Parabolic equations as Friedrichs systems

Nenad Antonić

Department of Mathematics
Faculty of Science
University of Zagreb

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...and Marko Erceg and Ivana Crnjac

Symmetric positive systems

K. O. Friedrichs: *Symmetric hyperbolic linear differential equations, Commun. Pure Appl. Math.* **7** (1954) 345–392.
Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).

A generalisation:
K. O. Friedrichs: *Symmetric positive linear differential equations, Commun. Pure Appl. Math.* **11** (1958), 333–418.

Goals:
– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

– unified treatment of equations and systems of different type,

– more recently: better numerical properties.

All of Gårding's theory of general elliptic equations, or Lerray's of general hyperbolic equations, is not covered.

The development of theory is nowadays mostly motivated by the needs in development of numerical methods.

Friedrichs' system (KOF1958)

Assumptions:
$d, r \in \mathbf{N}$, $\Omega \subseteq \mathbf{R}^d$ open and bounded with Lipschitz boundary Γ;

$\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, and $\mathbf{B} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{C}))$ satisfying

(F1) matrix functions $\mathbf{A}_k$ are hermitian: $\mathbf{A}_k = \mathbf{A}_k^*$;

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad (\text{ae on } \Omega)\,.$$

The operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{C}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{C}^r)$

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u}$$

is called *the symmetric positive operator* (*the Friedrichs operator*), and

$$\mathcal{L}\mathsf{u} = \mathsf{f}$$

the symmetric positive system (*the Friedrichs system*).

Symmetric hyperbolic systems (KOF1954)

$$\sum_{k=1}^{d} \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{D}\mathsf{u} = \mathsf{f}$$

In divergence form:

$$\sum_{k=1}^{d} \partial_k (\mathbf{A}^k \mathsf{u}) + (\mathbf{D} - \partial_k \mathbf{A}^k)\mathsf{u} = \mathsf{f}$$

It is symmetric if all matrices $\mathbf{A}^k$ are real and symmetric; and uniformly hyperbolic if there is a $\boldsymbol{\xi} \in \mathbf{R}^d$ such that for any $\mathbf{x} \in \mathrm{Cl}\,\Omega$ the matrix $\xi_k \mathbf{A}^k(\mathbf{x})$ is positive definite.

Such systems can easily be transformed into the form of a Friedrichs system.

It is known that the wave equation, the Maxwell and the Dirac system can be written as an equivalent symmetric hyperbolic system, thus as a Friedrichs system as well.

An example – scalar elliptic equation

Let $\Omega \subseteq \mathbf{R}^2$ be a bounded region, $\mu > 0$ and $f \in \mathrm{L}^2(\Omega)$ given; the equation

$$-\triangle u + \mu u = f$$

can be written as a first-order system

$$\begin{cases} \mathsf{p} + \nabla u = 0 \\ \mu u + \mathrm{div}\mathsf{p} = f \end{cases} ,$$

which is a Friedrichs system with the choice of

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

With some adjustments, the same holds for the equation

$$-\mathsf{div}\,(\mathbf{A}\nabla u) + \mathsf{div}\,(u\mathsf{b}) + cu = f\ ,$$

where we take $\mathbf{A} \in \mathrm{W}^{2,\infty}(\Omega; \mathrm{Psym})$, $\mathsf{b} \in \mathrm{W}^{1,\infty}(\Omega; \mathbf{R}^d)$ and $c \in \mathrm{L}^\infty(\Omega)$.

Example – heat equation

Take $\Omega \subseteq \mathbf{R}^d$ open and bounded, with Lipschitz boundary Γ, and $T > 0$.
Define: $\Omega_T := \langle 0, T\rangle \times \Omega$, $\Gamma_d := \langle 0, T\rangle \times \Gamma$, $\Gamma_0 := \{0\} \times \Gamma$ and $\Gamma_T := \{T\} \times \Gamma$.
Consider the heat equation with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \mathsf{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ \quad u = 0 \text{ on } \Gamma_d \\ u_{|_{\Gamma_0}} = 0\,. \end{cases}$$

It can be written as a Friedrichs system in the form:

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}}\mathsf{u}_d + c u_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d = f \end{cases}$$

(note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$, where $\mathsf{u}_d = -\mathbf{A}\nabla u$, and $u_{d+1} = u$). Indeed

$$\begin{bmatrix} \mathbf{0} & 0 \\ 0^\top & 1 \end{bmatrix} \partial_t \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} + \sum_{i=1}^{d} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 1 \\ \vdots & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 1 \end{bmatrix} \partial_{x^i} \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} + \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ -(\mathbf{A}^{-1}\mathsf{b})^\top & c \end{bmatrix} \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

The condition (F1) holds. The positivity condition $\mathbf{B} + \mathbf{B}^\top \geqslant 2\mu_0\mathbf{I}$ is fulfilled if and only if $c - \frac{1}{4}\mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{b}$ is uniformly positive.

Boundary conditions

Boundary conditions are enforced via a matrix valued boundary field:

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^{d} \nu_k \mathbf{A}_k \in \mathrm{L}^{\infty}(\Gamma; \mathrm{M}_r(\mathbf{C}))\,,$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)$ is the outward unit normal on Γ, and

$$\mathbf{M} \in \mathrm{L}^{\infty}(\Gamma; \mathrm{M}_r(\mathbf{C})).$$

Boundary condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_{\Gamma}} = 0$$

is sufficient for treatment of different types of usual boundary conditions.

Assumptions on boundary matrix $\mathbf{M}$

We assume (for ae $\mathbf{x} \in \Gamma$) [KOF1958]

(FM1) $$(\forall\, \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FM2) $$\mathbf{C}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,.$$

Such $\mathbf{M}$ is called *the admissible boundary condition*.

The boundary problem: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{C}^r)$ find u such that

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases}.$$

Does the above work for standard problems in continuum physics?

Elliptic equation – different boundary conditions

$$
\begin{array}{ccc}
\mathbf{M} & \mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} & (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix}_{|\Gamma} = 0 \\
\begin{bmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 2\nu_1 \\ 0 & 0 & 2\nu_2 \\ 0 & 0 & 0 \end{bmatrix} & u_{|\Gamma} = 0 \\
\begin{bmatrix} 0 & 0 & \nu_1 \\ 0 & 0 & \nu_2 \\ -\nu_1 & -\nu_2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\nu_1 & 2\nu_2 & 0 \end{bmatrix} & \boldsymbol{\nu} \cdot (\nabla u)_{|\Gamma} = 0 \\
\begin{bmatrix} 0 & 0 & \nu_1 \\ 0 & 0 & \nu_2 \\ -\nu_1 & -\nu_2 & 2\alpha \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\nu_1 & 2\nu_2 & 2\alpha \end{bmatrix} & \boldsymbol{\nu} \cdot (\nabla u)_{|\Gamma} + \alpha u_{|\Gamma} = 0
\end{array}
$$

All above matrices $\mathbf{M}$ satisfy (FM).

$$
\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.
$$

Different ways to enforce boundary conditions

Instead of

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u} = 0 \quad \text{on } \Gamma\,,$$

Lax proposed boundary conditions with

$$\mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma\,,$$

where $N = \{N(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ is a family of subspaces of $\mathbf{C}^r$.

Boundary problem:

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ \mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma \end{cases}.$$

Assumptions on N

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$) [PDL]

(FX1) $N(\mathbf{x})$ is non-negative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$:

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,;$$

(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which (properly) contains $N(\mathbf{x})$;

or [RSP&LS1966]

Let $N(\mathbf{x})$ and $\tilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ satisfy (for ae $\mathbf{x} \in \Gamma$)

(FV1)

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0$$

$$(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0$$

(FV2)

$$\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}\,.$$

Equivalence of different descriptions of boundary conditions

Theorem. *It holds*

$$\textit{(FM1)–(FM2)} \quad \Longleftrightarrow \quad \textit{(FX1)–(FX2)} \quad \Longleftrightarrow \quad \textit{(FV1)–(FV2)}\,,$$

with

$$N(\mathbf{x}) := \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big)\,.$$

∎

In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].

Classical results on well-posedness

Friedrichs:
– uniqueness of the classical solution,
– existence of a *weak* solution (under some additional assumptions).

Contributions (and particular cases):
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
– the meaning of traces for functions in the graph space,
– weak well-posedness results under additional assumptions (on $\mathbf{A}_{\boldsymbol{\nu}}$),
– regularity of solution,
– numerical treatment.

Shortcommings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

However, since the beginning of 21^{st} century the numerical advantages of FS have overshadowed that.

Classical theory of Friedrichs systems

- Friedrichs systems
- Boundary conditions for Friedrichs systems

Abstract formulation of Friedrichs systems

- Operator-theoretic formulation
- Kreĭn space formalism and different boundary conditions
- Bijective realisations

Parabolic equations (non-stationary diffusion)

- Heat equation as a Friedrichs system
- Two-field theory

Homogenisation of Friedrichs systems

- Homogenisation
- Stationary diffusion and heat equation, as particular cases

New approach...

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, *Comm. Partial Diff. Eq.* **32** (2007) 317–341.

– abstract setting (operators on Hilbert spaces),

– intrinsic criterion for the bijectivity of *Friedrichs'* operator,

– avoiding the question of traces for functions in the graph space,

– investigation of different formulations of boundary conditions,

. . . and new open questions.

Effectively, they considered only the real case.

Assumptions

Let L be real (complex) Hilbert space (L' is (anti)dual of L), $\mathcal{D} \subseteq L$ a dense subspace, and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ linear unbounded operators satisfying

(T1) $$(\forall\, \varphi, \psi \in \mathcal{D}) \quad \langle\, T\varphi \mid \psi \,\rangle_L = \langle\, \varphi \mid \tilde{T}\psi \,\rangle_L \,,$$

(T2) $$(\exists\, c > 0)(\forall\, \varphi \in \mathcal{D}) \quad \|(T + \tilde{T})\varphi\|_L \leqslant c\|\varphi\|_L \,,$$

(T3) $$(\exists\, \mu_0 > 0)(\forall\, \varphi \in \mathcal{D}) \quad \langle\, (T + \tilde{T})\varphi \mid \varphi \,\rangle_L \geqslant 2\mu_0\|\varphi\|_L^2 \,.$$

$(T, \widetilde{T})$ is referred to as a joint pair of abstract Friedrichs operators.

Recall the Friedrichs operator:
$\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{C}^r)$ and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ are defined by

$$T\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{B}\mathsf{u} \,,$$

$$\tilde{T}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + (\mathbf{B}^* + \sum_{k=1}^{d} \partial_k\mathbf{A}_k)\mathsf{u} \,,$$

where $\mathbf{A}_k$ and $\mathbf{B}$ are as above (they satisfy (F1)–(F2)).

Then T and $\tilde{T}$ satisfy (T1)–(T3). . . . fits in this framework.

Extension of operators, starting from $(T, \tilde{T}) = (T_1, \tilde{T}_1)$

$\mathcal{D}$ is an inner product space when equipped with the *graph norm* stemming from

$$\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle_L + \langle T\cdot \mid T\cdot \rangle_L .$$

By W_0 denote the completion of $\mathcal{D}$ in the graph norm, the same for $\tilde{T}$ by (T2). $W_0 \leqslant L$ by (T1), and both T and $\tilde{T}$ extend to bounded operators from W_0 to L, which we denote by $(T_2, \tilde{T}_2)$.

The following embeddings are dense and continuous (we have a Gel'fand triplet):

$$W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0' .$$

Let $T_3 := \tilde{T}_2' \in \mathcal{L}(L; W_0')$ be the Banach adjoint of $\tilde{T}_2 : W_0 \longrightarrow L$, and $\tilde{T}_3 := T_2'$. Thus we have defined $(T_3, \tilde{T}_3)$.

Note that the *graph space*

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\} \leqslant L$$

is a Hilbert space with respect to $\langle \cdot \mid \cdot \rangle_T$.

$(T_4, \tilde{T}_4)$ are defined as restrictions of T_3 and $\tilde{T}_3$ to W.

This produces the maximal pair of abstract Friedrichs operators $(T_4, \widetilde{T}_4)$, mapping $T_4, \widetilde{T}_4 : W \to L$, which are associated to the initial pair $(T, \widetilde{T})$.

Well-posedness for abstract Friedrichs operator (our goal)

Find sufficient conditions for a subspace $W_0 \leqslant V \leqslant W$ such that $T_{4|_V} : V \longrightarrow L$ is an isomorphism.

As the continuity in the graph norm holds for any restriction to a closed subspace V of W, the key question is bijectivity.

If T is the classical Friedrichs operator $\mathcal{L}$, then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ we have

$$
{}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,.
$$

With the assumptions:

(FV1)
$$
\begin{aligned}
(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad & \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,\\
(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad & \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0\,,
\end{aligned}
$$

(FV2)
$$
\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^\perp \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^\perp\,,
$$

we are lead to consider the following subspaces V and $\tilde{V}$ in the functional framework:

(V1)
$$
\begin{aligned}
(\forall\, u \in V) \qquad & {}_{W'}\langle\, Du, u\,\rangle_W \geqslant 0\,,\\
(\forall\, v \in \tilde{V}) \qquad & {}_{W'}\langle\, Dv, v\,\rangle_W \leqslant 0\,,
\end{aligned}
$$

(V2)
$$
V = D(\tilde{V})^0\,, \qquad \tilde{V} = D(V)^0\,.
$$

Boundary operator

Sufficient coditions were obtained by [EGC2007] and [AB2010] using the *boundary operator* $D \in \mathcal{L}(W; W')$:

$$ {}_{W'}\langle\, Du, v \,\rangle_W := \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,, \qquad u, v \in W \,. $$

D is symmetric: ${}_{W'}\langle\, Du, v \,\rangle_W = \overline{{}_{W'}\langle\, Dv, u \,\rangle_W}$ and satisfies

$$ \begin{aligned} &\ker D = W_0 \\ &\operatorname{im} D = W_0^0 := \{g \in W' : (\forall\, u \in W_0) \quad {}_{W'}\langle\, g, u \,\rangle_W = 0\} \,. \end{aligned} $$

For a given joint pair of abstract FO $(T, \widetilde{T})$, a pair $(V, \widetilde{V})$ of subspaces of W is said to allow the (V)-boundary conditions relative to $(T, \widetilde{T})$ when:

(V1) the boundary operator has opposite sign on V and on $\widetilde{V}$, in the sense that

$$ \begin{aligned} &(\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u \,\rangle_W \geqslant 0 \,, \\ &(\forall\, v \in \widetilde{V}) \qquad {}_{W'}\langle\, Dv, v \,\rangle_W \leqslant 0 \,; \end{aligned} $$

(V2) the image via D of either space has, as annihilator, the other space, namely

$$ V = D(\widetilde{V})^0 \qquad \text{and} \qquad \widetilde{V} = D(V)^0 \,. $$

Well-posedness theorem

Theorem. *Under assumptions* $(T1) - (T3)$ *and* $(V1) - (V2)$, *the operators* $T_{|_V} : V \longrightarrow L$ *and* $\tilde{T}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$ *are isomorphisms.* ∎

In the real case [EGC2007].

What happens with other types of boundary conditions (FX) and (FM)?

Indefinite inner product characterisation

For $u, v \in W$ define

$$[\, u \mid v \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,,$$

which is an indefinite inner product on W, and we consider subspaces V and $\tilde{V}$ satisfying:

(V1)
$$(\forall\, v \in V) \quad [\, v \mid v \,] \geqslant 0 \,,$$
$$(\forall\, v \in \tilde{V}) \quad [\, v \mid v \,] \leqslant 0 \,;$$

(V2)
$$V = \tilde{V}^{[\perp]} \,, \qquad \tilde{V} = V^{[\perp]} \,.$$

($^{[\perp]}$ stands for $[\,\cdot \mid \cdot\,]$-orthogonal complement)

Correspondence — maximal b.c.

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$)

(FX1) $$(\forall \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which contains $N(\mathbf{x})$,

subspace V is maximal non-negative in $(W, [\,\cdot \mid \cdot\,])$:

(X1) V is non-negative in $(W, [\,\cdot \mid \cdot\,])$: $\quad (\forall v \in V) \quad [\,v \mid v\,] \geqslant 0\,,$

(X2) there is no non-negative subspace in $(W, [\,\cdot \mid \cdot\,])$ containing V .

Correspondence — admissible b.c.

admissible boundary condition: there exists a matrix function $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{C})$ such that (for ae $\mathbf{x} \in \Gamma$)

(FM1) $$(\forall \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FM2) $$\mathbf{C}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,.$$

abstract admissible boundary condition: there exists $M \in \mathcal{L}(W; W')$ such that

(M1) $$(\forall u \in W) \quad {}_{W'}\langle\, (M + M^*)u, u \,\rangle_W \geqslant 0\,,$$

(M2) $$W = \ker(D - M) + \ker(D + M)\,.$$

Similarly as for boundary operator D, starting from a classical Friedrichs operator $\mathcal{L}$, operator M is defined for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathrm{Cl}\,\Omega; \mathbf{R}^r)$ by

$${}_{W'}\langle\, M\mathsf{u}, \mathsf{v} \,\rangle_W = \int_\Gamma \mathbf{M}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,.$$

Equivalence of different descriptions of b.c.

Theorem. *(**classical**)* *It holds*

(FM1)–(FM2) $\quad\Longleftrightarrow\quad$ *(FV1)–(FV2)* $\quad\Longleftrightarrow\quad$ *(FX1)–(FX2)*,

with

$$N(\mathbf{x}) := \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big)\,.$$

∎

Theorem. *[EGC2007, AB2010]* *It holds*

(M1)–(M2) $\quad\Longleftrightarrow\quad$ *(V1)–(V2)* $\quad\Longleftrightarrow\quad$ *(X1)–(X2)*,

with

$$V := \ker(D - M)\,.$$

∎

Theorem. *[Ern, Guermond, Caplain, 2007]* *Let $(T, \widetilde{T})$ be a joint pair of Friedrichs systems and let $(V, \widetilde{V})$ satisfy (V1)–(V2). Then $T_4|_V : V \to L$ and $\widetilde{T}_4|_{\widetilde{V}} : \widetilde{V} \to L$ are closed bijective realisations of T and $\widetilde{T}$, respectively.* ■

Can we say something more about extensions T_4, $\widetilde{T}_4$, and conditions (V)?

Theorem. *$(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators iff*

(i) $T \subseteq \widetilde{T}^*$ *and* $\widetilde{T} \subseteq T^*$*;*

(ii) $\overline{T + \widetilde{T}}$ *is a bounded self-adjoint operator in L with strictly positive bottom;*

(iii) dom $\overline{T}$ = dom $\overline{\widetilde{T}} = W_0$ *and* dom T^* = dom $\widetilde{T}^* = W$*.* ■

In fact: $T_4 = \widetilde{T}^*$ and $\widetilde{T}_4 = T^*$.

Bijective realisations with signed boundary map

Theorem. *Let $(T, \widetilde{T})$ be a pair of operators on the Hilbert space L satisfying conditions $(\mathrm{T1})$–$(\mathrm{T2})$, and let $(V, \widetilde{V})$ be a pair of subspaces of L. Then (V2) is equivalent to*

i) $W_0 \subseteq V \subseteq W, \;\; W_0 \subseteq \widetilde{V} \subseteq W$,

ii) V *and* $\widetilde{V}$ *are closed in* W, *and*

iii) $(\widetilde{T}^*|_V)^* \;=\; T^*|_{\widetilde{V}}, \;\; (T^*|_{\widetilde{V}})^* \;=\; \widetilde{T}^*|_V$. ∎

We are seeking bijective closed operators $S \equiv \widetilde{T}^*|_V$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^* ,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$.

In the following we work with closed T and $\widetilde{T}$.

Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space L. For a closed $T \subseteq S \subseteq \widetilde{T}^*$ such that $(\mathrm{dom}\, S, \mathrm{dom}\, S^*)$ satisfies $(V1)$ we call (S, S^*) an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.

Heat equation (recall)

... with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \operatorname{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ u = 0 \text{ on } \Gamma_d \cup \Gamma_0 \,, \end{cases}$$

...as a Friedrichs system:

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}}\mathsf{u}_d + cu_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d = f \end{cases}$$

(note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$, where $\mathsf{u}_d = -\mathbf{A}\nabla u$, and $u_{d+1} = u$).

More precisely, we take $f \in \mathrm{L}^2(\Omega_T)$, $c \in \mathrm{L}^\infty(\Omega_T)$, $\mathsf{b} \in \mathrm{L}^\infty(\Omega_T; \mathbf{R}^d)$ and $\mathbf{A} \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$. Furthermore, we suppose that there are constants $\beta \geqslant \alpha > 0$ such that $\mathbf{A}(\mathbf{x}, t)$ is a symmetric matrix with eigenvalues between α and β, almost everywhere on Ω_T.

The positivity condition $\mathbf{C} + \mathbf{C}^\top \geqslant 2\mu_0\mathbf{I}$ is fulfilled if and only if the Schur complement $c - \frac{1}{4}\mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{b}$ is uniformly positive, i.e. if there exists a constant $\gamma > 0$ such that $c - \frac{1}{4}\mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{b} \geqslant \gamma$ on Ω_T.

Friedrichs operator and the graph space

The operator T is given by

$$T \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d \\ \partial_t u_{d+1} + \operatorname{\mathsf{div}}_{\mathbf{x}} \mathsf{u}_d + c u_{d+1} - \mathbf{A}^{-1}\mathsf{b} \cdot \mathsf{u}_d \end{bmatrix},$$

while the corresponding graph space is

$$\begin{aligned} W &= \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1}) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \\ &\qquad\qquad \& \quad \partial_t u_{d+1} + \operatorname{\mathsf{div}}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(\Omega_T) \Big\} \\ &= \Big\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \Big\} \\ &= \Big\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \Big\} . \end{aligned}$$

Properties of the last component

Lemma. *The projection* $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top \mapsto u_{d+1}$ *is a continuous linear operator from* W *to* $W(0,T)$, *which is continuously embedded into* $\mathrm{C}([0,T];\mathrm{L}^2(\Omega))$. ∎

The space

$$W(0,T) = \left\{ u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) : \partial_t u \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega)) \right\},$$

is a Banach space when equipped by norm

$$\|\mathsf{u}\|_{W(0,T)} = \sqrt{\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\partial_t u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))}}.$$

Finally

Let us return to the initial-boundary value problem for the heat equation

$$\begin{cases} \partial_t u - \mathrm{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ \qquad u = 0 \text{ on } \Gamma_d \cup \Gamma_0 \ , \end{cases}$$

to be represented as a boundary-value problem for a Friedrichs system.

Take

$$V = \Big\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. on } \Omega \Big\} ,$$
$$\widetilde{V} = \Big\{ \mathsf{v} \in W : v_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v_{d+1}(\cdot,T) = 0 \text{ a.e. on } \Omega \Big\} .$$

Do they satisfy (V1)–(V2)? Technical...

Theorem. *The above V and $\widetilde{V}$ satisfy (V1)–(V2), and therefore the operator $T_{|_V} : V \longrightarrow L$ is an isomorphism.* ∎

Two-field theory...

Consider the heat equation with $\mathsf{b} = 0$ and $c = 0$:

$$\begin{cases} \partial_t u - \mathsf{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) = f \text{ in } \Omega_T \\ u = 0 \text{ on } \Gamma_d \cup \Gamma_0 \ . \end{cases}$$

The above cannot be applied for $c = 0$.

However, we can apply the two field theory: developed by Ern and Guermond for elliptic problems.

The vector unknown has to be split into two parts, i.e. the matrices need to be of the form

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{0} & \mathsf{B}^k \\ (\mathsf{B}^k)^\top & a^k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix} ,$$

where $\mathsf{B}^k \in \mathbf{R}^d$ are constant vectors, $a^k \in \mathrm{W}^{1,\infty}(\Omega_T)$, $\mathbf{C}^d \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$ and $c^{d+1} \in \mathrm{L}^\infty(\Omega_T)$, $k \in 1..(d+1)$.

For the heat equation the matrices have this form!

. . . with partial coercivity

Instead of coercivity (positivity) condition (F2), the following is required:

$$(\exists\, \mu_1 > 0)(\forall\, \boldsymbol{\xi} = (\boldsymbol{\xi}_d, \xi_{d+1}) \in \mathbf{R}^{d+1})$$

$$\Big(\mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d+1} \partial_k \mathbf{A}_k\Big)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant 2\mu_1 |\boldsymbol{\xi}_d|^2 \qquad (\text{a.e. on } \Omega)\,,$$

$$(\exists\, \mu_2 > 0)(\forall\, \mathsf{u} \in V \cup \widetilde{V})$$

$$\sqrt{\langle\, \mathcal{L}\mathsf{u} \mid \mathsf{u}\,\rangle_{\mathrm{L}^2(\Omega_T;\mathbf{R}^{d+1})}} + \|\mathsf{B}u_{d+1}\|_{\mathrm{L}^2(\Omega_T;\mathbf{R}^d)} \geqslant \mu_2 \|u_{d+1}\|_{\mathrm{L}^2(\Omega_T)}\,,$$

where $\mathsf{B}u_{d+1} := \sum_{k=1}^{d+1} \mathsf{B}^k \partial_k u_{d+1} = \nabla_{\mathbf{x}} u_{d+1}$.

For our system both conditions are trivially fulfilled.

Therefore, we have the well-posedness result.

Boundary operator and well-posedness result

Recall *boundary operator* $D \in \mathcal{L}(W, W')$: for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ we define

$$ {}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) \,. $$

Theorem. *Let (F1)–(F2) hold, and let subspaces V and $\widetilde{V}$ of W satisfy*

(*V1*)
$$ (\forall\, \mathsf{u} \in V) \qquad {}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W \geqslant 0 \,, $$
$$ (\forall\, \mathsf{v} \in \widetilde{V}) \qquad {}_{W'}\langle\, D\mathsf{v}, \mathsf{v}\,\rangle_W \leqslant 0 \,, $$

(*V2*)
$$ V = D(\widetilde{V})^0 \,, \qquad \widetilde{V} = D(V)^0 \,, $$

where ^{0} stands for the annihilator. Then the operator $T_{|_V} : V \longrightarrow L$ is an isomorphism and for every $\mathsf{u} \in V$ the following estimate holds:

$$ \|\mathsf{u}\|_T \leqslant \sqrt{\frac{1}{\alpha^2} + 1}\, \|T\mathsf{u}\|_L $$

∎

Classical theory of Friedrichs systems

- Friedrichs systems
- Boundary conditions for Friedrichs systems

Abstract formulation of Friedrichs systems

- Operator-theoretic formulation
- Kreĭn space formalism and different boundary conditions
- Bijective realisations

Parabolic equations (non-stationary diffusion)

- Heat equation as a Friedrichs system
- Two-field theory

Homogenisation of Friedrichs systems

- Homogenisation
- Stationary diffusion and heat equation, as particular cases

Homogenisation setting

Krešimir Burazin, Marko Vrdoljak: *Homogenisation theory for Friedrichs systems*, Comm. Pure Appl. Analysis **13** (2014) 1017–1044.

A sequence of Friedrichs systems $T_n \mathsf{u}_n = \mathsf{f}\,,\ \mathsf{f} \in L$.

Here u_n naturally belongs to the graph space of T_n.

Our assumptions must secure that every u_n belongs to the same space, with clearly identified topology that shall be used...

- $\mathbf{A}_k$ are symmetric constant matrices in $\mathrm{M}_r(R)$, $k \in 1..d$
- $$\mathbf{C} \in \mathcal{M}_r(\alpha, \beta; \Omega) = \Big\{\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R})) : (\forall\, \boldsymbol{\xi} \in \mathbf{R}^d)$$
  $$\mathbf{C}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \alpha|\boldsymbol{\xi}|^2 \ \&\ \mathbf{C}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \tfrac{1}{\beta}|\mathbf{C}\boldsymbol{\xi}|^2\Big\}.$$ and

$$T_0 \mathsf{u} = \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) = \sum_{k=1}^{d} \mathbf{A}_k \partial_k \mathsf{u}\,,$$

so that $T := T_0 + \mathbf{C}$ is the Friedrichs operator. Its graph space

$$W := \{\mathsf{u} \in L : T\mathsf{u} \in L\} = \{\mathsf{u} \in L : T_0\mathsf{u} \in L\}\,.$$

Moreover, we have equivalence of norms ($\gamma = \sqrt{\max\{2, 1 + 2\beta^2\}}$):

$$\|\mathsf{u}\|_T \leqslant \gamma \|\mathsf{u}\|_{T_0} \leqslant \gamma^2 \|\mathsf{u}\|_T\,, \quad \text{for any } \mathbf{C}\,.$$

Boundary operator and a priori bound

The boundary operator D corresponding to the operator T does not depend on particular $\mathbf{C}$ from $\mathcal{M}_r(\alpha, \beta; \Omega)$.
If V is a subspace of W that satisfies (V), well-posedness result implies that $T_{|_V} : V \longrightarrow L$ is an isomorphism, with

$$\|\mathsf{u}\|_{T_0} \leqslant \gamma \|\mathsf{u}\|_T \leqslant \gamma \sqrt{\frac{1}{\alpha^2} + 1}\, \|T\mathsf{u}\|_L \,, \quad \mathsf{u} \in V \,.$$

Therefore, for fixed T_0 and V satisfying (V), we have a priori bound

$$(\exists\, c > 0)(\forall\, \mathbf{C} \in \mathcal{M}_r(\alpha, \beta; \Omega))(\forall\, \mathsf{u} \in V) \quad \|\mathsf{u}\|_{T_0} \leq c \|(\mathcal{L}_0 + \mathbf{C})\mathsf{u}\|_L \,.$$

Note that constant c depends only on T_0, α and β.

In the sequel $\mathcal{L}_0 = \sum_{k=1}^d \mathbf{A}_k \partial_k$ and V are fixed.

H-convergence

A sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha,\beta;\Omega)$ H-converges to $\mathbf{C}\in\mathcal{M}_r(\alpha',\beta';\Omega)$ with respect to T_0 and V if, for any $\mathsf{f}\in L$, the sequence (u_n) in V defined by $\mathsf{u}_n := T_n^{-1}\mathsf{f}\in V$, with $T_n=\mathcal{L}_0+\mathbf{C}_n$, satisfies

$$\begin{aligned} \mathsf{u}_n &\rightharpoonup \mathsf{u} \quad \text{in } L\,,\\ \mathbf{C}_n\mathsf{u}_n &\rightharpoonup \mathbf{C}\mathsf{u} \quad \text{in } L\,, \end{aligned}$$

where $\mathsf{u}=T^{-1}\mathsf{f}\in V$, with $T=\mathcal{L}_0+\mathbf{C}$.

As $T_0\mathsf{u}_n+\mathbf{C}_n\mathsf{u}_n=f=T_0\mathsf{u}+\mathbf{C}\mathsf{u}$, the second convergence implies $T_0\mathsf{u}_n\rightharpoonup T_0\mathsf{u}$ in L, which gives the weak convergence $\mathsf{u}_n\rightharpoonup\mathsf{u}$ in W.

Theorem. *Let $F=\{\mathsf{f}_n : n\in\mathbf{N}\}$ be a dense countable family in $\mathrm{L}^2(\Omega;\mathbf{R}^r)$, $\mathbf{C},\mathbf{D}\in\mathcal{M}_r(\alpha,\beta;\Omega)$, and $\mathsf{u}_n,\mathsf{v}_n\in V$ solutions of $(T_0+\mathbf{C})\mathsf{u}_n=\mathsf{f}_n$ and $(T_0+\mathbf{D})\mathsf{v}_n=\mathsf{f}_n$, respectively. Furthermore, let*

$$d(\mathbf{C},\mathbf{D}) := \sum_{n=1}^{\infty} 2^{-n}\frac{\|\mathsf{u}_n-\mathsf{v}_n\|_{\mathrm{H}^{-1}(\Omega;\mathbf{R}^r)}+\|\mathbf{C}\mathsf{u}_n-\mathbf{D}\mathsf{v}_n\|_{\mathrm{H}^{-1}(\Omega;\mathbf{R}^r)}}{\|\mathsf{f}_n\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}}\,.$$

Then the function $d:\mathcal{M}_r(\alpha,\beta;\Omega)\times\mathcal{M}_r(\alpha,\beta;\Omega)\longrightarrow\mathbf{R}$ forms a metric on the set $\mathcal{M}_r(\alpha,\beta;\Omega)$, and the H-convergence is equivalent to the sequential convergence in this metric space. ∎

Compactness

Additional assumptions: for every sequence $\mathbf{C}_n \in \mathcal{M}_r(\alpha, \beta; \Omega)$ and every $\mathsf{f} \in L$, the sequence $\mathsf{u}_n \in V$ defined by $\mathsf{u}_n := (T_0 + \mathbf{C}_n)^{-1}\mathsf{f}$ satisfies the following: if (u_n) weakly converges to u in W, then also

(K1) $$_{W'}\langle D\mathsf{u}_n, \mathsf{u}_n \rangle_W \longrightarrow {}_{W'}\langle D\mathsf{u}, \mathsf{u} \rangle_W \, ,$$

or

(K2) $$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \quad \langle T_0\mathsf{u}_n \mid \varphi\mathsf{u}_n \rangle_L \longrightarrow \langle T_0\mathsf{u} \mid \varphi\mathsf{u} \rangle_L \, .$$

Theorem. *For fixed T_0 and V, if family $\mathcal{M}_r(\alpha, \beta; \Omega)$ satisfies (K1) and (K2), then it is compact with respect to H-convergence, i.e. from any sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha, \beta; \Omega)$ one can extract a H-converging subsequence whose limit belongs to $\mathcal{M}_r(\alpha, \beta; \Omega)$.* ∎

The proof follows the original proof of Spagnolo in the case of parabolic G-convergence.

Stationary diffusion equation as Friedrichs system

$$-\text{div}\,(\mathbf{A}\nabla u) + cu = f$$

in $\Omega \subseteq \mathbf{R}^d$, with $f \in \mathrm{L}^2(\Omega)$, $\mathbf{A} \in \mathcal{M}_d(\alpha', \beta'; \Omega)$ and $c \in \mathrm{L}^\infty(\Omega)$ with $\frac{1}{\beta'} \leqslant c \leqslant \frac{1}{\alpha'}$, for some $\beta' \geq \alpha' > 0$. $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, for $k = 1, \dots, d$

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ 0 & c \end{bmatrix} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_{d+1}(\mathbf{R}))\,,$$

$$T\mathsf{u} = \sum_{k=1}^{d} \mathbf{A}_k \partial_k \mathsf{u} + \mathbf{C}\mathsf{u} = \mathsf{f}$$

$$T_0 \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla u_{d+1} \\ \text{div}\,\mathsf{u}_d \end{bmatrix}, \quad \mathbf{C}\mathsf{u} = \begin{bmatrix} \mathbf{A}^{-1}\mathsf{u}_d \\ cu_{d+1} \end{bmatrix}.$$

Graph space $\dots W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$

Boundary conditions

Dirichlet

$$V_D = \widetilde{V}_D := \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)\,,$$

Neumann

$$V_N = \widetilde{V}_N := \left\{ \left(\mathsf{u}_d, u_{d+1}\right)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = 0 \right\},$$

Robin

$$V_R := \left\{ \left(\mathsf{u}_d, u_{d+1}\right)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = a {u_{d+1}}_{|_\Gamma} \right\},$$

$$\widetilde{V}_R := \left\{ \left(\mathsf{u}_d, u_{d+1}\right)^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = -a {u_{d+1}}_{|_\Gamma} \right\}.$$

Properties (K1) and (K2)

(K1) For any sequence (u_n) in V

$$\mathsf{u}_n \rightharpoonup \mathsf{u} \quad \Longrightarrow \quad {}_{W'}\langle\, D\mathsf{u}_n, \mathsf{u}_n \,\rangle_W \longrightarrow {}_{W'}\langle\, D\mathsf{u}, \mathsf{u} \,\rangle_W$$

$$\begin{aligned} {}_{W'}\langle\, D\mathsf{u}, \mathsf{u} \,\rangle_W &= 2\, {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \boldsymbol{\nu}\cdot\mathsf{u}_d, u_{d+1} \,\rangle_{\mathrm{H}^{\frac{1}{2}}} \\ &= \begin{cases} 0 & \ldots \text{ Dirichlet or Neumann} \\ 2a\|u_{d+1}\|^2_{\mathrm{L}^2(\Gamma)} & \ldots \text{ Robin } \ldots W = \mathrm{L}^2_{\mathrm{div}}(\Omega)\times\mathrm{H}^1(\Omega) \end{cases} \end{aligned}$$

✓

(K2) For any sequence (u_n) in V and any $\varphi \in \mathrm{C}^\infty_c(\Omega)$

$$\mathsf{u}_n \rightharpoonup \mathsf{u} \quad \Longrightarrow \quad \langle\, T_0\mathsf{u}_n \mid \varphi\mathsf{u}_n \,\rangle_L \longrightarrow \langle\, T_0\mathsf{u} \mid \varphi\mathsf{u} \,\rangle_L$$

Compactness by compensation

$$\begin{aligned}\langle\, T_0 \mathsf{u}_n \mid \varphi \mathsf{u}_n \,\rangle_L &= \int_\Omega \sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u}_n \cdot \varphi \mathsf{u}_n \, d\mathbf{x}\,, \qquad p = q = d+1 \\ &= -\frac{1}{2} \int_\Omega \partial_k \varphi \sum_{k=1}^d \overbrace{\mathbf{A}_k \mathsf{u}_n \cdot \mathsf{u}_n}^{Q \circ \mathsf{u}_n} d\mathbf{x}\end{aligned}$$

Theorem. *(Quadratic theorem) For $\mathbf{A}_k \in \mathrm{M}_{q,p}(\mathbf{R})$ let*

$$\Lambda := \left\{ \boldsymbol{\lambda} \in \mathbf{R}^p \,:\, (\exists\, \boldsymbol{\xi} \neq \mathbf{0}) \quad \sum_{k=1}^d \xi_k \mathbf{A}_k \boldsymbol{\lambda} = \mathbf{0} \right\},$$

while $Q(\boldsymbol{\lambda}) := \mathbf{Q}\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}$, such that $Q = 0$ on Λ. If

$$\mathsf{u}_n \rightharpoonup \mathsf{u} \quad \textit{weakly in} \quad \mathrm{L}^2(\Omega; \mathbf{R}^p)\,,$$

and

$$\sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u}_n \quad \textit{is relatively compact in} \quad \mathrm{H}^{-1}(\Omega; \mathbf{R}^q)\,,$$

then $Q \circ \mathsf{u}_n \rightharpoonup Q \circ \mathsf{u}$ in $\mathcal{D}'(\Omega)$. ∎

Proof of (K2)

$$\sum_{k=1}^{d} \xi_k \mathbf{A}_k \boldsymbol{\lambda} = \begin{bmatrix} \lambda_{d+1}\xi_1 \\ \vdots \\ \lambda_{d+1}\xi_d \\ \sum_{k=1}^{d} \lambda_k \xi_k \end{bmatrix} \implies \Lambda \ldots \lambda_{d+1} = 0$$

$$Q(\lambda) = \mathbf{A}_i \boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = 2\lambda_i \lambda_{d+1} = 0\,, \quad \lambda \in \Lambda$$

✓

Comparison with classical H-convergence

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c_n \end{bmatrix} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega)$$

$$\iff \left\{ \begin{array}{lll} \mathbf{C}_n(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} & \geq & \alpha|\boldsymbol{\xi}|^2 \\ \mathbf{C}_n(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} & \geq & \frac{1}{\beta}|\mathbf{C}_n(\mathbf{x})\boldsymbol{\xi}|^2 \end{array} \right.$$

$$\iff \left\{ \begin{array}{rcl} \alpha \leq & c_n(\mathbf{x}) & \leq \beta \\ \mathbf{A}^n(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} & \geq & \frac{1}{\beta}|\boldsymbol{\xi}|^2 \\ \mathbf{A}^n(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} & \geq & \alpha|\mathbf{A}^n(\mathbf{x})\boldsymbol{\xi}|^2 \end{array} \right.$$

At a subsequence $\mathbf{C}_n \xrightarrow{H} \mathbf{C}$, by compactness theorem.

- Has the limit $\mathbf{C}$ the same structure?
- Can we make a connection with H-converging (in classical sense) subsequence of $(\mathbf{A}^n)$?

Characterisation of the H-limit

Theorem. *For the Friedrichs system corresponding to the stationary diffusion equation, a sequence* $(\mathbf{C}_n)$ *in* $\mathcal{M}_{d+1}(\alpha, \beta; \Omega)$ *of the form*

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & 0 \\ 0^\top & c_n \end{bmatrix} .$$

H*-converges with respect to* $\mathcal{L}_0$ *and* V_D *if and only if* $(\mathbf{A}^n)$ *classically* H*-converges to some* $\mathbf{A}$ *and* (c_n) L^∞ *weakly* $*$ *converges to some* c*. In that case, the* H*-limit is the matrix function*

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0^\top & c \end{bmatrix} ,$$

∎

Heat equation as Friedrichs system

$\Omega \subseteq \mathbf{R}^d$ open and bounded set with Lipschitz boundary Γ, $T > 0$ and $\Omega_T := \Omega \times \langle 0, T \rangle$

$$\partial_t u_n - \mathsf{div}_{\mathbf{x}}(\mathbf{A}^n \nabla_{\mathbf{x}} u_n) + c u_n = f \qquad \text{in } \Omega_T \,,$$

$$\mathsf{u}_n = \begin{bmatrix} \mathsf{u}_{d\,n} \\ u_{d+1\,n} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}^n \nabla_{\mathbf{x}} u_n \\ u_n \end{bmatrix} .$$

The matrices $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, $k = 1, \dots d$, $\mathbf{A}_{d+1} = \mathbf{e}_{d+1} \otimes \mathbf{e}_{d+1}$ and

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c \end{bmatrix}$$

$$T_0 \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} \\ \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}} \mathsf{u}_d \end{bmatrix} .$$

Graph space

$$W = \left\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \right\} .$$

Compactness result

Dirichlet boundary conditions with zero initial value:

$$V = \Big\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. on } \Omega \Big\},$$

$$\widetilde{V} = \Big\{ \mathsf{v} \in W : v^u \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v^u(\cdot,T) = 0 \text{ a.e. on } \Omega \Big\}.$$

(K1):

$$_{W'}\langle D\mathsf{u}, \mathsf{u} \rangle_W = \|u_{d+1}(\cdot,T)\|^2_{\mathrm{L}^2(\Omega)} .$$

(K2): similarly to stationary diffusion equation: $\Lambda = \{\boldsymbol{\lambda} \in \mathbf{R}^{d+1} : \lambda_{d+1} = 0\}$ ✓

$\implies \quad \mathcal{M}_{d+1}(\alpha,\beta;\Omega)$ is compact with H-topology for given $\mathcal{L}_0$ and V

Comparison with classical parabolic H-convergence... similarly as for stationary diffusion equation.

G-convergence

Instead of $\mathbf{C}_n \in \mathcal{M}_r(\alpha, \beta; \Omega)$ we take

$$\mathcal{C}_n \in \mathcal{F}(\alpha, \beta; \Omega) := \Big\{ \mathcal{C} \in \mathcal{L}(L) : (\forall\, \mathsf{u} \in L) \\ \langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L \geq \alpha \|\mathsf{u}\|_L^2 \quad \& \quad \langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L \geq \frac{1}{\beta} \|\mathcal{C}\mathsf{u}\|_L^2 \Big\} .$$

For a given sequence $\mathcal{C}_n \in \mathcal{F}(\alpha, \beta; \Omega)$, the sequence of isomorphisms $T_n := T_0 + \mathcal{C}_n : V \to L$ G-converges to an isomorphism $T := T_0 + \mathcal{C} : V \to L$, for some $\mathcal{C} \in \mathcal{F}(\alpha', \beta'; \Omega)$ if

$$(\forall\, \mathsf{f} \in L) \quad T_n^{-1}\mathsf{f} \rightharpoonup T^{-1}\mathsf{f} \text{ in } W .$$

Theorem. *For fixed T_0 and V, if family $\mathcal{F}(\alpha, \beta; \Omega)$ satisfies (K1), then for any sequence $(\mathcal{C}_n)$ in $\mathcal{F}(\alpha, \beta; \Omega)$ there exists a subsequence of $T_n := T_0 + \mathcal{C}_n$ which G-converges to $T := T_0 + \mathcal{C}$ with $\mathcal{C} \in \mathcal{F}(\alpha, \beta; \Omega)$.* ∎

Thank you for your attention!