Parabolic equations as Friedrichs systems

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Joint work with Krešimir Burazin and Marko Vrdoljak,
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Classical theory of Friedrichs systems
  Friedrichs systems
  Boundary conditions for Friedrichs systems

Abstract formulation of Friedrichs systems
  Operator-theoretic formulation
  Kreĭn space formalism and different boundary conditions
  Bijective realisations

Parabolic equations (non-stationary diffusion)
  Heat equation as a Friedrichs system
  Two-field theory

Homogenisation of Friedrichs systems
  Homogenisation
  Stationary diffusion and heat equation, as particular cases
Symmetric positive systems


Unified treatment of linear hyperbolic systems like Maxwell’s, Dirac’s, or higher order equations (e.g. the wave equation).

A generalisation:

Goals:
– treating the equations of mixed type, such as the Tricomi equation:

\[ y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \]

– unified treatment of equations and systems of different type,

– more recently: better numerical properties.

All of Gårding’s theory of general elliptic equations, or Lerray’s of general hyperbolic equations, is not covered.

The development of theory is nowadays mostly motivated by the needs in development of numerical methods.
Friedrichs’ system (KOF1958)

Assumptions:
\( d, r \in \mathbb{N}, \Omega \subseteq \mathbb{R}^d \) open and bounded with Lipschitz boundary \( \Gamma \);
\( \mathbf{A}_k \in W^{1,\infty}(\Omega; M_r(\mathbb{C})), k \in 1..d, \text{ and } \mathbf{B} \in L^\infty(\Omega; M_r(\mathbb{C})) \) satisfying

(F1) matrix functions \( \mathbf{A}_k \) are hermitian: \( \mathbf{A}_k = \mathbf{A}_k^* \);

(F2) \( (\exists \mu_0 > 0) \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geq 2\mu_0 I \) \( \text{ (ae on } \Omega) \).

The operator \( \mathcal{L} : L^2(\Omega; \mathbb{C}^r) \rightarrow \mathcal{D}'(\Omega; \mathbb{C}^r) \)

\[ \mathcal{L}u := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k u) + \mathbf{B}u \]

is called the symmetric positive operator (the Friedrichs operator), and

\[ \mathcal{L}u = f \]

the symmetric positive system (the Friedrichs system).
Symmetric hyperbolic systems (KOF1954)

\[
\sum_{k=1}^{d} A^k \partial_k u + Du = f
\]

In divergence form:

\[
\sum_{k=1}^{d} \partial_k (A^k u) + (D - \partial_k A^k)u = f
\]

It is symmetric if all matrices $A^k$ are real and symmetric; and uniformly hyperbolic if there is a $\xi \in \mathbb{R}^d$ such that for any $x \in \text{Cl} \, \Omega$ the matrix $\xi_k A^k(x)$ is positive definite.

Such systems can easily be transformed into the form of a Friedrichs system.

It is known that the wave equation, the Maxwell and the Dirac system can be written as an equivalent symmetric hyperbolic system, thus as a Friedrichs system as well.
An example – scalar elliptic equation

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded region, $\mu > 0$ and $f \in L^2(\Omega)$ given; the equation

$$-\Delta u + \mu u = f$$

can be written as a first-order system

$$\begin{cases} p + \nabla u = 0 \\ \mu u + \text{div} p = f \end{cases},$$

which is a Friedrichs system with the choice of

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

With some adjustments, the same holds for the equation

$$-\text{div} (A \nabla u) + \text{div} (ub) + cu = f,$$

where we take $A \in W^{2,\infty}(\Omega; \text{Psym})$, $b \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ and $c \in L^\infty(\Omega)$. 
Example – heat equation

Take $\Omega \subseteq \mathbb{R}^d$ open and bounded, with Lipschitz boundary $\Gamma$, and $T > 0$.
Define: $\Omega_T := \langle 0, T \rangle \times \Omega$, $\Gamma_d := \langle 0, T \rangle \times \Gamma$, $\Gamma_0 := \{0\} \times \Gamma$ and $\Gamma_T := \{T\} \times \Gamma$.
Consider the heat equation with zero initial and Dirichlet boundary condition:

$$\begin{cases}
    \partial_t u - \text{div}_x(A \nabla_x u) + b \cdot \nabla_x u + cu = f \text{ in } \Omega_T \\
    u = 0 \text{ on } \Gamma_d \\
    u|_{\Gamma_0} = 0.
\end{cases}$$

It can be written as a Friedrichs system in the form:

$$\begin{cases}
    \nabla_x u_{d+1} + A^{-1} u_d = 0 \\
    \partial_t u_{d+1} + \text{div}_x u_d + cu_{d+1} - A^{-1} b \cdot u_d = f
\end{cases}$$

(note that we use $u = (u_d, u_{d+1})^\top$, where $u_d = -A \nabla u$, and $u_{d+1} = u$). Indeed

$$\begin{bmatrix}
    0 & 0 \\
    0 & 1
\end{bmatrix}
\partial_t \begin{bmatrix} u_d \\ u \end{bmatrix} + \sum_{i=1}^d \begin{bmatrix}
    0 & \cdots & 0 & \cdots & 0 \\
    \vdots & \ddots & 0 & \cdots & 0 \\
    0 & \cdots & 0 & \cdots & 1 \\
    \vdots & \ddots & 0 & \cdots & 0 \\
    0 & \cdots & 1 & \cdots & 1
\end{bmatrix}
\partial_{x_i} \begin{bmatrix} u_d \\ u \end{bmatrix} + \begin{bmatrix}
    A^{-1} \\
    -(A^{-1} b)^\top \cdot c
\end{bmatrix} \begin{bmatrix} u_d \\ u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. $$

The condition (F1) holds. The positivity condition $B + B^\top \geq 2\mu_0 I$ is fulfilled if and only if $c - \frac{1}{4} A^{-1} b \cdot b$ is uniformly positive.
Boundary conditions

Boundary conditions are enforced via a matrix valued boundary field:

\[ A_{\nu} := \sum_{k=1}^{d} \nu_k A_k \in L^{\infty}(\Gamma; M_r(C)) , \]

where \( \nu = (\nu_1, \nu_2, \cdots, \nu_d) \) is the outward unit normal on \( \Gamma \), and

\[ M \in L^{\infty}(\Gamma; M_r(C)). \]

Boundary condition

\[ (A_{\nu} - M)u|_{\Gamma} = 0 \]

is sufficient for treatment of different types of usual boundary conditions.
Assumptions on boundary matrix $M$

We assume (for ae $x \in \Gamma$) \[ (\forall \xi \in C^r) \ (M(x) + M(x)^*) \xi \cdot \xi \geq 0 , \]

\[ (FM1) \]

\[ (FM2) \]

\[ C^r = \ker \left( A_\nu(x) - M(x) \right) + \ker \left( A_\nu(x) + M(x) \right) . \]

Such $M$ is called the admissible boundary condition.

The boundary problem: for given $f \in L^2(\Omega; C^r)$ find $u$ such that

\[
\begin{cases}
    Lu = f \\
    (A_\nu - M)u|_{\Gamma} = 0
\end{cases}
\]

Does the above work for standard problems in continuum physics?
Elliptic equation – different boundary conditions

\[ \mathbf{M} \quad \mathbf{A}_\nu - \mathbf{M} \quad (\mathbf{A}_\nu - \mathbf{M}) \begin{bmatrix} \mathbf{p} \\ \mathbf{u} \end{bmatrix} |_{\Gamma} = 0 \]

\[
\begin{bmatrix}
0 & 0 & -\nu_1 \\
0 & 0 & -\nu_2 \\
\nu_1 & \nu_2 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 2\nu_1 \\
0 & 0 & 2\nu_2 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ \mathbf{u} |_{\Gamma} = 0 \]

\[
\begin{bmatrix}
0 & 0 & \nu_1 \\
0 & 0 & \nu_2 \\
-\nu_1 & -\nu_2 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
2\nu_1 & 2\nu_2 & 0
\end{bmatrix}
\]

\[ \mathbf{\nu} \cdot (\nabla \mathbf{u}) |_{\Gamma} = 0 \]

\[
\begin{bmatrix}
0 & 0 & \nu_1 \\
0 & 0 & \nu_2 \\
-\nu_1 & -\nu_2 & 2\alpha
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
2\nu_1 & 2\nu_2 & 2\alpha
\end{bmatrix}
\]

\[ \mathbf{\nu} \cdot (\nabla \mathbf{u}) |_{\Gamma} + \alpha \mathbf{u} |_{\Gamma} = 0 \]

All above matrices \( \mathbf{M} \) satisfy (FM).

\[
\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.
\]
Different ways to enforce boundary conditions

Instead of

\[(A_\nu - M)u = 0 \quad \text{on} \quad \Gamma,\]

Lax proposed boundary conditions with

\[u(x) \in N(x), \quad x \in \Gamma,\]

where \(N = \{N(x) : x \in \Gamma\}\) is a family of subspaces of \(C^r\).

Boundary problem:

\[
\begin{cases}
Lu = f \\
u(x) \in N(x), \quad x \in \Gamma
\end{cases}
\]
Assumptions on $N$

 maximal boundary conditions: (for ae $x \in \Gamma$) \[\text{[PDL]}\]

(FX1) \[N(x)\] is non-negative with respect to $A_\nu(x)$:
$$(\forall \xi \in N(x)) \quad A_\nu(x)\xi \cdot \xi \geq 0;$$

(FX2) there is no non-negative subspace with respect to $A_\nu(x)$, which (properly) contains $N(x)$;

or \[\text{[RSP&LS1966]}\]

Let $N(x)$ and $\tilde{N}(x) := (A_\nu(x)N(x))^\perp$ satisfy (for ae $x \in \Gamma$)

(FV1) \[\forall \xi \in N(x) \quad A_\nu(x)\xi \cdot \xi \geq 0\]

(FV2) \[\tilde{N}(x) = (A_\nu(x)N(x))^\perp \quad \text{and} \quad N(x) = (A_\nu(x)\tilde{N}(x))^\perp.\]
Equivalence of different descriptions of boundary conditions

**Theorem.** It holds

\[(FM1) – (FM2) \iff (FX1) – (FX2) \iff (FV1) – (FV2),\]

with

\[N(x) := \ker(A_{\nu}(x) - M(x)).\]

In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].
Classical results on well-posedness

Friedrichs:
– uniqueness of the classical solution,
– existence of a weak solution (under some additional assumptions).

Contributions (and particular cases):
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...  
– the meaning of traces for functions in the graph space,
– weak well-posedness results under additional assumptions (on $A_\nu$),
– regularity of solution,
– numerical treatment.

Shortcomings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

However, since the beginning of $21^{st}$ century the numerical advantages of FS have overshadowed that.
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New approach...


– abstract setting (operators on Hilbert spaces),

– intrinsic criterion for the bijectivity of *Friedrichs’* operator,

– avoiding the question of traces for functions in the graph space,

– investigation of different formulations of boundary conditions,

... and new open questions.

Effectively, they considered only the real case.
Assumptions

Let $L$ be real (complex) Hilbert space ($L'$ is (anti)dual of $L$), $\mathcal{D} \subseteq L$ a dense subspace, and $T, \tilde{T} : \mathcal{D} \rightarrow L$ linear unbounded operators satisfying

(T1) \hspace{1cm} (\forall \varphi, \psi \in \mathcal{D}) \hspace{1cm} \langle T\varphi \mid \psi \rangle_L = \langle \varphi \mid \tilde{T}\psi \rangle_L ,

(T2) \hspace{1cm} (\exists c > 0)(\forall \varphi \in \mathcal{D}) \hspace{1cm} \| (T + \tilde{T})\varphi \|_L \leq c\|\varphi\|_L ,

(T3) \hspace{1cm} (\exists \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \hspace{1cm} \langle (T + \tilde{T})\varphi \mid \varphi \rangle_L \geq 2\mu_0\|\varphi\|_L^2 .

$(T, \tilde{T})$ is referred to as a joint pair of abstract Friedrichs operators.

Recall the Friedrichs operator:

$\mathcal{D} := C_c^\infty (\Omega; \mathbb{C}^r), \hspace{1cm} L = L^2(\Omega; \mathbb{C}^r)$ and $T, \tilde{T} : \mathcal{D} \rightarrow L$ are defined by

$T u := \sum_{k=1}^{d} \partial_k(A_k u) + B u ,$

$\tilde{T} u := - \sum_{k=1}^{d} \partial_k(A_k u) + (B^* + \sum_{k=1}^{d} \partial_k A_k) u ,$

where $A_k$ and $B$ are as above (they satisfy (F1)–(F2)).

Then $T$ and $\tilde{T}$ satisfy (T1)–(T3). . . fits in this framework.
Extension of operators, starting from \((T, \tilde{T}) = (T_1, \tilde{T}_1)\)

\(\mathcal{D}\) is an inner product space when equipped with the \textit{graph norm} stemming from

\[
\langle \cdot | \cdot \rangle_T := \langle \cdot | \cdot \rangle_L + \langle T \cdot | T \cdot \rangle_L .
\]

By \(W_0\) denote the completion of \(\mathcal{D}\) in the graph norm, the same for \(\tilde{T}\) by (T2). \(W_0 \leq L\) by (T1), and both \(T\) and \(\tilde{T}\) extend to bounded operators from \(W_0\) to \(L\), which we denote by \((T_2, \tilde{T}_2)\).

The following embeddings are dense and continuous (we have a Gel’fand triplet):

\[
W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'.
\]

Let \(T_3 := \tilde{T}_2' \in \mathcal{L}(L; W_0')\) be the Banach adjoint of \(\tilde{T}_2 : W_0 \rightarrow L\), and \(\tilde{T}_3 := T_2'.\) Thus we have defined \((T_3, \tilde{T}_3)\).

Note that the \textit{graph space}

\[
W := \{ u \in L : Tu \in L \} = \{ u \in L : \tilde{T}u \in L \} \leq L
\]

is a Hilbert space with respect to \(\langle \cdot | \cdot \rangle_T\).

\((T_4, \tilde{T}_4)\) are defined as restrictions of \(T_3\) and \(\tilde{T}_3\) to \(W\).

This produces the \textit{maximal} pair of abstract Friedrichs operators \((T_4, \tilde{T}_4)\), mapping \(T_4, \tilde{T}_4 : W \rightarrow L\), which are associated to the initial pair \((T, \tilde{T})\).
Well-posedness for abstract Friedrichs operator (our goal)

Find sufficient conditions for a subspace $W_0 \leq V \leq W$ such that $T_4|_V : V \longrightarrow L$ is an isomorphism.

As the continuity in the graph norm holds for any restriction to a closed subspace $V$ of $W$, the key question is bijectivity.

If $T$ is the classical Friedrichs operator $\mathcal{L}$, then for $u, v \in C_c^\infty(\mathbb{R}^d; \mathbb{C}^r)$ we have

$$w'(Du, v)_W = \int_{\Gamma} A_\nu(x)u|_\Gamma(x) \cdot v|_\Gamma(x)dS(x).$$

With the assumptions:

(FV1) \hspace{1cm} (\forall \xi \in N(x)) \; A_\nu(x)\xi \cdot \xi \geq 0,

(FV2) \hspace{1cm} (\forall \xi \in \tilde{\mathcal{N}}(x)) \; A_\nu(x)\xi \cdot \xi \leq 0,

we are lead to consider the following subspaces $V$ and $\tilde{V}$ in the functional framework:

(V1) \hspace{1cm} (\forall u \in V) \; w'(Du, u)_W \geq 0,

(V2) \hspace{1cm} \forall v \in \tilde{V} \; w'(Dv, v)_W \leq 0,

(V2) \hspace{1cm} V = D(\tilde{V})^0, \quad \tilde{V} = D(V)^0.$

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Sufficient coditions were obtained by [EGC2007] and [AB2010] using the boundary operator $D \in \mathcal{L}(W; W')$:

$$W'\langle Du, v \rangle_W := \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L, \quad u, v \in W.$$  

$D$ is symmetric: $W'\langle Du, v \rangle_W = W'\langle Dv, u \rangle_W$ and satisfies

$$\ker D = W_0$$
$$\text{im } D = W_0^0 := \{ g \in W' : (\forall u \in W_0) \quad W'\langle g, u \rangle_W = 0 \}.$$  

For a given joint pair of abstract FO $(T, \tilde{T})$, a pair $(V, \tilde{V})$ of subspaces of $W$ is said to allow the (V)-boundary conditions relative to $(T, \tilde{T})$ when:

(V1) the boundary operator has opposite sign on $V$ and on $\tilde{V}$, in the sense that

$$\forall u \in V \quad W'\langle Du, u \rangle_W \geq 0,$$

$$\forall v \in \tilde{V} \quad W'\langle Dv, v \rangle_W \leq 0;$$

(V2) the image via $D$ of either space has, as annihilator, the other space, namely

$$V = D(\tilde{V})^0 \quad \text{and} \quad \tilde{V} = D(V)^0.$$
Theorem. Under assumptions (T1) – (T3) and (V1) – (V2), the operators \( T|_V : V \rightarrow L \) and \( \tilde{T}|_{\tilde{V}} : \tilde{V} \rightarrow L \) are isomorphisms.

In the real case [EGC2007].

What happens with other types of boundary conditions (FX) and (FM)?
Indefinite inner product characterisation

For $u, v \in W$ define

$$[u \mid v] := W\langle Du, v \rangle_W = \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L,$$

which is an indefinite inner product on $W$, and we consider subspaces $V$ and $\tilde{V}$ satisfying:

(V1) $(\forall v \in V) \ [v \mid v] \geq 0$, 
    $(\forall v \in \tilde{V}) \ [v \mid v] \leq 0$;

(V2) $V = \tilde{V}^{\perp}$, \quad $\tilde{V} = V^{\perp}$.

($^{\perp}$ stands for $[\cdot \mid \cdot]$-orthogonal complement)
maximal boundary conditions: (for ae $x \in \Gamma$)

(FX1) \hspace{1cm} (\forall \xi \in N(x)) \hspace{1cm} A_\nu(x)\xi \cdot \xi \geq 0,

(FX2) \hspace{1cm} \text{there is no non-negative subspace with respect to} \hspace{1cm} A_\nu(x), \text{which contains} \hspace{1cm} N(x),

subspace $V$ is maximal non-negative in $(W, [\cdot | \cdot])$:

(X1) \hspace{1cm} V \text{ is non-negative in } (W, [\cdot | \cdot]): \hspace{1cm} (\forall v \in V) \hspace{1cm} [v | v] \geq 0,

(X2) \hspace{1cm} \text{there is no non-negative subspace in } (W, [\cdot | \cdot]) \text{ containing } V.
Correspondence — admissible b.c.

**admissible boundary condition:** there exists a matrix function \( M : \Gamma \rightarrow M_r(C) \) such that (for \( a e x \in \Gamma \))

\[
\text{(FM1)} \quad (\forall \xi \in C^r) \quad (M(x) + M(x)^*)\xi \cdot \xi \geq 0,
\]

\[
\text{(FM2)} \quad C^r = \text{ker}\left(A_{\nu}(x) - M(x)\right) + \text{ker}\left(A_{\nu}(x) + M(x)\right).
\]

**abstract admissible boundary condition:** there exists \( M \in \mathcal{L}(W; W') \) such that

\[
\text{(M1)} \quad (\forall u \in W) \quad W'\langle (M + M^*)u, u \rangle_W \geq 0,
\]

\[
\text{(M2)} \quad W = \text{ker}(D - M) + \text{ker}(D + M).
\]

Similarly as for boundary operator \( D \), starting from a classical Friedrichs operator \( \mathcal{L} \), operator \( M \) is defined for \( u, v \in C^\infty_c(\partial \Omega; \mathbb{R}^r) \) by

\[
W'\langle Mu, v \rangle_W = \int_{\Gamma} M(x)u_{|\Gamma}(x) \cdot v_{|\Gamma}(x) dS(x).
\]
Equivalence of different descriptions of b.c.

**Theorem.** *(classical)*  It holds

\[(FM1)-(FM2) \iff (FV1)-(FV2) \iff (FX1)-(FX2),\]

with

\[N(x) := \ker\left(A_\nu(x) - M(x)\right).\]

**Theorem.** *[EGC2007, AB2010]*  It holds

\[(M1)-(M2) \iff (V1)-(V2) \iff (X1)-(X2),\]

with

\[V := \ker(D - M).\]
**Theorem.** [Ern, Guermond, Caplain, 2007] Let \((T, \tilde{T})\) be a joint pair of Friedrichs systems and let \((V, \tilde{V})\) satisfy (V1)–(V2). Then \(T_4|_V : V \to L\) and \(\tilde{T}_4|_{\tilde{V}} : \tilde{V} \to L\) are closed bijective realisations of \(T\) and \(\tilde{T}\), respectively.

Can we say something more about extensions \(T_4, \tilde{T}_4\), and conditions \((V)\)?

**Theorem.** \((T, \tilde{T})\) is a joint pair of abstract Friedrichs operators iff

\(i\) \(T \subseteq \tilde{T}^*\) and \(\tilde{T} \subseteq T^*\);

\(ii\) \(T + \tilde{T}\) is a bounded self-adjoint operator in \(L\) with strictly positive bottom;

\(iii\) \(\text{dom } T = \text{dom } \tilde{T} = W_0\) and \(\text{dom } T^* = \text{dom } \tilde{T}^* = W\).

In fact: \(T_4 = \tilde{T}^*\) and \(\tilde{T}_4 = T^*\).
Theorem. Let \((T, \tilde{T})\) be a pair of operators on the Hilbert space \(L\) satisfying conditions (T1)-(T2), and let \((V, \tilde{V})\) be a pair of subspaces of \(L\). Then (V2) is equivalent to

i) \(W_0 \subseteq V \subseteq W,\) \(W_0 \subseteq \tilde{V} \subseteq W,\)

ii) \(V\) and \(\tilde{V}\) are closed in \(W\), and

iii) \((\tilde{T}^*|_{V})^* = T^*|_{\tilde{V}},\) \((T^*|_{\tilde{V}})^* = \tilde{T}^*|_{V}.\)

We are seeking bijective closed operators \(S \equiv \tilde{T}^*|_{V}\) such that

\[\overline{T} \subseteq S \subseteq \tilde{T}^* ,\]

and thus also \(S^*\) is bijective and \(\overline{T} \subseteq S^* \subseteq T^* .\)

In the following we work with closed \(T\) and \(\tilde{T} .\)

Let \((T, \tilde{T})\) be a joint pair of closed abstract Friedrichs operators on the Hilbert space \(L\). For a closed \(T \subseteq S \subseteq \tilde{T}^*\) such that \((\text{dom } S, \text{dom } S^*)\) satisfies (V1) we call \((S, S^*)\) an adjoint pair of bijective realisations with signed boundary map relative to \((T, \tilde{T}) .\)
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Heat equation (recall)

... with zero initial and Dirichlet boundary condition:

\[
\begin{cases} 
\partial_t u - \text{div}_x (A \nabla_x u) + b \cdot \nabla_x u + cu = f \text{ in } \Omega_T \\
u = 0 \text{ on } \Gamma_d \cup \Gamma_0, 
\end{cases}
\]

...as a Friedrichs system:

\[
\begin{cases} 
\nabla_x u_{d+1} + A^{-1} u_d = 0 \\
\partial_t u_{d+1} + \text{div}_x u_d + cu_{d+1} - A^{-1} b \cdot u_d = f 
\end{cases}
\]

(note that we use \( u = (u_d, u_{d+1})^\top \), where \( u_d = -A \nabla u \), and \( u_{d+1} = u \)).

More precisely, we take \( f \in L^2(\Omega_T) \), \( c \in L^\infty(\Omega_T) \), \( b \in L^\infty(\Omega_T; \mathbb{R}^d) \) and \( A \in L^\infty(\Omega_T; M_d(\mathbb{R})) \). Furthermore, we suppose that there are constants \( \beta \geq \alpha > 0 \) such that \( A(x, t) \) is a symmetric matrix with eigenvalues between \( \alpha \) and \( \beta \), almost everywhere on \( \Omega_T \).

The positivity condition \( C + C^\top \succeq 2\mu_0 I \) is fulfilled if and only if the Schur complement \( c - \frac{1}{4} A^{-1} b \cdot b \) is uniformly positive, i.e. if there exists a constant \( \gamma > 0 \) such that \( c - \frac{1}{4} A^{-1} b \cdot b \succeq \gamma \) on \( \Omega_T \).
Friedrichs operator and the graph space

The operator $T$ is given by

$$T \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_x u_{d+1} + A^{-1} u_d \\ \partial_t u_{d+1} + \text{div}_x u_d + cu_{d+1} - A^{-1} b \cdot u_d \end{bmatrix},$$

while the corresponding graph space is

$$W = \left\{ u \in L^2(\Omega_T; \mathbb{R}^{d+1}) : \nabla_x u_{d+1} \in L^2(\Omega_T; \mathbb{R}^d) \right\}$$

$$\& \quad \partial_t u_{d+1} + \text{div}_x u_d \in L^2(\Omega_T)$$

$$= \left\{ u \in L^2_{\text{div}}(\Omega_T) : \nabla_x u_{d+1} \in L^2(\Omega_T; \mathbb{R}^d) \right\}$$

$$= \left\{ u \in L^2_{\text{div}}(\Omega_T) : u_{d+1} \in L^2(0, T; H^1(\Omega)) \right\}.$$
Properties of the last component

**Lemma.** *The projection* \( u = (u_d, u_{d+1})^\top \mapsto u_{d+1} \) *is a continuous linear operator from* \( W \) *to* \( W(0, T) \), *which is continuously embedded into* \( C([0, T]; L^2(\Omega)) \).

The space

\[
W(0, T) = \left\{ u \in L^2(0, T; H^1(\Omega)) : \partial_t u \in L^2(0, T; H^{-1}(\Omega)) \right\},
\]

is a Banach space when equipped by norm

\[
\|u\|_{W(0,T)} = \sqrt{\|u\|_{L^2(0,T;H^1(\Omega))}^2 + \|\partial_t u\|_{L^2(0,T;H^{-1}(\Omega))}^2}.
\]
Finally

Let us return to the initial-boundary value problem for the heat equation

\[
\begin{cases}
\partial_t u - \text{div}_x (A \nabla_x u) + b \cdot \nabla_x u + cu = f \quad \text{in } \Omega_T \\
u = 0 \quad \text{on } \Gamma_d \cup \Gamma_0,
\end{cases}
\]

to be represented as a boundary-value problem for a Friedrichs system. Take

\[
V = \left\{ u \in W : u_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \right\},
\]

\[
\tilde{V} = \left\{ v \in W : v_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad v_{d+1}(\cdot, T) = 0 \text{ a.e. on } \Omega \right\}.
\]

Do they satisfy (V1)–(V2)? Technical...

**Theorem.** The above $V$ and $\tilde{V}$ satisfy (V1)–(V2), and therefore the operator $T|_V : V \rightarrow L$ is an isomorphism.
Two-field theory... 

Consider the heat equation with $b = 0$ and $c = 0$:

\[
\begin{aligned}
\begin{cases}
\partial_t u - \text{div}_x (A \nabla_x u) = f \text{ in } \Omega_T \\
u = 0 \text{ on } \Gamma_d \cup \Gamma_0.
\end{cases}
\end{aligned}
\]

The above cannot be applied for $c = 0$.

However, we can apply the two field theory: developed by Ern and Guermond for elliptic problems.

The vector unknown has to be split into two parts, i.e. the matrices need to be of the form

\[
A^k = \begin{bmatrix} 0 & B^k \end{bmatrix}^\top \quad \text{and} \quad C = \begin{bmatrix} C^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix},
\]

where $B^k \in \mathbb{R}^d$ are constant vectors, $a^k \in W^{1,\infty}(\Omega_T)$, $C^d \in L^\infty(\Omega_T; M_d(\mathbb{R}))$ and $c^{d+1} \in L^\infty(\Omega_T)$, $k \in 1..(d + 1)$.

For the heat equation the matrices have this form!
Instead of coercivity (positivity) condition (F2), the following is required:

\[(\exists \mu_1 > 0)(\forall \xi = (\xi_d, \xi_{d+1}) \in \mathbb{R}^{d+1}) \]

\[
\left( C + C^\top + \sum_{k=1}^{d+1} \partial_k A_k \right) \xi \cdot \xi \geq 2\mu_1 |\xi_d|^2 \quad (\text{a.e. on } \Omega) ,
\]

\[(\exists \mu_2 > 0)(\forall u \in V \cup \tilde{V}) \]

\[
\sqrt{\langle Lu | u \rangle_{L^2(\Omega_T; \mathbb{R}^{d+1})} + \|Bu_{d+1}\|_{L^2(\Omega_T; \mathbb{R}^d)} \geq \mu_2 \|u_{d+1}\|_{L^2(\Omega_T)}},
\]

where \(Bu_{d+1} := \sum_{k=1}^{d+1} B^k \partial_k u_{d+1} = \nabla_x u_{d+1}\).

For our system both conditions are trivially fulfilled.

Therefore, we have the well-posedness result.
Recall *boundary operator* $D \in \mathcal{L}(W, W')$: for $u, v \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^r)$ we define

$$W'(Du, v)_W = \int_\Gamma A_\nu(x)u|_\Gamma(x) \cdot v|_\Gamma(x)dS(x).$$

**Theorem.** Let (F1)–(F2) hold, and let subspaces $V$ and $\tilde{V}$ of $W$ satisfy

(V1) \quad (\forall u \in V) \quad W'(Du, u)_W \geq 0,

(V2) \quad (\forall v \in \tilde{V}) \quad W'(Dv, v)_W \leq 0,

where $^0$ stands for the annihilator. Then the operator $T|_V : V \rightarrow L$ is an isomorphism and for every $u \in V$ the following estimate holds:

$$\|u\|_T \leq \sqrt{\frac{1}{\alpha^2} + 1 \|Tu\|_L}$$
Classical theory of Friedrichs systems
  Friedrichs systems
  Boundary conditions for Friedrichs systems

Abstract formulation of Friedrichs systems
  Operator-theoretic formulation
  Kreĭn space formalism and different boundary conditions
  Bijective realisations

Parabolic equations (non-stationary diffusion)
  Heat equation as a Friedrichs system
  Two-field theory

Homogenisation of Friedrichs systems
  Homogenisation
  Stationary diffusion and heat equation, as particular cases
Homogenisation setting


A sequence of Friedrichs systems $T_n u_n = f$, $f \in L$.

Here $u_n$ naturally belongs to the graph space of $T_n$.

Our assumptions must secure that every $u_n$ belongs to the same space, with clearly identified topology that shall be used...

- $A_k$ are symmetric constant matrices in $M_r(R)$, $k \in 1..d$

- $C \in M_r(\alpha, \beta; \Omega) = \left\{ C \in L^\infty(\Omega; M_r(R)) : (\forall \xi \in \mathbb{R}^d) \right.$
  \[ C\xi \cdot \xi \geq \alpha |\xi|^2 \text{ and } C\xi \cdot \xi \geq \frac{1}{\beta} |C\xi|^2 \}$.

\[ T_0 u = \sum_{k=1}^d \partial_k (A_k u) = \sum_{k=1}^d A_k \partial_k u , \]

so that $T := T_0 + C$ is the Friedrichs operator. Its graph space

\[ W := \{ u \in L : Tu \in L \} = \{ u \in L : T_0 u \in L \} . \]

Moreover, we have equivalence of norms ($\gamma = \sqrt{\max\{2, 1 + 2\beta^2\}}$):

\[ \|u\|_T \leq \gamma \|u\|_{T_0} \leq \gamma^2 \|u\|_T , \quad \text{for any } C . \]
Boundary operator and a priori bound

The boundary operator $D$ corresponding to the operator $T$ does not depend on particular $C$ from $\mathcal{M}_T(\alpha, \beta; \Omega)$.

If $V$ is a subspace of $W$ that satisfies (V), well-posedness result implies that $T|_V : V \rightarrow L$ is an isomorphism, with

$$\|u\|_{T_0} \leq \gamma \|u\|_T \leq \gamma \sqrt{\frac{1}{\alpha^2} + 1} \|Tu\|_L, \quad u \in V.$$ 

Therefore, for fixed $T_0$ and $V$ satisfying (V), we have a priori bound

$$(\exists c > 0)(\forall C \in \mathcal{M}_T(\alpha, \beta; \Omega))(\forall u \in V) \quad \|u\|_{T_0} \leq c \|(L_0 + C)u\|_L.$$ 

Note that constant $c$ depends only on $T_0$, $\alpha$ and $\beta$.

In the sequel $L_0 = \sum_{k=1}^{d} A_k \partial_k$ and $V$ are fixed.
**H-convergence**

A sequence \((C_n)\) in \(\mathcal{M}_r(\alpha, \beta; \Omega)\) **H-converges** to \(C \in \mathcal{M}_r(\alpha', \beta'; \Omega)\) with respect to \(T_0\) and \(V\) if, for any \(f \in L\), the sequence \((u_n)\) in \(V\) defined by \(u_n := T_n^{-1}f \in V\), with \(T_n = \mathcal{L}_0 + C_n\), satisfies

\[
\begin{align*}
  u_n & \rightharpoonup u \quad \text{in } L, \\
  C_n u_n & \rightharpoonup C u \quad \text{in } L,
\end{align*}
\]

where \(u = T^{-1}f \in V\), with \(T = \mathcal{L}_0 + C\).

As \(T_0 u_n + C_n u_n = f = T_0 u + C u\), the second convergence implies \(T_0 u_n \rightharpoonup T_0 u\) in \(L\), which gives the weak convergence \(u_n \rightharpoonup u\) in \(W\).

**Theorem.** Let \(F = \{f_n : n \in \mathbb{N}\}\) be a dense countable family in \(L^2(\Omega; \mathbb{R}^r)\), \(C, D \in \mathcal{M}_r(\alpha, \beta; \Omega)\), and \(u_n, v_n \in V\) solutions of \((T_0 + C)u_n = f_n\) and \((T_0 + D)v_n = f_n\), respectively. Furthermore, let

\[
d(C, D) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{H^{-1}(\Omega; \mathbb{R}^r)} + \|C u_n - D v_n\|_{H^{-1}(\Omega; \mathbb{R}^r)}}{\|f_n\|_{L^2(\Omega; \mathbb{R}^r)}}.
\]

Then the function \(d: \mathcal{M}_r(\alpha, \beta; \Omega) \times \mathcal{M}_r(\alpha, \beta; \Omega) \rightarrow \mathbb{R}\) forms a metric on the set \(\mathcal{M}_r(\alpha, \beta; \Omega)\), and the \(H\)-convergence is equivalent to the sequential convergence in this metric space.
Compactness

Additional assumptions: for every sequence $C_n \in M_r(\alpha, \beta; \Omega)$ and every $f \in L$, the sequence $u_n \in V$ defined by $u_n := (T_0 + C_n)^{-1}f$ satisfies the following: if $(u_n)$ weakly converges to $u$ in $W$, then also

\[(K1) \quad W'\langle Du_n, u_n \rangle_W \longrightarrow W'\langle Du, u \rangle_W,\]

or

\[(K2) \quad (\forall \varphi \in C_c^\infty(\Omega)) \quad \langle T_0u_n | \varphi u_n \rangle_L \longrightarrow \langle T_0u | \varphi u \rangle_L.\]

**Theorem.** For fixed $T_0$ and $V$, if family $M_r(\alpha, \beta; \Omega)$ satisfies $(K1)$ and $(K2)$, then it is compact with respect to $H$-convergence, i.e. from any sequence $(C_n)$ in $M_r(\alpha, \beta; \Omega)$ one can extract a $H$-converging subsequence whose limit belongs to $M_r(\alpha, \beta; \Omega)$.

The proof follows the original proof of Spagnolo in the case of parabolic $G$-convergence.
Stationary diffusion equation as Friedrichs system

\[-\text{div} (A \nabla u) + cu = f\]

in \(\Omega \subseteq \mathbb{R}^d\), with \(f \in L^2(\Omega)\), \(A \in \mathcal{M}_d(\alpha', \beta'; \Omega)\) and \(c \in L^\infty(\Omega)\) with \(\frac{1}{\beta'} \leq c \leq \frac{1}{\alpha'}\), for some \(\beta' \geq \alpha' > 0\). \(A_k = e_k \otimes e_{d+1} + e_{d+1} \otimes e_k \in \mathcal{M}_{d+1}(\mathbb{R})\), for \(k = 1, \ldots, d\)

\[C = \begin{bmatrix} A^{-1} & 0 \\ 0 & c \end{bmatrix} \in L^\infty(\Omega; \mathcal{M}_{d+1}(\mathbb{R})) ,
\]

\[Tu = \sum_{k=1}^d A_k \partial_k u + Cu = f\]

\[T_0 \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla u_{d+1} \\ \text{div} u_d \end{bmatrix} , \quad Cu = \begin{bmatrix} A^{-1} u_d \\ cu_{d+1} \end{bmatrix}.
\]

Graph space \(\ldots W = L^2_{\text{div}}(\Omega) \times H^1(\Omega)\)
Boundary conditions

**Dirichlet**

\[ V_D = \tilde{V}_D := L^2_{\text{div}}(\Omega) \times H^1_0(\Omega), \]

**Neumann**

\[ V_N = \tilde{V}_N := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = 0\}, \]

**Robin**

\[ V_R := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = a u_{d+1} |_{\Gamma}\}, \]

\[ \tilde{V}_R := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = -a u_{d+1} |_{\Gamma}\}. \]
Properties (K1) and (K2)

(K1) For any sequence \((u_n)\) in \(V\)

\[
\begin{align*}
\quad & u_n \rightharpoonup u \implies W'\langle Du_n, u_n \rangle_W \to W'\langle Du, u \rangle_W \\
\end{align*}
\]

\[
W'\langle Du, u \rangle_W = 2 H^{-\frac{1}{2}} \langle \nu \cdot u_d, u_{d+1} \rangle_{H^\frac{1}{2}} \\
= \begin{cases} 
0 & \ldots \text{Dirichlet or Neumann} \\
2a \|u_{d+1}\|_{L^2(\Gamma)}^2 & \ldots \text{Robin} \ldots W = L^2_{\text{div}}(\Omega) \times H^1(\Omega)
\end{cases}
\]

(K2) For any sequence \((u_n)\) in \(V\) and any \(\varphi \in C_c^\infty(\Omega)\)

\[
\begin{align*}
\quad & u_n \rightharpoonup u \implies \langle T_0 u_n \mid \varphi u_n \rangle_L \to \langle T_0 u \mid \varphi u \rangle_L \\
\end{align*}
\]
Compactness by compensation

\[ \langle T_0 u_n \mid \varphi u_n \rangle_L = \int_{\Omega} \sum_{k=1}^{d} A_k \partial_k u_n \cdot \varphi u_n \, dx, \quad p = q = d + 1 \]

\[ = -\frac{1}{2} \int_{\Omega} \partial_k \varphi \sum_{k=1}^{d} A_k u_n \cdot u_n \, dx \]

**Theorem.** *(Quadratic theorem)* For \( A_k \in M_{q,p}(\mathbb{R}) \) let

\[ \Lambda := \left\{ \lambda \in \mathbb{R}^p : (\exists \xi \neq 0) \sum_{k=1}^{d} \xi_k A_k \lambda = 0 \right\}, \]

while \( Q(\lambda) := Q \lambda \cdot \lambda \), such that \( Q = 0 \) on \( \Lambda \). If

\[ u_n \rightharpoonup u \quad \text{weakly in} \quad L^2(\Omega; \mathbb{R}^p), \]

and

\[ \sum_{k=1}^{d} A_k \partial_k u_n \quad \text{is relatively compact in} \quad H^{-1}(\Omega; \mathbb{R}^q), \]

then \( Q \circ u_n \rightharpoonup Q \circ u \quad \text{in} \quad \mathcal{D}'(\Omega). \)
Proof of (K2)

\[ \sum_{k=1}^{d} \xi_k A_k \lambda = \begin{bmatrix} \lambda_{d+1} \xi_1 \\ \vdots \\ \lambda_{d+1} \xi_d \\ \sum_{k=1}^{d} \lambda_k \xi_k \end{bmatrix} \implies \Lambda \ldots \lambda_{d+1} = 0 \]

\[ Q(\lambda) = A_i \lambda \cdot \lambda = 2\lambda_i \lambda_{d+1} = 0, \quad \lambda \in \Lambda \]
Comparison with classical $H$-convergence

\[ C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0^\top & c_n \end{bmatrix} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega) \]

\[ \iff \begin{cases} C_n(x)\xi \cdot \xi \geq \alpha |\xi|^2 \\ C_n(x)\xi \cdot \xi \geq \frac{1}{\beta} |C_n(x)\xi|^2 \end{cases} \]

\[ \iff \begin{cases} \alpha \leq c_n(x) \leq \beta \\ A^n(x)\xi \cdot \xi \geq \frac{1}{\beta} |\xi|^2 \\ A^n(x)\xi \cdot \xi \geq \alpha |A^n(x)\xi|^2 \end{cases} \]

At a subsequence $C_n \xrightarrow{H} C$, by compactness theorem.

– Has the limit $C$ the same structure?

– Can we make a connection with $H$-converging (in classical sense) subsequence of $(A^n)$?
Theorem. For the Friedrichs system corresponding to the stationary diffusion equation, a sequence \((C_n)\) in\( M_{d+1}(\alpha, \beta; \Omega)\) of the form

\[
C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0^\top & c_n \end{bmatrix}.
\]

\(H\)-converges with respect to \(L_0\) and \(V_D\) if and only if \((A^n)\) classically \(H\)-converges to some \(A\) and \((c_n)\) \(L^\infty\) weakly * converges to some \(c\). In that case, the \(H\)-limit is the matrix function

\[
C = \begin{bmatrix} A^{-1} & 0 \\ 0^\top & c \end{bmatrix},
\]

\[\square\]
Heat equation as Friedrichs system

\[ \Omega \subseteq \mathbb{R}^d \] open and bounded set with Lipschitz boundary \( \Gamma \), \( T > 0 \) and
\[ \Omega_T := \Omega \times (0, T) \]

\[ \partial_t u_n - \text{div}_x (A^n \nabla_x u_n) + c u_n = f \quad \text{in} \ \Omega_T, \]

\[ u_n = \begin{bmatrix} u_{dn} \\ u_{d+1,n} \end{bmatrix} = \begin{bmatrix} -A^n \nabla_x u_n \\ u_n \end{bmatrix}. \]

The matrices \( A_k = e_k \otimes e_{d+1} + e_{d+1} \otimes e_k \in M_{d+1}(\mathbb{R}) \), \( k = 1, \ldots, d \), \( A_{d+1} = e_{d+1} \otimes e_{d+1} \) and
\[ C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0 & c \end{bmatrix} \]

\[ T_0 \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_x u_{d+1} \\ \partial_t u_{d+1} + \text{div}_x u_d \end{bmatrix}. \]

Graph space
\[ W = \left\{ u \in L^2_{\text{div}}(\Omega_T) : u_{d+1} \in L^2(0, T; H^1(\Omega)) \right\}. \]
Compactness result

Dirichlet boundary conditions with zero initial value:

\[ V = \left\{ u \in W : u_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \right\}, \]

\[ \tilde{V} = \left\{ v \in W : v^u \in L^2(0, T; H^1_0(\Omega)), \quad v^u(\cdot, T) = 0 \text{ a.e. on } \Omega \right\}. \]

(K1):

\[ W' \langle Du, u \rangle_W = \|u_{d+1}(\cdot, T)\|^2_{L^2(\Omega)}. \]

(K2): similarly to stationary diffusion equation:

\[ \Lambda = \{ \lambda \in \mathbb{R}^{d+1} : \lambda_{d+1} = 0 \} \]

\[ \Rightarrow \quad \mathcal{M}_{d+1}(\alpha, \beta; \Omega) \text{ is compact with } H\text{-topology for given } L_0 \text{ and } V \]

Comparison with classical parabolic H-convergence... similarly as for stationary diffusion equation.
Instead of \( C_n \in \mathcal{M}_r(\alpha, \beta; \Omega) \) we take
\[
C_n \in \mathcal{F}(\alpha, \beta; \Omega) := \left\{ C \in \mathcal{L}(L) : (\forall u \in L)
\left\langle Cu | u \right\rangle_L \geq \alpha \|u\|^2_L \quad \& \quad \left\langle Cu | u \right\rangle_L \geq \frac{1}{\beta} \|Cu\|^2_L \right\}.
\]

For a given sequence \( C_n \in \mathcal{F}(\alpha, \beta; \Omega) \), the sequence of isomorphisms \( T_n := T_0 + C_n : V \to L \) \( G \)-converges to an isomorphism \( T := T_0 + C : V \to L \), for some \( C \in \mathcal{F}(\alpha', \beta'; \Omega) \) if
\[
(\forall f \in L) \quad T_n^{-1}f \rightharpoonup T^{-1}f \text{ in } W.
\]

**Theorem.** For fixed \( T_0 \) and \( V \), if family \( \mathcal{F}(\alpha, \beta; \Omega) \) satisfies (K1), then for any sequence \( (C_n) \) in \( \mathcal{F}(\alpha, \beta; \Omega) \) there exists a subsequence of \( T_n := T_0 + C_n \) which \( G \)-converges to \( T := T_0 + C \) with \( C \in \mathcal{F}(\alpha, \beta; \Omega) \).
Thank you for your attention!