H-measures, H-distributions and applications

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http://riemann.math.hr/weconmapp/

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Start from $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$, and take the Fourier transform:

$$\widehat{\varphi u_n}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} (\varphi u_n)(\mathbf{x}) d\mathbf{x}$$

As φu_n is supported on a fixed compact set K, so $|\widehat{\varphi u_n}(\boldsymbol{\xi})| \leqslant C$.
Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\boldsymbol{\xi}) \longrightarrow 0$ pointwise.
By the Lebesgue dominated convergence theorem applied on bounded sets, we get
$\widehat{\varphi u_n} \longrightarrow 0$ strong, i.e. strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$.

On the other hand, by the Plancherel theorem: $\|\widehat{\varphi u_n}\|_{\mathrm{L}^2(\mathbf{R}^d)} = \|\varphi u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}$.

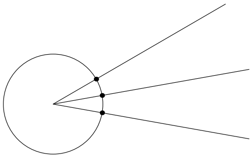
If $\varphi u_n \not\rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then $\widehat{\varphi u_n} \not\rightharpoonup 0$; some information must go to infinity.

Limit is a measure

How does it go to infinity in various directions? Take $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, and consider:

$$\lim_n \int_{\mathbf{R}^d} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)|\widehat{\varphi u_n}|^2 d\boldsymbol{\xi} = \int_{\mathrm{S}^{d-1}} \psi(\boldsymbol{\xi}) d\nu_\varphi(\boldsymbol{\xi}) \ .$$

The limit is a linear functional in ψ, thus an integral over the sphere of some nonnegative Radon measure (a bounded sequence of Radon measures has an accumulation point), which depends on φ. How does it depend on φ?



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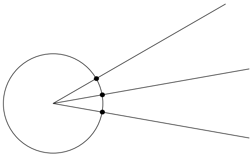

Theorem. (u^n) *a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi}) \ .$$

Why a parabolic variant?

- Parabolic pde-s are:
  - well studied, and we have good theory for them
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Terminology: *classical* as opposed to *parabolic or variant* H-measures.
The sphere we replace by:

$$\sigma^4(\tau,\boldsymbol{\xi}) := (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4 = 1\ , \text{ or}$$
$$\sigma_1^2(\tau,\boldsymbol{\xi}) := |\tau| + (2\pi|\boldsymbol{\xi}|)^2 = 1\ .$$

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finally we chose the ellipse

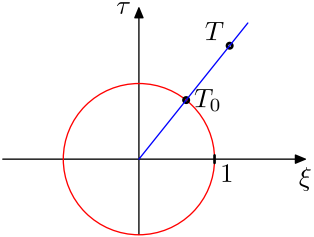
$$\rho^2(\tau, \boldsymbol{\xi}) := |\boldsymbol{\xi}/2|^2 + \sqrt{(\boldsymbol{\xi}/2)^4 + \tau^2} = 1 \ .$$

Notation.
For simplicity (2D): $(t, x) = (x^0, x^1) = \mathbf{x}$ and $(\tau, \xi) = (\xi_0, \xi_1) = \boldsymbol{\xi}$.

We use the Fourier transform in both space and time variables.

Rough geometric idea

Take a sequence $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^2)$, and integrate $|\widehat{\varphi \mathsf{u}_n}|^2$ along rays and project onto S^1



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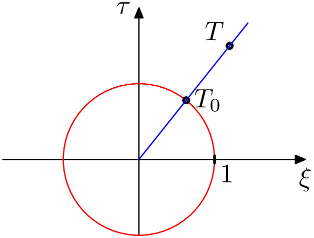

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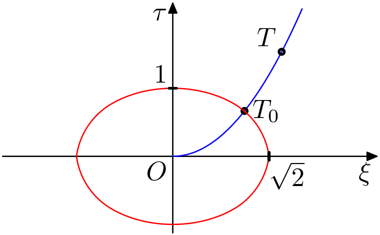

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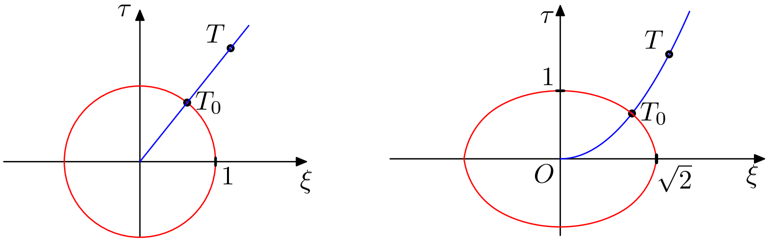

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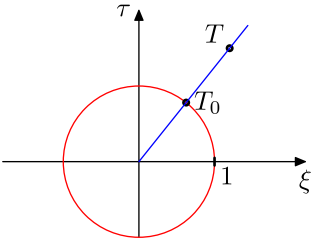

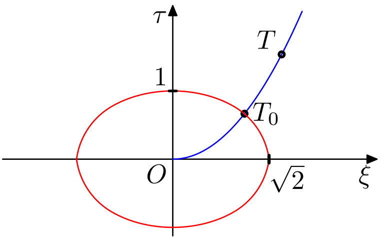

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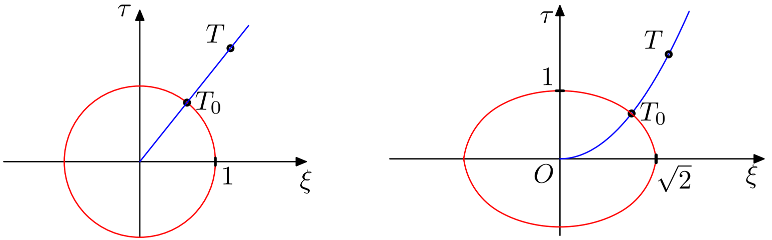

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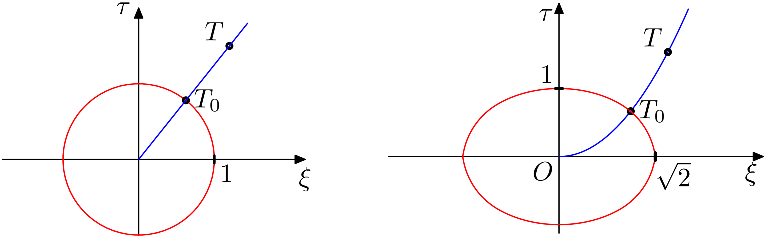

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Analytic picture

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
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Now we can state the main theorem.

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there exists its subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times S^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and

$$\psi \in \mathrm{C}(S^{d-1})$$

one has

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ p \ ) \, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

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Oscillation (classical H-measures)

$$u_n(\mathbf{x}) := v(n\mathbf{x}) \rightharpoonup 0$$

$v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$ periodic function (with the unit period in each of variables), with the zero mean value.

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The associated H-measure

$$\mu(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}} |v_{\mathsf{k}}|^2 \lambda(\mathbf{x})\, \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}),$$

v_{k} Fourier coefficients of v ($v(\mathbf{x}) = \sum_{\mathsf{k} \in \mathbf{Z}^d} v_{\mathsf{k}} e^{2\pi i \mathsf{k} \cdot \mathbf{x}}$).

Dual variable *preserves* information on the direction of propagation (of oscillation).

Oscillation (parabolic H-measures)

Let $v \in \mathrm{L}^2(\mathbf{R}^{1+d})$ be a periodic function

$$v(t, \mathbf{x}) = \sum_{(\omega, \mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega,\mathsf{k}} \, e^{2\pi i(\omega t + \mathsf{k} \cdot \mathbf{x})} \,,$$

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For $\alpha, \beta \in \mathbf{R}^+$, we have a sequence of periodic functions with period tending to zero:

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Their Fourier transforms are:

$$\hat{u}_n(\tau, \boldsymbol{\xi}) = \sum_{(\omega, \mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega, \mathsf{k}} \, \delta_{n^\alpha \omega}(\tau) \delta_{n^\beta \mathsf{k}}(\boldsymbol{\xi}) \,.$$

Oscillation (cont.)

(u_n) is a pure sequence, and the corresponding parabolic H-measure $\mu(t,\mathbf{x},\tau,\boldsymbol{\xi})$ is

$$\lambda(t,\mathbf{x}) \begin{cases} \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \omega\neq 0}} \left|\hat{v}_{\omega,\mathsf{k}}\right|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}) + \sum_{\mathsf{k}\in\mathbf{Z}^d} \left|\hat{v}_{0,\mathsf{k}}\right|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}), & \alpha > 2\beta \\ \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \mathsf{k}\neq 0}} \left|\hat{v}_{\omega,\mathsf{k}}\right|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}) + \sum_{\omega\in\mathbf{Z}} \left|\hat{v}_{\omega,0}\right|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}), & \alpha < 2\beta \\ \displaystyle\sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} \left|\hat{v}_{\omega,\mathsf{k}}\right|^2 \delta_{\left(\frac{\omega}{\rho^2(\omega,\mathsf{k})},\frac{\mathsf{k}}{\rho(\omega,\mathsf{k})}\right)}(\tau,\boldsymbol{\xi}), & \alpha = 2\beta, \end{cases}$$

where λ denotes the Lebesgue measure.

Concentration (classical H-measures)

$$u_n(\mathbf{x}) := n^{\frac{d}{2}} v\,(n\mathbf{x})\,, \qquad \Big(v \in \mathrm{L}^2(\mathbf{R}^d)\Big)\,.$$

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The associated H-measure is of the form $\delta_0(\mathbf{x})\nu(\boldsymbol{\xi})$, where ν is measure on S^{d-1} with surface density

$$\nu(\boldsymbol{\xi}) = \int_0^\infty |\hat{v}(t\boldsymbol{\xi})|^2 t^{d-1} dt,$$

i.e.

$$\mu(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbf{R}^d} |\hat{v}(\boldsymbol{\eta})|^2 \delta_{\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}}(\boldsymbol{\xi})\delta_0(\mathbf{x})\, d\boldsymbol{\eta},$$

where $\hat{v}$ denotes the Fourier transformation of v.

Concentration (parabolic H-measures)

For $v \in \mathrm{L}^2(\mathbf{R}^{1+d})$ and $\alpha, \beta \in \mathbf{R}^+$

$$u_n(t, \mathbf{x}) := n^{\alpha+\beta d} v(n^{2\alpha} t, n^{2\beta} \mathbf{x}),$$

is bounded in $\mathrm{L}^2(\mathbf{R}^{1+d})$ with the norm $\|u_n\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \|v\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}$ which does not depend on n, and weakly converges to zero.

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(u_n) is a pure sequence, with the parabolic H-measure $\langle \boldsymbol{\mu}, \phi \boxtimes \psi \rangle =$

$$\phi(0,0) \begin{cases} \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(\frac{\sigma}{|\sigma|}, 0) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}^d} |\hat{v}(0, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|})\, d\boldsymbol{\eta}, & \alpha > 2\beta \\ \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}} |\hat{v}(\sigma, 0)|^2 \psi(\frac{\sigma}{|\sigma|}, 0)\, d\sigma, & \alpha < 2\beta \\ \displaystyle\int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi\left(\frac{\sigma}{\rho^2(\sigma, \boldsymbol{\eta})}, \frac{\boldsymbol{\eta}}{\rho(\sigma, \boldsymbol{\eta})}\right) d\sigma d\boldsymbol{\eta}, & \alpha = 2\beta. \end{cases}$$

From examples we learn ...

Actually, any non-negative Radon measure on $\Omega \times P^{d-1}$, of total mass A^2, can be described as a parabolic H-measure of some sequence $u_n \longrightarrow 0$, with $\|u_n\|_{\mathrm{L}^2} \leqslant A + \varepsilon$.

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Other research in this direction:
Panov (IHP:AN, 2011): ultraparabolic H-measures
Ivec & Mitrović (CPAA, 2011)
Lazar & Mitrović (MathComm, 2011):
Erceg & Ivec (2017): fractional H-measures

Symmetric systems — localisation principle

$$\partial_k(\mathbf{A}^k \mathsf{u}) + \mathbf{B}\mathsf{u} = \mathsf{f} \ \ , \ \mathbf{A}^k \in \mathrm{C}_b(\mathbf{R}^d; \mathrm{M}_{r\times r}) \text{ Hermitian}$$

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$$\partial_k\big(\mathbf{A}^k \mathsf{u}^n\big) \longrightarrow 0 \ \text{ in space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

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It is a generalisation of compactness by compensation to variable coefficients.

Localisation principle for parabolic H-measures

In the parabolic case the details become more involved.

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Anisotropic Sobolev spaces ($s \in \mathbf{R}$; $k_p(\tau, \boldsymbol{\xi}) := (1 + \sigma^4(\tau, \boldsymbol{\xi}))^{1/4}$)

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{S}' : k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d}) \right\} .$$

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Theorem. **(localisation principle)** Let $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$, uniformly compactly supported in t, satisfy ($s \in \mathbf{N}$)

$$\sqrt{\partial_t}^{\,s}(\mathsf{u}_n \cdot \mathsf{b}) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathsf{u}_n \cdot \mathsf{a}_{\boldsymbol{\alpha}}) \longrightarrow 0 \quad \text{in} \quad \mathrm{H}_{\mathrm{loc}}^{-\frac{s}{2}, -s}(\mathbf{R}^{1+d}) ,$$

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For parabolic H-measure $\boldsymbol{\mu}$ associated to sequence (u_n) one has

$$\boldsymbol{\mu}\left((\sqrt{2\pi i \tau})^s \overline{\mathsf{b}} + \sum_{|\boldsymbol{\alpha}|=s} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}}\, \overline{\mathsf{a}}_{\boldsymbol{\alpha}} \right) = 0.$$

How to use such a relation? — the heat equation

$$\begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}\nabla u_n) = \mathsf{div}\,\mathsf{f}_n \\ u_n(0) = \gamma_n \ , \end{cases}$$

$\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d};\mathbf{R}^d)$, $\gamma_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$

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Localise in time: take θu_n, for $\theta \in \mathrm{C}^1_c(\mathbf{R}^+)$, ...
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Furthermore, $\sqrt{\partial_t}\,(u_n) := \left(\sqrt{2\pi i \tau}\,\widehat{u_n}\right)^{\vee} \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d})$.

The heat equation (cont.)

Take

$$\tilde{\mathsf{v}}_n = (v_n^0, \mathsf{v}_n, \mathsf{f}_n) := (\sqrt{\partial_t} u_n, \nabla u_n, \mathsf{f}_n) \longrightarrow 0$$

in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{R}^{1+2d})$, which (on a subsequence) defines H-measure

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \mu_0 & \boldsymbol{\mu}_{01} & \boldsymbol{\mu}_{02} \\ \boldsymbol{\mu}_{10} & \boldsymbol{\mu} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{20} & \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_f \end{bmatrix}.$$

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The localisation principle gives us:

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After some calculation (linear algebra) ...

Expression for H-measure — from given data

$$\mathsf{tr}\boldsymbol{\mu} = \frac{(2\pi\boldsymbol{\xi})^2}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi},$$

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However, the oscillation in initial data dies out (the equation is hypoelliptic). Only the right hand side affects the H-measure of solutions.

The situation is different for the Schrödinger equation and for the vibrating plate equation.

Small amplitude homogenisation: setting of the problem

A sequence of parabolic problems

$$(*) \qquad \begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}^n \nabla u_n) = f \\ u_n(0,\cdot) = u_0 \ . \end{cases}$$

where $\mathbf{A}^n$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(Q; \mathrm{M}_{d\times d})$, which is bounded from below; for small γ function $\mathbf{A}^n$ is analytic in γ:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma \mathbf{B}^n(t,\mathbf{x}) + \gamma^2 \mathbf{C}^n(t,\mathbf{x}) + o(\gamma^2) \ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$).

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where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$).
Then (after passing to a subsequence if needed)

$$\mathbf{A}^n_\gamma \xrightarrow{H} \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma \mathbf{B}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2) \ ;$$

the limit being measurable in $t, \mathbf{x}$, and analytic in γ.

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

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Indeed, take $u \in \mathrm{L}^2([0,T];\mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle;\mathrm{H}^{-1}(\Omega))$, and define $f_\gamma := \partial_t u - \mathrm{div}\,(\mathbf{A}_\gamma^\infty \nabla u)$, and $u_0 := u(0,\cdot) \in \mathrm{L}^2(\Omega)$.

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Next, solve $(*)$ with $\mathbf{A}_\gamma^n$, f_γ and u_0, the solution u_γ^n.
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Because of H-convergence, we have the weak convergences in $\mathrm{L}^2(Q)$:

$$(\dagger)\qquad \begin{aligned} \mathsf{E}_\gamma^n &:= \nabla u_\gamma^n \rightharpoonup \nabla u \\ \mathsf{D}_\gamma^n &:= \mathbf{A}_\gamma^n \mathsf{E}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u\ . \end{aligned}$$

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Expansions in Taylor serieses (similarly for f_γ and u_γ^n):

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Inserting (†) and equating the terms with equal powers of γ:

$$\mathsf{E}_0^n = \nabla u\,, \qquad \mathsf{D}_0^n = \mathbf{A}_0 \nabla u$$
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For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0 \mathsf{E}_2^n + \mathbf{B}^n \mathsf{E}_1^n + \mathbf{C}^n \nabla u \rightharpoonup \lim \mathbf{B}^n \mathsf{E}_1^n = \mathbf{C}_0 \nabla u \,,$$

and this is the limit we still have to compute.

Periodic homogenisation — an example

In the periodic case the explicit formulae for the homogenisation limit are known [BLP].
Together with Fourier analysis:
leading terms in expansion for the small amplitude homogenisation limit.

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Then $\mathbf{A}_n$ H-converges to a constant $\mathbf{A}_\infty$ defined by

$$\mathbf{A}_\infty \mathsf{h} = \int_Z \mathbf{A}(\tau, \mathbf{y})(\mathsf{h} + \nabla w(\tau, \mathbf{y}))\, d\tau d\mathbf{y} .$$

For given h, w is a solution of some BVP, depending on ρ.

Three different cases depending on ρ

$\rho \in \langle 0, 2 \rangle$: $w(\tau, \cdot)$ is the unique solution of

$$-\mathsf{div}\,(\mathbf{A}(\tau,\cdot)(\mathsf{h} + \nabla w(\tau,\cdot))) = 0$$
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$\rho \in \langle 2, \infty\rangle$: define $\widetilde{\mathbf{A}}(y) = \int_0^1 \mathbf{A}(\tau,\mathbf{y})\,d\tau$ and w as the solution of

$$-\mathsf{div}\,(\widetilde{\mathbf{A}}(\mathsf{h} + \nabla w)) = 0$$
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Periodic small-amplitude homogenisation

A sequence of small perturbations of a constant coercive matrix $\mathbf{A}_0 \in \mathrm{M}_{d\times d}$:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma\mathbf{B}^n(t,\mathbf{x})\,,$$

where $\mathbf{B}^n(t,\mathbf{x}) = \mathbf{B}(n^\rho t, n\mathbf{x})$, $\mathbf{B}$ is Z-periodic L^∞ matrix function satisfying $\int_Z \mathbf{B}\,d\tau d\mathbf{y} = 0$.

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For γ small enough, (eventually passing to a subsequence) we have H-convergence to a limit depending analytically on γ:

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and a formula for $\mathbf{A}^\infty_\gamma$:

$$\begin{aligned}\mathbf{A}^\infty_\gamma \mathsf{h} &= \int_Z (\mathbf{A}_0 + \gamma\mathbf{B}))(\mathsf{h} + \nabla w_\gamma)\,d\tau d\mathbf{y} \\ &= \mathbf{A}_0\mathsf{h} + \int_Z \mathbf{A}_0 \nabla w_\gamma + \gamma \int_Z \mathbf{B}\mathsf{h} + \gamma \int_Z \mathbf{B}\nabla w_\gamma = \mathbf{A}_0 \mathsf{h} + \gamma \int_Z \mathbf{B}\nabla w_\gamma\,.\end{aligned}$$

Periodic small-amplitude homogenisation (cont.)

In the last equality the second term equals zero by Gauss' theorem, as w_γ is a periodic function. Similarly for the third term.

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so we conclude that $\mathbf{B}_0 = \mathbf{0}$ and $\mathbf{C}_0 \mathsf{h} = \int_Z \mathbf{B}\nabla w_1$.

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From this formula, using the Fourier series, one can calculate the second-term approximation $\mathbf{C}_0$. Off course, we must treat separately each one of the above three cases for ρ.

The case $\rho \in \langle 0, 2\rangle$ on the limit

Fix $\tau \in [0, 1]$; the BVP with coefficient $\mathbf{A}_0 + \gamma\mathbf{B}$ instead of $\mathbf{A}$ and the above expression for w, we see that w_1 solves

$$(\ddagger) \quad -\mathsf{div}\,(\mathbf{A}_0 \nabla w_1(\tau, \cdot)) = \mathsf{div}\,(\mathbf{B}\mathsf{h})\,,\ w_1(\tau, \cdot) \in \mathrm{H}^1(Y)\,,\ \int_Y w_1(\tau, \mathbf{y})\,d\mathbf{y} = 0$$

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Expanding w_1 in the Fourier series gives us ($J = \mathbf{Z} \times (\mathbf{Z}^d \setminus \{\mathbf{0}\})$)

$$w_1 = \sum_{(l,\mathsf{k})\in J} a_{l\mathsf{k}} e^{2\pi i(l\tau + \mathsf{k}\cdot\mathbf{y})}\,,$$

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Straightforward calculation gives us

$$\nabla w_1 = \sum_J 2\pi i \mathsf{k}\, a_{l\mathsf{k}} e^{2\pi i(l\tau + \mathsf{k}\cdot\mathbf{y})}\,,$$

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The case $\rho \in \langle 0, 2\rangle$ on the limit (cont.)

For $\mathbf{B}$ denote $I := \mathbf{Z}^{d+1} \setminus \{\mathbf{0}\}$

$$\mathbf{B} = \sum_I \mathbf{B}_{l\mathsf{k}} e^{2\pi i(l\tau + \mathsf{k}\cdot\mathbf{y})} ,$$

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(‡) leads to a relation among corresponding Fourier coefficients

$$2\pi i \mathbf{A}_0 \mathsf{k}\cdot\mathsf{k}\, a_{l\mathsf{k}} = -\mathbf{B}_{l\mathsf{k}}\mathsf{h}\cdot\mathsf{k}\,, \quad (l,\mathsf{k}) \in \mathbf{Z}^{d+1}\,,$$

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Finally, we obtain

$$\begin{aligned} \mathbf{C}_0\mathsf{h} &= \int_Z \mathbf{B}\nabla w_1\, d\tau d\mathbf{y} \\ &= \int_Z \left(\sum_I \mathbf{B}_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}\right)\left(\sum_J (2\pi i \mathsf{k}') a_{l'\mathsf{k}'} e^{2\pi i (l'\tau + \mathsf{k}'\cdot\mathbf{y})}\right) d\tau d\mathbf{y} \end{aligned}$$

The case $\rho \in \langle 0, 2 \rangle$ on the limit (cont.)

Due to orthogonality, for the non-vanishing terms in the above product of two series we have $l' = -l$ and $\mathsf{k}' = -\mathsf{k}$. Therefore,

$$\begin{aligned}\mathbf{C}_0 \mathsf{h} &= -2\pi i \sum_J \mathbf{B}_{l\mathsf{k}} \mathsf{k} a_{-l,-\mathsf{k}} \\ &= -\sum_J \mathbf{B}_{l\mathsf{k}} \mathsf{k} \frac{\mathbf{B}_{-l,-\mathsf{k}} \mathsf{h} \cdot \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}} \mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} \mathsf{h}\,,\end{aligned}$$

where the last equality holds since $\mathbf{B}$ is a real matrix function i.e. $\overline{\mathbf{B}_{l\mathsf{k}}} = \mathbf{B}_{-l,-\mathsf{k}}$.

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$$= -\sum_J \mathbf{B}_{l\mathsf{k}} \mathsf{k} \frac{\mathbf{B}_{-l,-\mathsf{k}} \mathsf{h} \cdot \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}} \mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} \mathsf{h}\,,$$

where the last equality holds since $\mathbf{B}$ is a real matrix function i.e. $\overline{\mathbf{B}_{l\mathsf{k}}} = \mathbf{B}_{-l,-\mathsf{k}}$. We conclude

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}} \mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}}\,.$$

The case $\rho = 2$ on the limit

The calculation is similar to the first case. The only difference appears in the equation for $w_1 = \sum_{(l,\mathsf{k})\in I} a_{l\mathsf{k}} e^{2\pi i(l\tau+\mathsf{k}\cdot\mathbf{y})}$:

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implying the following relation for the Fourier coefficients

$$(l - 2\pi i \mathbf{A}_0 \mathsf{k}\cdot\mathsf{k} a_{l\mathsf{k}}) = \mathbf{B}_{l\mathsf{k}}\mathsf{h}\cdot\mathsf{k}\,, \quad (l,\mathsf{k}) \in I\,,$$

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and the formula for the second order approximation of the H-limit:

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}}\mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}}\mathsf{k}}{\frac{l}{2\pi i} + \mathbf{A}_0\mathsf{k}\cdot\mathsf{k}}\,.$$

The case $\rho \in \langle 2, \infty \rangle$ on the limit

In this case w_1 does not depend on τ. Introducing

$$\widetilde{\mathbf{B}}(\mathbf{y}) := \int_0^1 \mathbf{B}(\tau, \mathbf{y})\, d\tau$$

this case actually has the same behaviour as the one in elliptic setting, giving the formula

$$\mathbf{C}_0 = -\sum_{\mathbf{Z}^d \setminus \{\mathbf{0}\}} \frac{\widetilde{\mathbf{B}}_{\mathsf{k}}\mathsf{k} \otimes \widetilde{\mathbf{B}}_{\mathsf{k}}\mathsf{k}}{\mathbf{A}_0\mathsf{k} \cdot \mathsf{k}} \,.$$

Parabolic small-amplitude homogenisation—general case

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u_1^n satisfies the equation $(*)$ with coefficients $\mathbf{A}_0$, div $(\mathbf{B}^n\nabla u)$ on the right hand side and the homogeneous innitial condition.

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$$\widehat{\nabla u_1^n}(\tau, \boldsymbol{\xi}) = -\frac{(2\pi)^2\,(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\,(\widehat{\mathbf{B}^n\nabla u})(\tau, \boldsymbol{\xi})}{2\pi i\tau + (2\pi)^2\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}}.$$

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(the precise argument involves localisation principle and some calculations ...)

Expression for the quadratic correction

As $(\boldsymbol{\xi} \otimes \boldsymbol{\xi})/(2\pi i\tau + (2\pi)^2 \mathbf{A}_0\boldsymbol{\xi} \cdot \boldsymbol{\xi})$ is constant along branches of paraboloids $\tau = c\boldsymbol{\xi}^2, c \in \overline{\mathbf{R}}$, we have ($\varphi \in \mathrm{C}_c^\infty(Q)$)

$$\lim_n \Big\langle\, \varphi\mathbf{B}^n \mid \nabla u_1^n \,\Big\rangle = -\lim_n \Big\langle\, \widehat{\varphi\mathbf{B}^n} \mid \frac{(2\pi)^2\,(\boldsymbol{\xi} \otimes \boldsymbol{\xi})\,(\widehat{\mathbf{B}^n\nabla u})}{2\pi i\tau + (2\pi)^2\mathbf{A}_0\boldsymbol{\xi} \cdot \boldsymbol{\xi}} \,\Big\rangle$$
$$= -\Big\langle \boldsymbol{\mu}, \varphi\frac{(2\pi)^2\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \nabla u}{-2\pi i\tau + (2\pi)^2\mathbf{A}_0\boldsymbol{\xi} \cdot \boldsymbol{\xi}} \Big\rangle,$$

where $\boldsymbol{\mu}$ is the parabolic variant H-measure associated to $(\mathbf{B}^n)$, a measure with four indices (the first two of them not being contracted above).

Expression for the quadratic correction (cont.)

By varying function $u \in \mathrm{C}^1(Q)$ (e.g. choosing ∇u constant on $\langle 0, T\rangle \times \omega$, where $\omega \subseteq \Omega$) we get

$$\int_{\langle 0,T\rangle \times \omega} C_0^{ij}(t,\mathbf{x})\phi(t,\mathbf{x})dtd\mathbf{x} = -\Big\langle \boldsymbol{\mu}^{ij}, \phi \frac{(2\pi)^2 \boldsymbol{\xi} \otimes \boldsymbol{\xi}}{-2\pi i\tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \Big\rangle,$$

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For the periodic example of small-amplitude homogenisation, we get the same results by applying the variant H-measures, as with direct calculations performed above.

Homogenisation of a model based on the Stokes equation: stationary case

(Tartar, 1976 and 1984)
$\Omega \subseteq \mathbf{R}^3$ open set, $\mathsf{u}_n \longrightarrow \mathsf{u}_0$ in $\mathrm{H}^1_{\mathrm{loc}}(\Omega; \mathbf{R}^3)$

$$\begin{cases} -\nu \triangle \mathsf{u}_n + \mathsf{u}_n \times \operatorname{rot}(\mathsf{v}_0 + \lambda \mathsf{v}_n) + \nabla p_n = \mathsf{f}_n \\ \operatorname{div} \mathsf{u}_n = 0 \,. \end{cases}$$

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Not a realistic model, but contains the terms: $\mathsf{u} \times \mathsf{rot}\,\mathsf{A}$
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Theorem. *There is a subsequence and* $\mathbf{M} \geqslant 0$*, depending on the choice of the subsequence, such that the limit* u_0 *satisfies:*

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and it holds:

$$\nu |\nabla \mathsf{u}_n|^2 \longrightarrow \nu |\nabla \mathsf{u}_0|^2 + \lambda^2 \mathbf{M} \mathsf{u}_0 \cdot \mathsf{u}_0 \qquad \text{in } \mathcal{D}'(\Omega)\ .$$

∎

Explicit formula via H-measures

Can $\mathbf{M}$ be computed directly from $\mathsf{v}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega; \mathbf{R}^3)$ (also bounded in $\mathrm{L}^3(\Omega; \mathbf{R}^3)$)?

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$$\mathbf{M} = \frac{1}{\nu} \left\langle\!\left\langle \boldsymbol{\mu}, (\mathsf{v}^2 - (\mathsf{v} \cdot \boldsymbol{\xi})^2) \boldsymbol{\xi} \otimes \boldsymbol{\xi} \right\rangle\!\right\rangle \; .$$

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$\mathbf{M}$ is not only a measure, but a function.

What in the time-dependent case?

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M. Lazar and myself — wrote it down (technical difference in the scaling).

Time dependent case

On $\mathbf{R}^3$ (we need good estimates for the pressure).

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Tartar's model from 1985:

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Assume that

$$\begin{aligned} \mathsf{u}_n &\longrightarrow \mathsf{u}_0 \quad \text{in } \mathrm{L}^2([0,T]; \mathrm{H}^1(\mathbf{R}^3;\mathbf{R}^3)) \; , \\ \mathsf{u}_n &\xrightarrow{\;*\;} \mathsf{u}_0 \quad \text{in } \mathrm{L}^\infty([0,T]; \mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3)) \; . \end{aligned}$$

and (p_n) is bounded in $\mathrm{L}^2([0,T] \times \mathbf{R}^3)$.

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Oscillation in (v_n) generates oscillation in $(\nabla \mathsf{u}_n)$, which dissipates energy via viscosity.

This should be visible from macroscopic equation (equation satisfied by u_0).

Sufficient assumptions on v_n and f_n

$$\mathsf{f}_n = \operatorname{div} \mathbf{G}_n, \text{ with } \mathbf{G}_n \longrightarrow \mathbf{G}_0 \text{ in } \mathrm{L}^2([0,T] \times \mathbf{R}^3; \mathrm{M}_{3\times 3})$$

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$\mathsf{v}_n = \mathsf{a}_n + \mathsf{b}_n$, where

$\mathsf{a}_n \xrightarrow{\ *\ } 0$ in $\mathrm{L}^q([0,T]; \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{R}^3))$, for some $q > 2$,

$\mathsf{b}_n \xrightarrow{\ *\ } 0$ in $\mathrm{L}^\infty([0,T]; \mathrm{L}^r(\mathbf{R}^3; \mathbf{R}^3))$, for some $r > 3$.

Homogenised equation

Theorem. There is a subsequence and a function $\mathbf{M} \geqslant \mathbf{0}$ such that the limit u_0 satisfies:

$$\begin{cases} \partial_t \mathsf{u}_0 - \nu \triangle \mathsf{u}_0 + \mathsf{u}_0 \times \operatorname{rot} \mathsf{v}_0 + \lambda^2 \mathbf{M} \mathsf{u}_0 + \nabla p_0 = \mathsf{f}_0 \\ \operatorname{div} \mathsf{u}_0 = 0 \, , \end{cases}$$

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and that we have the convergence

$$\nu |\nabla \mathsf{u}_n|^2 \rightharpoonup \nu |\nabla \mathsf{u}_0|^2 + \lambda^2 \mathbf{M} \mathsf{u}_0 \cdot \mathsf{u}_0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^{1+3}) \, .$$

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There is a new term, $\mathbf{M}$, in the macroscopic equation.
How can it be computed?

Oscillating test functions

$$\begin{cases} -\partial_t \mathsf{w}_n - \nu \triangle \mathsf{w}_n + \mathsf{k} \times \operatorname{rot} \mathsf{v}_n + \nabla r_n = 0 \\ \operatorname{div} \mathsf{w}_n = 0 \,, \end{cases}$$

supplemented by requirements:

$\mathsf{w}_n \longrightarrow 0$ in $\mathrm{L}^2([0,T];\mathrm{H}^1(\mathbf{R}^3;\mathbf{R}^3))$, and
$\mathsf{w}_n \overset{*}{\longrightarrow} 0$ in $\mathrm{L}^\infty([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3))$.

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Sufficient to take homogeneous condition at $t = T$,
and (additional assumption) v_n bounded in $\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3))$.
This in particular gives r_n bounded in $\mathrm{L}^2([0,T] \times \mathbf{R}^3)$.

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$$\nu \int_{\mathbf{R}^{1+3}} \varphi |\nabla \mathsf{w}_n|^2 \, d\mathbf{y} \longrightarrow \int_{\mathbf{R}^{1+3}} \varphi \mathbf{M}\mathsf{k} \cdot \mathsf{k} \, d\mathbf{y} \ ,$$

$\mathbf{M} \in \mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^3;\mathrm{M}_{3\times 3\times)})$ and $\langle \mathbf{M}\mathsf{k} \mid \mathsf{k} \rangle \geqslant 0, \quad \mathsf{k} \in \mathbf{R}^3$.

Can we avoid w_n?

Theorem. Let $\boldsymbol{\mu}$ be a variant H-measure associated to a subsequence of (v_n).

$$\int_{\mathbf{R}^{1+3}} \mathbf{M}(t,\mathbf{x})\phi(t,\mathbf{x})dtd\mathbf{x} =$$

$$= 4\pi^2\nu\Big\langle \Big(\mathrm{tr}\boldsymbol{\mu}|\boldsymbol{\xi}|^2 - \boldsymbol{\mu}\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\Big)\frac{(\boldsymbol{\xi}\otimes\boldsymbol{\xi})}{\tau^2+\nu^2 4\pi^2|\boldsymbol{\xi}|^4}, \phi\boxtimes 1\Big\rangle,$$

with $\phi \in \mathrm{C}_c^\infty(\langle 0,T\rangle\times\mathbf{R}^3)$.

Proof.

For w_n we have (with $0 \leqslant \mathbf{M} \in \mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^3;\mathrm{M}_{3\times 3})))$:

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From estimates on r_n and v_n we get $\mathsf{w}_n' \longrightarrow 0$ in $\mathrm{L}^2(0,T; \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^3))$, and compactness lemma gives us $\mathsf{w}_n \to 0$ in $\mathrm{L}^2_{\mathrm{loc}}([0,T] \times \mathbf{R}^3)$.

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For w_n we have (with $0 \leqslant \mathbf{M} \in \mathrm{L}^2([0,T]; \mathrm{H}^{-1}(\mathbf{R}^3; \mathrm{M}_{3\times 3})))$:

$$\nu \int_{\mathbf{R}^{1+3}} \varphi |\nabla \mathsf{w}_n|^2 \, d\mathbf{y} \longrightarrow \int_{\mathbf{R}^{1+3}} \varphi \mathbf{M} \mathsf{k} \cdot \mathsf{k} \, d\mathbf{y} \; .$$

From estimates on r_n and v_n we get $\mathsf{w}_n' \longrightarrow 0$ in $\mathrm{L}^2(0,T; \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^3))$, and compactness lemma gives us $\mathsf{w}_n \to 0$ in $\mathrm{L}^2_{\mathrm{loc}}([0,T] \times \mathbf{R}^3)$.

Therefore:

$$\lim_n \int_{\mathbf{R}^{1+3}} |\varphi \nabla \mathsf{w}_n|^2 \, d\mathbf{y} = \lim_n \int_{\mathbf{R}^{1+3}} |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{y} \; .$$

Localise . . .

Localise by multiplying the auxilliary problem by $\varphi \in \mathrm{C}_c^\infty(\langle 0, T\rangle \times \mathbf{R}^3)$

$$-\partial_t(\varphi \mathsf{w}_n) - \nu \triangle(\varphi \mathsf{w}_n) + \mathsf{k} \times \mathsf{rot}\,(\varphi \mathsf{v}_n) = -\nabla(\varphi r_n) + \mathsf{q}_n \ ,$$

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$$\mathsf{q}_n = -(\partial_t\varphi)\mathsf{w}_n - \nu(\triangle\varphi)\mathsf{w}_n - 2\nu(\nabla \mathsf{w}_n)\nabla\varphi + \mathsf{k}\times(\nabla\varphi\times \mathsf{v}_n) + r_n\nabla\varphi \ ,$$

$\mathsf{q}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^{1+3})$ (and also strongly in $\mathrm{H}^{-\frac{1}{2},-1}(\mathbf{R}^{1+3})$).
As $\mathsf{w}_n \rightharpoonup 0$ in $\mathrm{L}^2([0,T];\mathrm{H}^1(\mathbf{R}^3))$, so localised w_n and $\nabla \mathsf{w}_n$ converge weakly in L^2.

Of course, localised v_n and r_n converge weakly in L^2 as well.
From boundedness of the support of φ, we have strong convergence in $\mathrm{H}^{-\frac{1}{2},-1}$.

The Fourier transform

$$(-2\pi i\tau + \nu 4\pi^2 \boldsymbol{\xi}^2)\widehat{\varphi \mathsf{w}_n} = -\mathsf{k} \times \left((2\pi i\boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\right) - 2\pi i\widehat{\varphi r_n}\boldsymbol{\xi} + \hat{\mathsf{q}}_n \ ,$$

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$\operatorname{div} \mathsf{w}_n = 0$, so $\boldsymbol{\xi} \cdot \hat{\mathsf{w}}_n = 0$; which does not hold for $\operatorname{div} (\varphi \mathsf{w}_n) = \nabla\varphi \cdot \mathsf{w}_n$. However, the RHS converges strongly in L^2 to 0, so in the Fourier space:

$$2\pi \boldsymbol{\xi} \cdot \widehat{\varphi \mathsf{w}_n} \longrightarrow 0 \ .$$

Projection by $P_{\boldsymbol{\xi}}$

After projection

$$\widehat{\varphi \mathsf{w}_n} = \frac{-P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big)\Big) + P_{\boldsymbol{\xi}} \hat{\mathsf{q}}_n}{-2\pi i \tau + \nu 4 \pi^2 \boldsymbol{\xi}^2} + \mathsf{d}_n \ ,$$

with $\mathsf{d}_n \longrightarrow 0$ in L^2.

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By Plancherel

$$\begin{aligned}
\lim_n \int_\Omega \nu|\nabla(\varphi \mathsf{w}_n)|^2\, d\mathbf{x} &= \lim_n \int_{\mathbf{R}}^{1+d} \nu 4\pi^2 |\widehat{(\varphi \mathsf{w}_n)}|^2 d\tau d\boldsymbol{\xi} \\
&= \lim_n \int_{\mathbf{R}}^{1+d} \nu 4\pi^2\boldsymbol{\xi}^2 \left| \frac{P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i\boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big) + \hat{\mathsf{q}}_n\Big)}{-2\pi i\tau + \nu 4\pi^2\boldsymbol{\xi}^2} \right|^2 d\tau d\boldsymbol{\xi} \\
&= \lim_n \int_{\mathbf{R}}^{1+d} \nu\boldsymbol{\xi}^2 \frac{\Big|P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i\boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big) + \hat{\mathsf{q}}_n\Big)\Big|^2}{\tau^2 + \nu 4\pi^2\boldsymbol{\xi}^4} d\tau d\boldsymbol{\xi}
\end{aligned}$$

Applying the Lemma (analysis)

$$\frac{|\boldsymbol{\xi}|\hat{\mathsf{q}}_n}{\sqrt{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4}} \to 0 \quad \text{in} \qquad \mathrm{L}^2(\mathbf{R}^{1+3}) \ .$$

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By $P_{\boldsymbol{\eta}}$

$$\Big| P_{\boldsymbol{\eta}}\Big(\mathsf{k} \times (\boldsymbol{\eta} \times \mathbf{a})\Big)\Big|^2 = (\mathsf{k} \cdot \boldsymbol{\eta})^2 \Big(|\mathsf{a}|^2 - |\mathsf{a} \cdot \boldsymbol{\eta}_0|^2\Big)$$

where $\boldsymbol{\eta}_0$ is the unit vector in the direction of $\boldsymbol{\eta}$.

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where $\boldsymbol{\eta}_0$ is the unit vector in the direction of $\boldsymbol{\eta}$.
Note that k and $\boldsymbol{\eta}$ are real, while only a is complex. Therefore:

$$\lim_n \int_\Omega \nu |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{x}$$

$$= \lim_n \int_{\mathbf{R}^3} \boldsymbol{\xi}^2 \frac{\Big(\mathsf{k} \cdot 2\pi i \boldsymbol{\xi}\Big)^2 \Big(|\widehat{\varphi \mathsf{v}_n}|^2 - \Big| \widehat{\varphi \mathsf{v}_n} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \Big|^2 \Big)}{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4} \, d\boldsymbol{\xi} \; .$$

Finally (after some algebra)

$$\lim_n \int_{\mathbf{R}^3} \boldsymbol{\xi}_0^2 \frac{\left(\mathsf{k}\cdot 2\pi i \boldsymbol{\xi}_0\right)^2 \left(|\widehat{\varphi \mathsf{v}_n}|^2 - \left|\widehat{\varphi \mathsf{v}_n} \cdot \frac{\boldsymbol{\xi}_0}{|\boldsymbol{\xi}_0|}\right|^2\right)}{\tau_0^2 + \nu 4\pi^2 \boldsymbol{\xi}_0^4}\, d\boldsymbol{\xi} =$$

$$= \frac{1}{\nu}\langle \mathsf{tr}\boldsymbol{\mu}, (\frac{\boldsymbol{\xi}_0 \cdot \mathsf{k}}{\tau_0^2 + \nu 4\pi^2 \boldsymbol{\xi}_0^4})^2 \varphi\overline{\varphi}\rangle$$

$$- \frac{1}{\nu}\langle \boldsymbol{\mu}, (\frac{\boldsymbol{\xi}_0 \cdot \mathsf{k}}{\tau_0^2 + \nu 4\pi^2 \boldsymbol{\xi}_0^4})^2 \varphi\overline{\varphi}\boldsymbol{\xi} \otimes \boldsymbol{\xi}\rangle \ .$$

Good bounds in the L^p case: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d) \ , \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

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Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|} ,$$

then for any $p \in \langle 1, \infty\rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty) .$$

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For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d_*$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

The main theorem

Theorem. [N.A. & D. Mitrović (2011)] *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$ *and* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^q(\mathbf{R}^d)$ *for some* $q \geqslant \max\{p', 2\}$, *then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$, *such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ *we have:*

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The H-distribution would correspond to a non-diagonal block for an H-measure.

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d\rangle$, *such that*

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Take an arbitrary (v_n) *bounded in* $\mathrm{L}^\infty(\mathbf{R}^d)$, *and by* μ *denote the* H*-distribution corresponding to a subsequence of* (u_n) *and* (v_n). *Then*

$$(\mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi})\mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}$ *being the symbol of the linear PDO with* C^κ_0 *coefficients.* ∎

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$$(\mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi})\mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}$ *being the symbol of the linear PDO with* C^κ_0 *coefficients.* ∎

In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\boldsymbol{\xi}|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\frac{\xi_j}{i|\boldsymbol{\xi}|}}$.

Note that

$$\int I_1(\phi)\partial_j g = \int (R_j \phi) g, \quad g \in \mathcal{S}(\mathbf{R}^d).$$

Using the density argument and that R_j is bounded from $\mathrm{L}^p(\mathbf{R}^d)$ to itself, we conclude $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in \mathrm{L}^p(\mathbf{R}^d)$.

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d \rangle$, *such that*

$$\operatorname{div}(\mathsf{a}(\mathbf{x}) u_n(\mathbf{x})) = f_n(\mathbf{x}) \ .$$

Take an arbitrary (v_n) *bounded in* $\mathrm{L}^\infty(\mathbf{R}^d)$, *and by* μ *denote the* H*-distribution corresponding to a subsequence of* (u_n) *and* (v_n). *Then*

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(an application suggested by Darko Mitrović) For scalar conservation law with discontinuous flux, the most up to date existence result for the equation

$$u_t + \operatorname{div} \mathsf{f}(t, \mathbf{x}, u) = 0$$

is obtained under the assumptions

$$\max_{\lambda \in \mathbf{R}} |\mathsf{f}(t, \mathbf{x}, \lambda)| \in L^{2+\varepsilon}(\mathbf{R}^d_+) \ .$$

Using the H-distributions, it is poossible to prove an existence result for the given equation under the assumption

$$\max_{\lambda \in \mathbf{R}} |\mathsf{f}(t, \mathbf{x}, \lambda)| \in L^{1+\varepsilon}(\mathbf{R}^d_+) \ .$$

Further variants

N.A. & I. Ivec (JMAA, 2016): extension to Lebesgue spaces with mixed norm

M. Lazar & D. Mitrović (DynPDE, 2012): applications to velocity averaging

M. Mišur & D. Mitrović (JFA, 2015): a form of compactness by compensation

J. Aleksić, S. Pilipović, I. Vojnović (Mediter. J. Maths, 2017): in $\mathcal{S}-\mathcal{S}'$ setting

F. Rindler (ARMA, 2015): microlocal compactness forms

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_{\mathrm{b}}(\Omega \times \mathbf{R}^d; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi})\Big) \psi(\omega_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \Big\langle \boldsymbol{\mu}_{sc}^{(\omega_n)}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \Big\rangle .$$

Measure $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ *we call the semiclassical measure with characteristic length* (ω_n) *corresponding to the (sub)sequence* (u_n)*.* ∎

Semiclassical measures

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}(\Omega \times \mathbf{R}^d; \mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

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Theorem.

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{sc}^{(\omega_n)} = \mathbf{0} \quad \& \quad (\mathsf{u}_n) \textit{ is } (\omega_n)-\textit{oscillatory} .$$

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Definition (u_n) is (ω_n)-oscillatory if
$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \quad \lim_{R\to\infty} \limsup_n \int_{|\boldsymbol{\xi}| \geqslant \frac{R}{\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} = 0\,.$

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∎

Localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\mathbf{P}_n \mathsf{u}_n := \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \, ,$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

Then we have

$$\mathbf{p} \boldsymbol{\mu}_{sc}^{\top} = \mathbf{0} \, ,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

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$$\operatorname{supp} \boldsymbol{\mu}_{sc} \subseteq \{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathbf{R}^d \;:\; \det \mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = 0\}\,,$$

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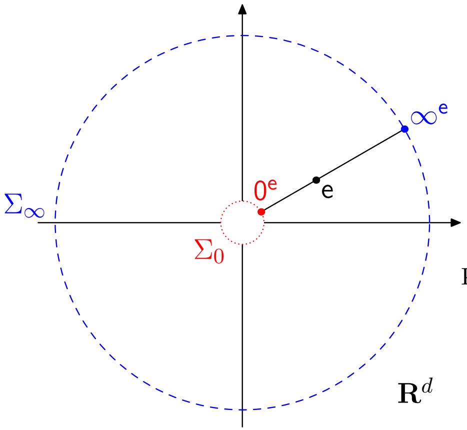
Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

Compatification of $\mathbf{R}^d \setminus \{0\}$



$$\Sigma_0 := \{0^{\mathsf{e}} \ : \ \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\Sigma_\infty := \{\infty^{\mathsf{e}} \ : \ \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Corollary. *a)* $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$. ∎

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exists a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_{\mathrm{b}}(\Omega \times \mathbf{R}^d; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

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One-scale H-measures

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N. A., Marko Erceg, Martin Lazar: *Localisation principle for one-scale H-measures*, submitted (`arXiv`).

Idea of the proof

Tartar's approach:

- $\mathsf{v}_n(\mathbf{x}, x^{d+1}) := \mathsf{u}_n(\mathbf{x}) e^{\frac{2\pi i x^{d+1}}{\omega_n}} \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega \times \mathbf{R}; \mathbf{C}^r)$
- $\boldsymbol{\nu}_H \in \mathcal{M}(\Omega \times \mathbf{R} \times \mathrm{S}^d; \mathrm{M_r}(\mathbf{C}))$
- $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}$ is obtained from $\boldsymbol{\nu}_H$ (suitable projection in x^{d+1} and ξ_{d+1})

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Our approach:

- First commutation lemma:

Lemma. *Let $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, and denote $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. Then the commutator can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K\,,$$

where K is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$, while $\tilde{C}_n \longrightarrow 0$ in the operator norm on $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$. ∎

- standard procedure: (a variant of) the kernel theorem, separability, . . .

Some properties of $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$

Theorem.

$$
\begin{array}{llcl}
a) & \boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}\ , & & \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0} \\
b) & \mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 & \iff & \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0} \\
c) & \mathrm{tr}\,\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 & \iff & (\mathsf{u}_n) \textit{ is } (\omega_n) - \textit{oscillatory}
\end{array}
$$

∎

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Theorem.

a) $\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$, $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0}$

b) $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$

c) $\mathrm{tr}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 \iff (\mathsf{u}_n)$ *is* $(\omega_n) -$ *oscillatory*

∎

Theorem. $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^d)$, $\tilde\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \to 0^+$,

a) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle = \langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle$,

b) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi \circ \boldsymbol{\pi}\rangle = \langle \boldsymbol{\mu}_H, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi\rangle$,

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

∎

Localisation principle

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

where

- $l \in 0..m$
- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

Localisation principle

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

where

- $l \in 0..m$
- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

Lemma. *a)* $(\mathrm{C}(\varepsilon_n))$ *is equivalent to*

$$(\forall\, \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)\,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \quad \Longrightarrow \quad (\varepsilon_n^{k-l} \mathsf{f}_n)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$. ∎

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem. [Tartar (2009)] *Under previous assumptions and $l=1$, one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) corresponding to (u_n) satisfies*

$$\operatorname{supp}\,(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in\mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem. [N.A., Erceg, Lazar (2015)] *Under previous assumptions, one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ε_n) *corresponding to* (u_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

Localisation principle - final generalisation

Theorem. *Take* $\varepsilon_n > 0$ *bounded,* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *and*

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *uniformly on compact sets, and* $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *satisfies* ($\mathrm{C}(\varepsilon_n)$).
Then for $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]$, *the corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0} \,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = 0 \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = \infty \end{cases}$$

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *we can take*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

∎

Localisation principle - final generalisation

Theorem. *Take $\varepsilon_n > 0$ bounded,* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *and*

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *uniformly on compact sets, and* $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *satisfies* ($\mathrm{C}(\varepsilon_n)$).
Then for $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]$, *the corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0} \,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = 0 \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = \infty \end{cases}$$

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *we can take*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

∎

As a corollary from the previous theorem we can derive localisation principles for H-measures and semiclassical measures.

Thank you for your attention.