cdot x}, \]

\[ \mu_H = \lambda(x) \boxdot \delta_{\frac{k}{|k|}}(\xi) \]

\[ \mu_{sc} = \lambda(x) \boxdot \begin{cases} 
\delta_0(\xi), & \lim_{n} n^{\alpha} \varepsilon_n = 0 \\
\delta_{ck}(\xi), & \lim_{n} n^{\alpha} \varepsilon_n = c \in (0, \infty) \\
0, & \lim_{n} n^{\alpha} \varepsilon_n = \infty \end{cases} \]
Example 1a revisited

\[ u_n(x) = e^{2\pi i n^\alpha k \cdot x}, \]

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\delta_0(\xi), & \lim_n n^\alpha \varepsilon_n = 0 \\
\delta_{ck}(\xi), & \lim_n n^\alpha \varepsilon_n = c \in (0, \infty) \\
0, & \lim_n n^\alpha \varepsilon_n = \infty 
\end{cases} \]

\[ \mu_{K_0,\infty} = \lambda(x) \mathbin{\boxtimes} \begin{cases} 
\delta_{\frac{k}{|k|}}(\xi), & \lim_n n^\alpha \varepsilon_n = 0 \\
\delta_{ck}(\xi), & \lim_n n^\alpha \varepsilon_n = c \in (0, \infty) \\
\delta_{\infty \frac{k}{|k|}}(\xi), & \lim_n n^\alpha \varepsilon_n = \infty 
\end{cases} \]
Example 1b revisited

The corresponding measures of $u_n + v_n$ for:

$$u_n(x) = e^{2\pi in^\alpha k \cdot x}, \quad v_n(x) = e^{2\pi in^\beta s \cdot x},$$

$$\mu_H = \lambda(x) \boxdot \left( \delta_{\frac{k}{|k|}} + \delta_{\frac{s}{|s|}} \right) (\xi)$$

$$\mu_{sc} = \lambda(x) \boxdot \begin{cases} 2\delta_0(\xi), & \text{lim}_{n} n^\beta \epsilon_n = 0 \\ (\delta_0 + \delta_{cs})(\xi), & \text{lim}_{n} n^\beta \epsilon_n = c \in \langle 0, \infty \rangle \\ \delta_0(\xi), & \text{lim}_{n} n^\beta \epsilon_n = \infty \& \text{lim}_{n} n^\alpha \epsilon_n = 0 \\ \delta_{ck}, & \text{lim}_{n} n^\beta \epsilon_n = c \in \langle 0, \infty \rangle \\ 0, & \text{lim}_{n} n^\alpha \epsilon_n = \infty \end{cases}$$
Example 1b revisited

The corresponding measures of $u_n + v_n$ for:

$$u_n(x) = e^{2\pi in^\alpha k \cdot x}, \quad v_n(x) = e^{2\pi in^\beta s \cdot x},$$

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\delta_0(\xi), & \lim_n n^\beta \varepsilon_n = \infty \& \lim_n n^\alpha \varepsilon_n = 0 \\
\delta_{ck}, & \lim_n n^\alpha \varepsilon_n = c \in \langle 0, \infty \rangle \\
0, & \lim_n n^\alpha \varepsilon_n = \infty \end{cases}$$

$$\mu_{K0,\infty} = \lambda(x) \Box \begin{cases} 
(\delta_{\frac{k}{|k|}} + \delta_{\frac{s}{|s|}})(\xi), & \lim_n n^\beta \varepsilon_n = 0 \\
(\delta_{\frac{k}{|k|}} + \delta_{cs})(\xi), & \lim_n n^\beta \varepsilon_n = c \in \langle 0, \infty \rangle \\
(\delta_{ck} + \delta_{\frac{s}{|s|}})(\xi), & \lim_n n^\beta \varepsilon_n = \infty \& \lim_n n^\alpha \varepsilon_n = 0 \\
(\delta_{ck} + \delta_{\frac{s}{|s|}})(\xi), & \lim_n n^\alpha \varepsilon_n = c \in \langle 0, \infty \rangle \\
(\delta_{\frac{k}{|k|}} + \delta_{\frac{s}{|s|}}), & \lim_n n^\alpha \varepsilon_n = \infty \end{cases}$$
A similar construction can be carried out by starting with parabolic H-measures instead of classical H-measures. The resulting objects will have two scales: one corresponding to $t$, and another to $x$. 
One-scale H-distributions

This construction requires much more work. The topological construction is not enough, as we also have to check the derivatives. However, the construction is feasible, and we obtain the new objects.
Localisation principle

Most of the known applications of H-measures depend in one way or the other on the localisation principle, which gives the information on the support of H-measure. It is indispensable even for the known applications of the propagation principle.
Localisation principle

Most of the known applications of H-measures depend in one way or the other on the localisation principle, which gives the information on the support of H-measure. It is indispensable even for the known applications of the propagation principle. A similar statement holds for semiclassical measures as well.
Localisation principle for H-measures (symmetric systems)

\[ \sum_{k=1}^{d} \partial_k (A^k u) + Bu = f, \quad A^k \in C_b(\Omega; M_{r\times r}) \text{ Hermitian} \]

Assume:

\[ u_n \xrightarrow{L^2} 0, \quad \text{and defines } \mu_H \]

\[ f_n \xrightarrow{H^{-1}_{loc}} 0. \]
Localisation principle for H-measures (symmetric systems)

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\sum_{k=1}^{d} \partial_k (A^k u) + Bu = f, \quad A^k \in C_b(\Omega; M_{r \times r}) \text{ Hermitian}
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Assume:

\[
u_n \xrightarrow{L^2} 0, \quad \text{and defines } \mu_H
\]

\[
f_n \xrightarrow{H^{-1}_{loc}} 0.
\]

**Theorem.** If \(u_n\) satisfies:

\[
\sum_{k=1}^{d} \partial_k (A^k u^n) \longrightarrow 0 \quad \text{in } H^{-1}_{loc}(\Omega; C^r),
\]

then for \(P(x, \xi) := \sum_{k=1}^{d} \xi_k A^k(x)\) on \(\Omega \times S^{d-1}\) one has:

\[
P(x, \xi) \mu_H^\top = 0.
\]
Localisation principle for H-measures (symmetric systems)

\[
\sum_{k=1}^{d} \partial_{k}(A^{k}u) + Bu = f, \quad A^{k} \in C_{b}(\Omega; M_{r \times r}) \text{ Hermitian}
\]

Assume:

\[
u_{n} \xrightarrow{L^{2}} 0, \quad \text{and defines } \mu_{H}
\]

\[
f_{n} \xrightarrow{H^{-1}_{loc}} 0.
\]

**Theorem.** If \( u_{n} \) satisfies:

\[
\sum_{k=1}^{d} \partial_{k}(A^{k}u^{n}) \longrightarrow 0 \quad \text{in } H^{-1}_{loc}(\Omega; C^{r}),
\]

then for \( P(x, \xi) := \sum_{k=1}^{d} \xi_{k}A^{k}(x) \) on \( \Omega \times S^{d-1} \) one has:

\[
P(x, \xi)\mu_{H}^{\top} = 0.
\]

Thus, the support of H-measure \( \mu \) is contained in the set

\[
\{(x, \xi) \in \Omega \times S^{d-1} : \det P(x, \xi) = 0\}
\]

of points where \( P \) is a singular matrix.
Localisation principle for H-measures (symmetric systems)

\[ \sum_{k=1}^{d} \partial_k (A^k u) + Bu = f , \quad A^k \in C_b(\Omega; M_{r \times r}) \text{ Hermitian} \]

Assume:
\[ u_n \xrightarrow{L^2} 0 , \quad \text{and defines } \mu_H \]
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**Theorem.** If \( u_n \) satisfies:

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\[ \{(x, \xi) \in \Omega \times S^{d-1} : \text{det} \ P(x, \xi) = 0\} \]
of points where \( P \) is a singular matrix.

It contains a generalisation of compactness by compensation to variable coefficients.
Localisation principle for H-measures (higher derivatives)

Let $\Omega \subseteq \mathbb{R}^d$ open, $m \in \mathbb{N}$, $u_n \rightharpoonup 0$ in $L^2_{loc}(\Omega; \mathbb{C}^r)$, $A^\alpha \in C(\Omega; M_r(\mathbb{C}))$ and

$$P u_n = \sum_{|\alpha|=m} \partial_\alpha (A^\alpha u_n) \rightharpoonup 0 \text{ in } H^{-m}_{loc}(\Omega; \mathbb{C}^r).$$

Then we have

$$p(x, \xi) \mu_H^\top = 0,$$

where $p(x, \xi) = \sum_{|\alpha|=m} \xi^\alpha A^\alpha(x)$ is the principle symbol of $P$. 
Localisation principle for parabolic $H$-measures

In the parabolic case the details become more involved.

Anisotropic Sobolev spaces ($s \in \mathbb{R}; k_p(\tau, \xi) := \sqrt[4]{1 + \sigma^4(\tau, \xi)}$)

$$H^{\frac{s}{2}, s}(\mathbb{R}^{1+d}) := \left\{ u \in S' : k_p^s \hat{u} \in L^2(\mathbb{R}^{1+d}) \right\}.$$
Localisation principle for parabolic H-measures

In the parabolic case the details become more involved.

Anisotropic Sobolev spaces \((s \in \mathbb{R}; \, k_p(\tau, \xi) := \sqrt[4]{1 + \sigma^4(\tau, \xi)})\)

\[
H^{s, s}_{2, 2}(\mathbb{R}^{1+d}) := \left\{ u \in S' : k_p^s \hat{u} \in L^2(\mathbb{R}^{1+d}) \right\}.
\]

**Theorem. (localisation principle)** Let \(u_n \longrightarrow 0\) in \(L^2(\mathbb{R}^{1+d}; \mathbb{C}^r)\), uniformly compactly supported in \(t\), satisfy \((s \in \mathbb{N})\)

\[
\sqrt{\partial_t}^s (u_n \cdot b) + \sum_{|\alpha| = s} \partial_x^\alpha (u_n \cdot a_\alpha) \longrightarrow 0 \quad \text{in} \quad H^{-\frac{s}{2}, -s}_{loc}(\mathbb{R}^{1+d}),
\]

where \(b, a_\alpha \in C_b(\mathbb{R}^{1+d}; \mathbb{C}^r)\), while \(\sqrt{\partial_t}\) is a pseudodifferential operator with polyhomogeneous symbol \(\sqrt{2\pi i \tau}\), i.e.

\[
\sqrt{\partial_t} u = \mathcal{F} \left( \sqrt{2\pi i \tau} \hat{u}(\tau) \right).
\]

For a parabolic H-measure \(\mu\) associated to (a sub)sequence (of) \((u_n)\) one has

\[
\mu \left( (\sqrt{2\pi i \tau})^s b + \sum_{|\alpha| = s} (2\pi i \xi)^\alpha \bar{a_\alpha} \right) = 0.
\]
Localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbb{R}^d$ open, $m \in \mathbb{N}$, $A^\alpha \in C(\Omega; M_r(\mathbb{C}))$, $\varepsilon_n \searrow 0$, $f_n \rightrightarrows 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$ and consider:

$$P_n u_n = \sum_{|\alpha| \leq m} \varepsilon_n^{|\alpha|} \partial \alpha (A^\alpha u_n) = f_n \quad \text{in } \Omega.$$ 

Furthermore, assume that $u_n \rightrightarrows 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$. 

Problem: $\mu_{\text{sc}} = 0$ is not enough for the strong convergence!
Localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbb{R}^d$ open, $m \in \mathbb{N}$, $A^\alpha \in C(\Omega; M_r(\mathbb{C}))$, $\varepsilon_n \searrow 0$, $f_n \rightarrow 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$ and consider:

$$P_n u_n = \sum_{|\alpha| \leq m} \varepsilon_n^{\alpha} \partial^{\alpha}(A^\alpha u_n) = f_n \text{ in } \Omega.$$ 

Furthermore, assume that $u_n \rightharpoonup 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$. Then we have

$$p(x, \xi) \mu_{sc}^\top = 0,$$

where $p(x, \xi) = \sum_{|\alpha| \leq m} \xi^\alpha A^\alpha(x)$, and $\mu_{sc}$ is semiclassical measure with characteristic length $(\varepsilon_n)$, corresponding to $(u_n)$. 

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Problem: $\mu_{sc} = 0$ is not enough for the strong convergence!
One-scale H-measures

Let $u_n \rightharpoonup 0$ in $L^2_{loc}(\Omega; \mathbb{C}^r)$, $\varepsilon_n \searrow 0$, $A^\alpha \in C(\Omega; M_r(\mathbb{C}))$

$$\sum_{l \leq |\alpha| \leq m} \varepsilon^{|\alpha|-l} \partial^\alpha (A^\alpha u_n) = f_n \quad \text{in} \ \Omega,$$

where $f_n \in H^{-m}_{loc}(\Omega; \mathbb{C}^r)$ such that

$$(\forall \varphi \in C_c^\infty(\Omega)) \quad \frac{\varphi f_n}{1 + \sum_{s=1}^m \varepsilon_n^{s-l} |\xi|^s} \rightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^d; \mathbb{C}^r) \quad (C(\varepsilon_n))$$
One-scale H-measures

Let \( u_n \to 0 \) in \( L^2_{\text{loc}}(\Omega; \mathbb{C}^r) \), \( \varepsilon_n \downarrow 0 \), \( A^\alpha \in C(\Omega; M_r(\mathbb{C})) \)

\[
\sum_{l \leq |\alpha| \leq m} \varepsilon_n^{|\alpha| - l} \partial_{\alpha} (A^\alpha u_n) = f_n \quad \text{in} \ \Omega,
\]

where \( f_n \in H^{-m}_{\text{loc}}(\Omega; \mathbb{C}^r) \) such that

\[
(\forall \varphi \in C_\infty^\infty(\Omega)) \quad \frac{\hat{\varphi f_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\xi|^s} \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d; \mathbb{C}^r) \quad (C(\varepsilon_n))
\]

**Lemma.**

a) \((C(\varepsilon_n))\) is equivalent to

\[
(\forall \varphi \in C_\infty^\infty(\Omega)) \quad \frac{\hat{\varphi f_n}}{1 + |\xi|^l + \varepsilon_n^{m-l} |\xi|^m} \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d; \mathbb{C}^r).
\]

b) \((\exists k \in l..m) \ f_n \to 0 \ \text{in} \ H^{-k}_{\text{loc}}(\Omega; \mathbb{C}^r) \quad \implies \quad (\varepsilon_n^{k-l} f_n) \ \text{satisfies} \ (C(\varepsilon_n)).\)
Localisation principle

\[
\sum_{l \leq |\alpha| \leq m} \varepsilon_{n}^{|\alpha|-l} \partial_{\alpha} (A^{\alpha} u_{n}) = f_{n} \quad \text{in } \Omega ,
\]

\[
(\forall \varphi \in C_{c}^{\infty} (\Omega)) \quad \frac{\widehat{\varphi f_{n}}}{1 + \sum_{s=l}^{m} \varepsilon_{n}^{s-l} |\xi|^{s}} \rightarrow 0 \quad \text{in } L^{2}(\mathbb{R}^{d}; \mathbb{C}^{r}) . \quad (C(\varepsilon_{n}))
\]

**Theorem.** [Tartar (2009)] Under previous assumptions and \( l = 1 \), 1-scale H-measure \( \mu_{K_{0},\infty} \) with characteristic length \( \varepsilon_{n} \) corresponding to \( (u_{n}) \) satisfies

\[
\text{supp } (p \mu_{K_{0},\infty}^{\top}) \subseteq \Omega \times \Sigma_{0} ,
\]

where

\[
p(x, \xi) := \sum_{1 \leq |\alpha| \leq m} (2\pi i)^{|\alpha|} \frac{\xi^{\alpha}}{|\xi| + |\xi|^{m}} A^{\alpha}(x) .
\]
Localisation principle

\[
\sum_{l \leq |\alpha| \leq m} \varepsilon_n^{|\alpha|-l} \partial_\alpha (A^\alpha u_n) = f_n \quad \text{in } \Omega,
\]

\[
(\forall \varphi \in C^\infty_c (\Omega)) \quad \frac{\hat{\varphi f_n}}{1 + \sum_{s=l}^{m} \varepsilon_n^{s-l} |\xi|^s} \rightarrow 0 \quad \text{in } L^2(\mathbb{R}^d; \mathbb{C}^r). \quad (C(\varepsilon_n))
\]

**Theorem.** Under previous assumptions, 1-scale H-measure \( \mu_{K_0, \infty} \) with characteristic length \( \varepsilon_n \) corresponding to \( (u_n) \) satisfies

\[
p_{\mu_{K_0, \infty}}^T = 0,
\]

where

\[
p(x, \xi) := \sum_{l \leq |\alpha| \leq m} (2\pi i)^{|\alpha|} \frac{\xi^\alpha}{|\xi|^l + |\xi|^m} A^\alpha(x).
\]
Localisation principle — final generalisation

**Theorem.** $\epsilon_n > 0$ bounded $u_n \to 0$ in $L^2_{\text{loc}}(\Omega; C^r)$ and

$$
\sum_{l \leq |\alpha| \leq m} \epsilon_n^{|\alpha|-l} \partial_\alpha (A^n_\alpha u_n) = f_n,
$$

where $A^n_\alpha \in C(\Omega; M_{r}(C))$, $A^n_\alpha \to A^\alpha$ uniformly on compact sets, and $f_n \in H_{\text{loc}}^{-m}(\Omega; C^r)$ satisfies $(C(\epsilon_n))$.

Then for $\omega_n \to 0$ such that $\lim_n \frac{\omega_n}{\epsilon_n} = c \in [0, \infty]$, corresponding 1-scale H-measure $\mu_{K_0,\infty}$ with characteristic length $\omega_n$ satisfies

$$p\mu_{K_0,\infty}^\top = 0,$$

where

$$p(x, \xi) := \begin{cases} 
\sum_{|\alpha|=l} \frac{\xi^\alpha}{|\xi|^l+|\xi|^m} A^\alpha(x), & \lim_n \frac{\omega_n}{\epsilon_n} = \infty \\
\sum_{l \leq |\alpha| \leq m} \left( \frac{2\pi i}{c} \right)^{|\alpha|} \frac{\xi^\alpha}{|\xi|^l+|\xi|^m} A^\alpha(x), & \lim_n \frac{\omega_n}{\epsilon_n} = c \in (0, \infty) \\
\sum_{|\alpha|=m} \frac{\xi^\alpha}{|\xi|^l+|\xi|^m} A^\alpha(x), & \lim_n \frac{\omega_n}{\epsilon_n} = 0 
\end{cases}$$
Localisation principle — final generalisation

**Theorem.** \( \varepsilon_n > 0 \) bounded \( u_n \to 0 \) in \( L^2_{\text{loc}}(\Omega; C^r) \) and

\[
\sum_{l \leq |\alpha| \leq m} \varepsilon_n^{|\alpha|-l} \partial_\alpha (A_\alpha^\alpha u_n) = f_n ,
\]

where \( A_\alpha^\alpha \in C(\Omega; M_r(C)) \), \( A_\alpha^\alpha \longrightarrow A^\alpha \) uniformly on compact sets, and \( f_n \in H^{-m}_{\text{loc}}(\Omega; C^r) \) satisfies \((C(\varepsilon_n))\).

Then for \( \omega_n \to 0 \) such that \( \lim_n \frac{\omega_n \varepsilon_n}{c} = c \in [0, \infty] \), corresponding 1-scale H-measure \( \mu_{K_0, \infty} \) with characteristic length \( \omega_n \) satisfies

\[
p^{\mu_{K_0, \infty}} = 0 ,
\]

where

\[
p(\mathbf{x}, \xi) := \begin{cases}
\sum_{|\alpha| = l} \frac{\xi^\alpha}{|\xi|^l + |\xi|^m} A^\alpha(x) , & \lim_n \frac{\omega_n \varepsilon_n}{c} = \infty \\
\sum_{l \leq |\alpha| \leq m} \left( \frac{2\pi i}{c} \right)^{|\alpha|} \frac{\xi^\alpha}{|\xi|^l + |\xi|^m} A^\alpha(x) , & \lim_n \frac{\omega_n \varepsilon_n}{c} = c \in \langle 0, \infty \rangle \\
\sum_{|\alpha| = m} \frac{\xi^\alpha}{|\xi|^l + |\xi|^m} A^\alpha(x) , & \lim_n \frac{\omega_n \varepsilon_n}{c} = 0
\end{cases}
\]

Moreover, if there exists \( \varepsilon_0 > 0 \) such that \( \varepsilon_n > \varepsilon_0 \), \( n \in \mathbb{N} \), we can take

\[
p(\mathbf{x}, \xi) := \sum_{|\alpha| = m} \frac{\xi^\alpha}{|\xi|^m} A^\alpha(x) .
\]
Localisation principle (H-measures and semiclassical measures)

- Using the preceding theorem and \( \mu_{K_0, \infty} = \mu_H \) on \( \Omega \times S^{d-1} \), we can obtained the known localisation principle for H-measures.
Using the preceding theorem and $\mu_{K_0,\infty} = \mu_H$ on $\Omega \times S^{d-1}$, we can obtained the known localisation principle for H-measures.

**Theorem.** Under the assumptions of the preceding theorem, we have

$$p(x, \xi)\mu_{sc}^\top = 0,$$

where

$$p(x, \xi) := \begin{cases} 
\sum_{|\alpha|=l} \xi^\alpha A^\alpha(x), & \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\
\sum_{l \leq |\alpha| \leq m} \left( \frac{2\pi i}{c} \right)^{|\alpha|} \xi^\alpha A^\alpha(x), & \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle \\
\sum_{|\alpha|=m} \xi^\alpha A^\alpha(x), & \lim_n \frac{\omega_n}{\varepsilon_n} = 0
\end{cases}$$