One-scale variants of H-measures

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Days of Analysis, Novi Sad, 7^{th} July 2014
(a conference in honour of prof. Stanković's anniversary)

Joint work with Marko Erceg and Martin Lazar

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Before 1990: Tools to describe passage from one scale to another in the models of continuum mechanics included compactness by compensation, Young measures, and defect measures.
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$$\widehat{\varphi u_n}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} (\varphi u_n)(\mathbf{x}) d\mathbf{x} \ .$$

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On the other hand, by the Plancherel theorem: $\|\widehat{\varphi u_n}\|_{\mathrm{L}^2(\mathbf{R}^d)} = \|\varphi u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}$.

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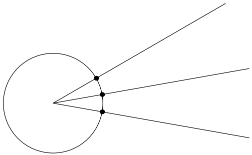

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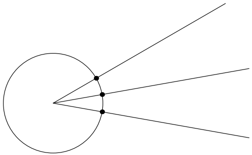

He took $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, and considered the limits of the integrals:

$$\lim_n \int_{\mathbf{R}^d} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)|\widehat{\varphi u_n}|^2 d\boldsymbol{\xi} = \int_{\mathrm{S}^{d-1}} \psi(\boldsymbol{\xi}) d\nu_\varphi(\boldsymbol{\xi}) \,.$$

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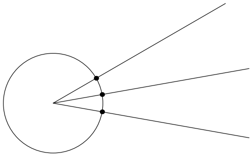

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(a bounded sequence of Radon measures has an accumulation point)
The crucial question was how does this limit depend on φ.

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_H \in \mathcal{M}_{\mathrm{b}}(\Omega\times\mathrm{S}^{d-1};\mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ such that for every $\varphi_1,\varphi_2\in\mathrm{C}_0(\Omega)$ and $\psi\in\mathrm{C}(\mathrm{S}^{d-1})$

$$\lim_{n'}\int_{\mathbf{R}^d}\widehat{\varphi_1\mathsf{u}_{n'}}(\boldsymbol{\xi})\otimes\widehat{\varphi_2\mathsf{u}_{n'}}(\boldsymbol{\xi})\psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\,d\boldsymbol{\xi}=\langle\boldsymbol{\mu}_H,\varphi_1\bar{\varphi}_2\boxtimes\psi\rangle\,.$$

Measure $\boldsymbol{\mu}_H$ we call the H-measure corresponding to the (sub)sequence (u_n).

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Above we use the notation

$$\mathsf{v} \cdot \mathsf{u} := \sum v_i \bar{u}_i \, , \quad (\mathsf{v} \otimes \mathsf{u})\mathsf{a} := (\mathsf{a} \cdot \mathsf{u})\mathsf{v} \text{ , while} \quad (f \boxtimes g)(\mathbf{x}, \boldsymbol{\xi}) := f(\mathbf{x}) g(\boldsymbol{\xi}) \, .$$

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Theorem.

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_H = \mathbf{0} \;.$$

Example 1: Oscillation

Take a periodic function $v \in \mathrm{L}^2(\mathbf{R}^d/\mathbf{Z}^d)$, extend it to $\mathbf{R}^d$, and write

$$v(\mathbf{x}) = \sum_{\mathsf{k} \in \mathbf{Z}^d} \hat{v}_{\mathsf{k}} e^{2\pi i \mathsf{k} \cdot \mathbf{x}} \; .$$

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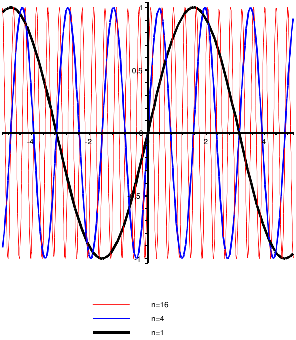
Assume that $\hat{v}_0 = 0$, and define $u_n(\mathbf{x}) = v(n\mathbf{x})$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$.



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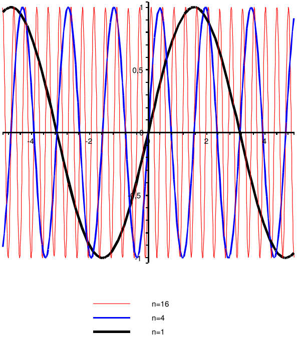


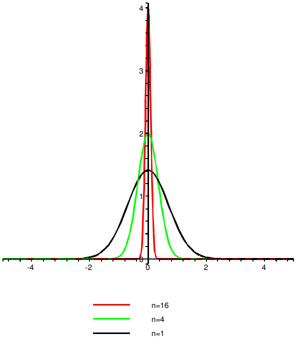
Associated H-measure

$$\mu_H = \sum_{\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}} |\hat{v}_{\mathsf{k}}|^2 \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) \lambda(\mathbf{x}) \; .$$

Example 2: Concentration

For $U \in \mathrm{L}^2(\mathbf{R}^d)$ define

$$u_n(x) = n^{\frac{d}{2}} U(nx) \ .$$



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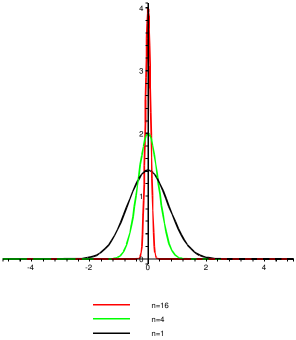

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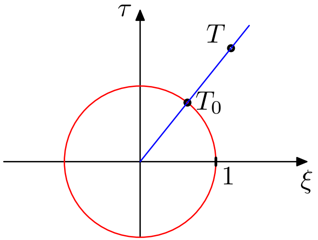
Parabolic H-measures — rough idea in comparison

Take a sequence $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^2)$, and integrate $|\widehat{\varphi \mathsf{u}_n}|^2$ along rays and project onto S^1



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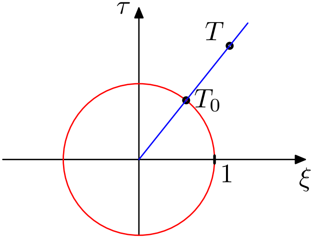

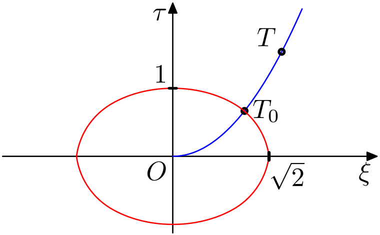

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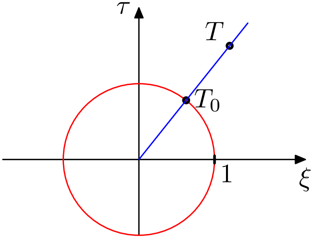

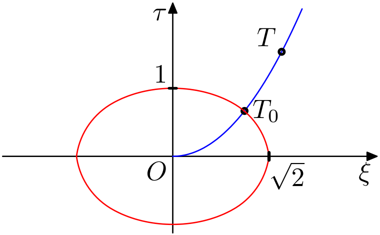

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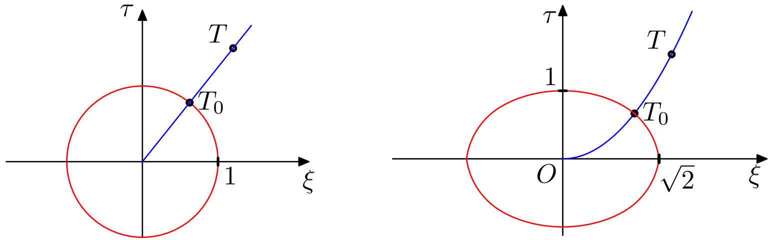

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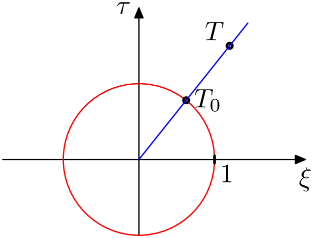

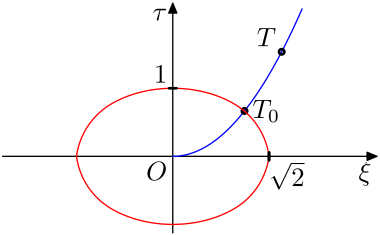

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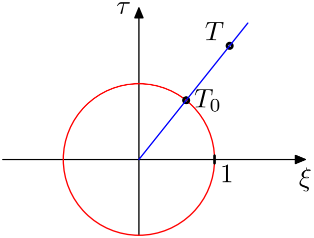

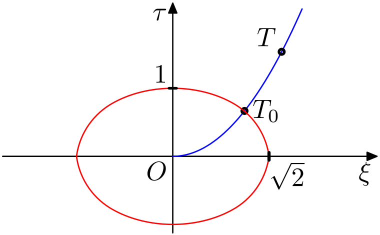

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The precise scaling is contained in the projections, not the surface.

Existence of parabolic H-measures

Theorem. If $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there exists its subsequence and a complex matrix Radon measure $\boldsymbol{\mu}_H$ on

$$\mathbf{R}^d \times \mathrm{S}^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and

$$\psi \in \mathrm{C}(\mathrm{S}^{d-1})$$

one has

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ p \ ) \, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, d\boldsymbol{\mu}_H(\mathbf{x}, \boldsymbol{\xi})$$

Existence of parabolic H-measures

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Existence of parabolic H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there exists its subsequence and a complex matrix Radon measure $\boldsymbol{\mu}_{HP}$ on

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Theorem.

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_H = \mathbf{0} \,.$$

Example 1: Oscillation

Periodic function (take $\hat{v}_{0,0} = 0$, as before):

$$v(t, \mathbf{x}) = \sum_{(\omega, \mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega, \mathsf{k}} \, e^{2\pi i (\omega t + \mathsf{k} \cdot \mathbf{x})} \, .$$

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For $\alpha, \beta \in \mathbf{R}^+$, a sequence of periodic functions with periods approaching zero:

$$u_n(t, \mathbf{x}) := v(n^\alpha t, n^\beta \mathbf{x}) = \sum_{(\omega, \mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega, \mathsf{k}}\, e^{2\pi i (n^\alpha \omega t + n^\beta \mathsf{k} \cdot \mathbf{x})} .$$

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Their Fourier transforms are:

$$\hat{u}_n(\tau, \boldsymbol{\xi}) = \sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}}\, \delta_{n^\alpha\omega}(\tau)\delta_{n^\beta\mathsf{k}}(\boldsymbol{\xi}) \,.$$

Example 1: Oscillation (cont.)

$$u_n(t,\mathbf{x}) := v(n^\alpha t, n^\beta \mathbf{x}) = \sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}}\, e^{2\pi i(n^\alpha \omega t + n^\beta \mathsf{k}\cdot\mathbf{x})}\,.$$

(u_n) is a pure sequence, and its variant H-measure $\mu_P(t,\mathbf{x},\tau,\boldsymbol{\xi})$ is

$$\lambda(t,\mathbf{x}) \begin{cases} \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \omega\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}) + \sum_{\mathsf{k}\in\mathbf{Z}^d} |\hat{v}_{0,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}), & \alpha > 2\beta \\ \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \mathsf{k}\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}) + \sum_{\omega\in\mathbf{Z}} |\hat{v}_{\omega,0}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}), & \alpha < 2\beta \\ \displaystyle\sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{\left(\frac{\omega}{\rho^2(\omega,\mathsf{k})},\frac{\mathsf{k}}{\rho(\omega,\mathsf{k})}\right)}(\tau,\boldsymbol{\xi}), & \alpha = 2\beta, \end{cases}$$

Example 2: Concentration

For $v \in \mathrm{L}^2(\mathbf{R}^{1+d})$ and $\alpha, \beta \in \mathbf{R}^+$

$$u_n(t, \mathbf{x}) := n^{\alpha+\beta d} v(n^{2\alpha} t, n^{2\beta} \mathbf{x}),$$

bounded in $\mathrm{L}^2(\mathbf{R}^{1+d})$ with constant norm $\|u_n\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \|v\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}$, and weakly converges to zero.

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(u_n) is pure, with variant H-measure $\langle \mu_P, \phi \boxtimes \psi \rangle =$

$$\phi(0,0) \begin{cases} \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(\frac{\sigma}{|\sigma|}, 0) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}^d} |\hat{v}(0, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|})\, d\boldsymbol{\eta}, & \alpha > 2\beta \\ \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}} |\hat{v}(\sigma, 0)|^2 \psi(\frac{\sigma}{|\sigma|}, 0)\, d\sigma, & \alpha < 2\beta \\ \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi\left(\frac{\sigma}{\rho^2(\sigma, \boldsymbol{\eta})}, \frac{\boldsymbol{\eta}}{\rho(\sigma, \boldsymbol{\eta})}\right) d\sigma d\boldsymbol{\eta}, & \alpha = 2\beta. \end{cases}$$

Other variants

E. Yu. Panov (2009): ultraparabolic H-measures
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H-measures can be tailored to the equations, following the above ideas (and surmounting technical details, which do appear).
The objects are quadratic in nature, and are suited essentially to linear problems.

H-distributions

Introduced by D. Mitrović and N.A. (2011)
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There is also independent work of F. Rindler on microlocal defect forms (preprint on arXiv).

Existence of H-distributions

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)\ , \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

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Theorem. *If $u_n \longrightarrow 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ and $v_n \xrightarrow{\ *\ } v$ in $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ for some $q \geqslant \max\{p', 2\}$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and $\mu_D \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ of order not more than $\kappa = [d/2] + 1$ in $\boldsymbol{\xi}$, such that for every $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ we have:*

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Of course, for $q \in \langle 1, \infty\rangle$ the weak $*$ convergence coincides with the weak convergence.

Some remarks

The question of replacing L^2 by L^p was already raised by Gérard (1991), as it was important for nonlinear problems.

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If (u_n), (v_n) are defined on $\Omega \subseteq \mathbf{R}^d$, extension by zero to $\mathbf{R}^d$ preserves the convergence, and we can apply the Theorem. μ_D is supported on $\mathrm{Cl}\,\Omega \times \mathrm{S}^{d-1}$.

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In the Theorem we distinguish $u_n \in \mathrm{L}^p(\mathbf{R}^d)$ and $v_n \in \mathrm{L}^q(\mathbf{R}^d)$. If $p \geqslant 2$, $p' \leqslant 2$ so we can take $q \geqslant 2$; this covers the L^2 case (including $u_n = v_n$).
Thus we can take $u_n, v_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, resulting in a distribution μ_D of order zero (a Radon measure, not necessary bounded), instead of a more general distribution.
The real improvement in Theorem is for $p < 2$.

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Thus we can take $u_n, v_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, resulting in a distribution μ_D of order zero (a Radon measure, not necessary bounded), instead of a more general distribution.
The real improvement in Theorem is for $p < 2$.

For applications, of interest is to extend the result to vector-valued functions. For $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu}_D = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

In contrast to H-measures, we cannot consider H-distributions corresponding to the same sequence, but only to a pair of sequences, and the H-distribution would correspond to a non-diagonal block for an H-measure.

One-scale H-measures

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Luc Tartar (1990) constructed a similar object on an example, but Gérard's construction was easier; later they jointly simplified it further.

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Further step would be to introduce multi-scale H-measures.

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A sample problem: consider $T > 0$, $\Omega \subseteq \mathbf{R}^d$, $U := \langle 0, T\rangle \times \Omega$, (u_n) in $\mathrm{H}^1_{\mathrm{loc}}(U)$,
$u_n \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}(U)} 0$, $\mathbf{A} \in \mathrm{W}^{1,\infty}(U)$, $f_n \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}(U)} 0$, and $\varepsilon_n \searrow 0$

$$\partial_t u_n - \varepsilon_n \mathsf{div}\,(\mathbf{A}\nabla u_n) = f_n\ .$$

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What can we say about solutions on the limit $n \to \infty$?

Semiclassical measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_{sc} \in \mathcal{M}_{\mathrm{b}}(\Omega\times\mathbf{R}^d;\mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ such that for every $\varphi_1,\varphi_2 \in \mathrm{C}_0(\Omega)$ and $\psi \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \psi(\varepsilon_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}_{sc}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \rangle\,.$$

Measure $\boldsymbol{\mu}_{sc}$ we call the semiclassical measure with characteristic length ε_n corresponding to the (sub)sequence (u_n).

Semiclassical measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_{sc} \in \mathcal{M}(\Omega \times \mathbf{R}^d; \mathrm{M}_r(\mathbf{C}))$ such that for every $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathcal{S}(\mathbf{R}^d)$

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$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{sc} = \mathbf{0} \quad \& \quad (\mathsf{u}_n) \text{ is } (\varepsilon_n)-\text{oscillatory} .$$

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$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{\mathsf{0}\}$, $\varepsilon_n \searrow 0$:

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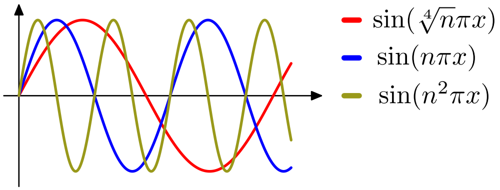

Example 1b: Oscillation — two characteristic lengths

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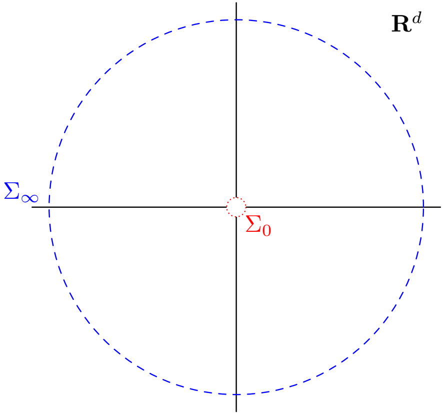
Compatification of $\mathbf{R}^d \setminus \{0\}$



$$\Sigma_0 := \{0^{\boldsymbol{\xi}_0} \;:\; \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}$$

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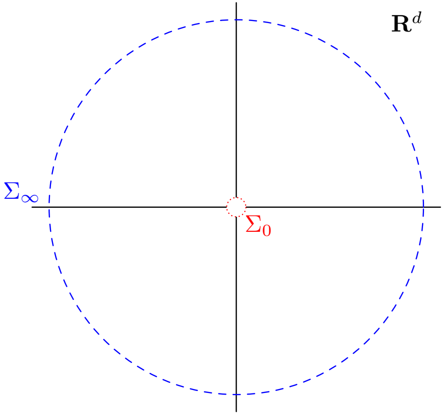

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We have:
a) $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

Existence and definition of one-scale H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, $\varepsilon_n \searrow 0$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_{sc} \in \mathcal{M}_{\mathrm{b}}(\Omega\times\mathbf{R}^d;\mathrm{M}_r(\mathbf{C}))$ such that for every $\varphi_1,\varphi_2 \in \mathrm{C}_0(\Omega)$ and $\psi \in \mathcal{S}(\mathbf{R}^d)$

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The corresponding measures of $u_n + v_n$ for:

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A similar construction can be carried out by starting with parabolic H-measures instead of classical H-measures.
The resulting objects will have two scales: one corresponding to t, and another to x.

One-scale H-distributions

This construction requires much more work. The topological construction is not enough, as we also have to check the derivatives.
However, the construction is feasible, and we obtain the new objects.

Localisation principle

Most of the known applications of H-measures depend in one way or the other on the localisation principle, which gives the information on the support of H-measure.

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It is indispensable even for the known applications of the propagation principle.
A similar statement holds for semiclassical measures as well.

Localisation principle for H-measures (symmetric systems)

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}) + \mathbf{B}\mathsf{u} = \mathsf{f} \ , \ \mathbf{A}^k \in \mathrm{C}_b(\Omega; \mathrm{M}_{r\times r}) \text{ Hermitian}$$

Assume:

$$\mathsf{u}_n \xrightharpoonup{\mathrm{L}^2} 0 \ , \qquad \text{and defines } \boldsymbol{\mu}_H$$

$$\mathsf{f}_n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \ .$$

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Theorem. If u_n satisfies:

$$\sum_{k=1}^{d} \partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \longrightarrow 0 \;\; \text{in}\, \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \, ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{k=1}^{d} \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\mu}_H^{\top} = \mathbf{0} \, .$$

Localisation principle for H-measures (symmetric systems)

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}) + \mathbf{B}\mathsf{u} = \mathsf{f} \;\;,\;\; \mathbf{A}^k \in \mathrm{C}_b(\Omega; \mathrm{M}_{r\times r}) \text{ Hermitian}$$

Assume:

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0 \;, \qquad \text{and defines } \boldsymbol{\mu}_H$$

$$\mathsf{f}_n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \;.$$

Theorem. If u_n satisfies:

$$\sum_{k=1}^{d} \partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \longrightarrow 0 \;\text{ in } \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \;,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{k=1}^{d} \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

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Thus, the support of H-measure $\boldsymbol{\mu}$ is contaned in the set
$\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{P}$ is a singular matrix.

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It contains a generalisation of compactness by compensation to variable coefficients.

Localisation principle for H-measures (higher derivatives)

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$ and

$$\mathbf{P}\mathsf{u}_n = \sum_{|\boldsymbol{\alpha}|=m} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) \longrightarrow 0 \text{ in } \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)\,.$$

Then we have

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}_H^{\top} = \mathbf{0}\,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$ is the principle simbol of $\mathbf{P}$.

Localisation principle for parabolic H-measures

In the parabolic case the details become more involved.

Anisotropic Sobolev spaces ($s \in \mathbf{R}$; $k_p(\tau, \boldsymbol{\xi}) := \sqrt[4]{1 + \sigma^4(\tau, \boldsymbol{\xi})}$)

$$\mathrm{H}^{\frac{s}{2}, s}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{S}' : k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d}) \right\} .$$

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Theorem. **(localisation principle)** Let $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$, uniformly compactly supported in t, satisfy ($s \in \mathbf{N}$)

$$\sqrt{\partial_t}^{\,s} (\mathsf{u}_n \cdot \mathsf{b}) + \sum_{|\boldsymbol{\alpha}| = s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} (\mathsf{u}_n \cdot \mathsf{a}_{\boldsymbol{\alpha}}) \longrightarrow 0 \quad \text{in} \quad \mathrm{H}_{\mathrm{loc}}^{-\frac{s}{2}, -s}(\mathbf{R}^{1+d}) \,,$$

where $\mathsf{b}, \mathsf{a}_{\boldsymbol{\alpha}} \in \mathrm{C}_b(\mathbf{R}^{1+d}; \mathbf{C}^r)$, while $\sqrt{\partial_t}$ is a pseudodifferential operator with polyhomogeneous symbol $\sqrt{2\pi i \tau}$, i.e.

$$\sqrt{\partial_t} u = \overline{\mathcal{F}} \left(\sqrt{2\pi i \tau} \, \hat{u}(\tau) \right) .$$

For a parabolic H-measure $\boldsymbol{\mu}$ associated to (a sub)sequence (of) (u_n) one has

$$\boldsymbol{\mu} \left((\sqrt{2\pi i \tau})^s \overline{\mathsf{b}} + \sum_{|\boldsymbol{\alpha}| = s} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}} \, \overline{\mathsf{a}}_{\boldsymbol{\alpha}} \right) = 0.$$

Localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$, $\varepsilon_n \searrow 0$, $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and consider:

$$P_n \mathsf{u}_n = \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,.$$

Furthermore, assume that $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

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Then we have

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\mu}_{sc}^{\top} = \mathbf{0} \,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

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Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

One-scale H-measures

Let $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\varepsilon_n \searrow 0$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_r(\mathbf{C}))$

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

where $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

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Lemma.
a) $(\mathrm{C}(\varepsilon_n))$ is equivalent to

$$(\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)\,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \quad \Longrightarrow \quad (\varepsilon_n^{k-l} \mathsf{f}_n)$ satisfies $(\mathrm{C}(\varepsilon_n))$.

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in\mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem. [Tartar (2009)] Under previous assumptions and $l=1$, 1-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ε_n corresponding to (u_n) satisfies

$$\operatorname{supp}\,(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

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Theorem. Under previous assumptions, 1-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ε_n corresponding to (u_n) satisfies

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

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Localisation principle — final generalisation

Theorem. $\varepsilon_n > 0$ bounded $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n\,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ uniformly on compact sets, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ satisfies $(\mathrm{C}(\varepsilon_n))$.
Then for $\omega_n \to 0$ such that $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in [0, \infty]$, corresponding 1-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ω_n satisfies

$$\mathbf{p} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

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Moreover, if there exists $\varepsilon_0 > 0$ such that $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, we can take

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Localisation principle (H-measures and semiclassical measures)

- Using the preceding theorem and $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_H$ on $\Omega \times \mathrm{S}^{d-1}$, we can obtained the known localisation principle for H-measures.

Localisation principle (H-measures and semiclassical measures)

- Using the preceding theorem and $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_H$ on $\Omega \times \mathrm{S}^{d-1}$, we can obtained the known localisation principle for H-measures.

  Theorem. Under the assumptions of the preceding theorem, we have

  $$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}_{sc}^{\top} = \mathbf{0}\,,$$

  where

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