Friedrichs systems with complex coefficients

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SISSA, Trieste, 27th October 2016

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Joint work with Krešimir Burazin, Ivana Crnjac, Marko Erceg and Marko Vrdoljak

Friedrichs' system (KOF1958)

Assumptions:
$d, r \in \mathbf{N}$, $\Omega \subseteq \mathbf{R}^d$ open and bounded with Lipschitz boundary Γ;

$\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^{\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$

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(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad (\text{ae on } \Omega)\,.$$

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The operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{C}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{C}^r)$

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

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Symmetric hyperbolic systems (KOF1954)

$$\sum_{k=1}^{d} \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f}$$

In divergence form:

$$\sum_{k=1}^{d} \partial_k (\mathbf{A}^k \mathsf{u}) + (\mathbf{B} - \partial_k \mathbf{A}^k)\mathsf{u} = \mathsf{f}$$

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It is symmetric if all matrices $\mathbf{A}^k$ are real and symmetric; and uniformly hyperbolic if there is a $\boldsymbol{\xi} \in \mathbf{R}^d$ such that for any $\mathbf{x} \in \mathrm{Cl}\,\Omega$ the matrix $\xi_k \mathbf{A}^k(\mathbf{x})$ is positive definite.

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Such systems can easily be transformed into Friedrichs' systems.

It is known that the wave equation and the Maxwell system can be written as an equivalent hyperbolic system.

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Goals:

– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

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The development of theory is nowadays mostly motivated by the needs in development of numerical methods.

An example – scalar elliptic equation

$\Omega \subseteq \mathbf{R}^2$, $\mu > 0$ and $f \in \mathrm{L}^2(\Omega)$ given.

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which is a Friedrichs system with the choice of

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Example – heat equation

... with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \operatorname{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ u = 0 \text{ on } \langle 0, T\rangle \times \Gamma \\ u(0,\cdot) = 0 \text{ on } \Omega \end{cases}$$

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...as a Friedrichs system:

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$$\begin{bmatrix} \mathbf{0} & 0 \\ 0^\top & 1 \end{bmatrix} \partial_t \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} + \sum_{i=1}^{d} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 1 \\ \vdots & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 1 \end{bmatrix} \partial_{x^i} \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} + \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ -(\mathbf{A}^{-1}\mathsf{b})^\top & c \end{bmatrix} \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

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The condition (F1) holds. The positivity condition $\mathbf{C} + \mathbf{C}^\top \geqslant 2\mu_0\mathbf{I}$ is fulfilled if and only if $c - \frac{1}{4}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b}$ is uniformly positive.

Boundary conditions

Boundary conditions are enforced via matrix valued boundary field:

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Boundary condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_{\Gamma}} = 0$$

is sufficient for treatment of different types of usual boundary conditions.

Assumptions on boundary matrix $\mathbf{M}$

We assume (for ae $\mathbf{x} \in \Gamma$) [KOF1958]

(FM1) $$(\forall\, \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

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Such $\mathbf{M}$ is called *the admissible boundary condition*.

The boundary problem: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{C}^r)$ find u such that

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases}.$$

Elliptic equation – different boundary conditions

$$\mathbf{M} \qquad\qquad \mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} \qquad\qquad (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix}_{|\Gamma} = 0$$

$$\begin{bmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 2\nu_1 \\ 0 & 0 & 2\nu_2 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_{|\Gamma} = 0$$

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All above matrices $\mathbf{M}$ satisfy (FM).

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Different ways to enforce boundary conditions

Instead of

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u} = 0 \quad \text{on } \Gamma\,,$$

Lax proposed boundary conditions with

$$\mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma\,,$$

where $N = \{N(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ is a family of subspaces of $\mathbf{C}^r$.

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$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ \mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma \end{cases}.$$

Assumptions on N

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$) [PDL]

(FX1) $N(\mathbf{x})$ is non-negative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$:

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,;$$

(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which (properly) contains $N(\mathbf{x})\,;$

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or [RSP&LS1966]

Let $N(\mathbf{x})$ and $\tilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ satisfy (for ae $\mathbf{x} \in \Gamma$)

(FV1)

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0$$

$$(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0$$

(FV2)

$$\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}\,.$$

Equivalence of different descriptions of boundary conditions

Theorem. *It holds*

$$\text{(FM1)–(FM2)} \quad \Longleftrightarrow \quad \text{(FX1)–(FX2)} \quad \Longleftrightarrow \quad \text{(FV1)–(FV2)},$$

with

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In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].

Classical results on well-posedness

Friedrichs:
– uniqueness of the classical solution
– existence of a *weak* solution (under some additional assumptions)

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– uniqueness of the classical solution
– existence of a *weak* solution (under some additional assumptions)

Contributions:
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...

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Contributions:
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, . . .
– the meaning of traces for functions in the graph space
– weak well-posedness results under additional assumptions (on $\mathbf{A}_{\boldsymbol{\nu}}$)
– regularity of solution
– numerical treatment

New approach...

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Diff. Eq.* **32** (2007) 317–341.

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. . . and new open questions.

They considered only the real case.

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L — real (complex) Hilbert space (L' is (anti)dual of L),
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(T3) $$(\exists\, \mu_0 > 0)(\forall\, \varphi \in \mathcal{D}) \quad \langle\, (T + \tilde{T})\varphi \mid \varphi\,\rangle_L \geqslant 2\mu_0\|\varphi\|_L^2\,.$$

The Friedrichs operator

Let $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{C}^r)$ and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ be defined by

$$T\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,,$$

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...fits in this framework.

Prolongations

$(\mathcal{D}, \langle \cdot \mid \cdot \rangle_T)$ is an inner product space, where

$$\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle_L + \langle T\cdot \mid T\cdot \rangle_L \,.$$

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Therefore $T = \tilde{T}^*_{|W_0}$, and analogously $\tilde{T} = T^*_{|W_0}$.
Abusing notation: $T, \tilde{T} \in \mathcal{L}(L; W_0')$ $\ldots$ (T1)–(T3)

Formulation of the problem

Lemma. *The graph space*

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\},$$

is a Hilbert space with respect to $\langle \cdot \mid \cdot \rangle_T$*.*

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Problem: for given $f \in L$ find $u \in W$ such that $Tu = f$.

Find sufficient conditions on $V \leqslant W$ such that $T_{|_V} : V \longrightarrow L$ is an isomorphism.

Boundary operator

Boundary operator $D \in \mathcal{L}(W; W')$:

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Lemma. D *is selfadjoint*

$$ {}_{W'}\langle\, Du, v \,\rangle_W = \overline{{}_{W'}\langle\, Dv, u \,\rangle_W} $$

and satisfies

$$ \begin{aligned} \ker D &= W_0 \\ \operatorname{im} D &= W_0^0 := \{ g \in W' : (\forall\, u \in W_0) \quad {}_{W'}\langle\, g, u \,\rangle_W = 0 \} \,. \end{aligned} $$

In particular, $\operatorname{im} D$ *is closed in* W'*.* ∎

For classical Friedrichs operator

If T is the Friedrichs operator $\mathcal{L}$, then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ we have

$$ {}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,. $$

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With the assumptions:

(FV1)
$$ \begin{aligned} (\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) &\quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,, \\ (\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) &\quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0\,, \end{aligned} $$

(FV2) $\quad \tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^\perp$ and $N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^\perp\,,$

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we are lead to consider subspaces V and $\tilde{V}$ in the functional framework:

(V1)
$$\begin{aligned} &(\forall u \in V) \qquad {}_{W'}\langle Du, u \rangle_W \geqslant 0 \,, \\ &(\forall v \in \tilde{V}) \qquad {}_{W'}\langle Dv, v \rangle_W \leqslant 0 \,, \end{aligned}$$

(V2)
$$V = D(\tilde{V})^0 \,, \qquad \tilde{V} = D(V)^0 \,.$$

Well-posedness theorem

$$[\, u \mid v \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,, \qquad u, v \in W$$

is an indefinite inner product on W, and we consider subspaces V and $\tilde{V}$ satisfying:

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Theorem. *Under assumptions* $(T1) - (T3)$ *and* $(V1) - (V2)$, *the operators* $T_{|_V} : V \longrightarrow L$ *and* $\tilde{T}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$ *are isomorphisms.* ∎

In the real case [AE&JLG&GC2007].

Correspondence — maximal b.c.

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$)

(FX1) $$(\forall \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

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subspace V is maximal non-negative in $(W, [\,\cdot \mid \cdot\,])$:

(X1) V is non-negative in $(W, [\,\cdot \mid \cdot\,])$: $\quad (\forall\, v \in V) \quad [\, v \mid v\,] \geqslant 0\,,$

(X2) there is no non-negative subspace in $(W, [\,\cdot \mid \cdot\,])$ containing V .

Correspondence — admissible b.c.

admissible boundary condition: there exists a matrix function $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{C})$ such that (for ae $\mathbf{x} \in \Gamma$)

(FM1) $$(\forall \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FM2) $$\mathbf{C}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,.$$

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abstract admissible boundary condition: there exists $M \in \mathcal{L}(W; W')$ such that

(M1) $$(\forall\, u \in W) \quad {}_{W'}\langle\, (M + M^*)u, u\,\rangle_W \geqslant 0\,,$$

(M2) $$W = \ker(D - M) + \ker(D + M)\,.$$

Equivalence of different descriptions of b.c.

Theorem. *(**classical**)* *It holds*

$$\text{(FM1)–(FM2)} \quad \Longleftrightarrow \quad \text{(FV1)–(FV2)} \quad \Longleftrightarrow \quad \text{(FX1)–(FX2)}\,,$$

with

$$N(\mathbf{x}) := \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big)\,.$$

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Theorem. *(**A. Ern, J.-L. Guermond, G. Caplain**)* *It holds*

$$\textit{(M1)–(M2)} \quad \begin{smallmatrix}\Longrightarrow \\ \longleftarrow\end{smallmatrix} \quad \textit{(V1)–(V2)} \quad \Longrightarrow \quad \textit{(X1)–(X2)},$$

with

$$V := \ker(D - M).$$

∎

This was obtained in the real case only.

(M1)–(M2) $\longleftarrow$ (V1)–(V2)

Theorem. *Let V and $\tilde{V}$ satisfy (V1)–(V2), and suppose that there exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$ such that*

$$\begin{aligned}(\forall\, v \in V) \quad & D(v - Pv) = 0\,,\\ (\forall\, v \in \tilde{V}) \quad & D(v - Qv) = 0\,,\\ & DPQ = DQP\,.\end{aligned}$$

Let us define $M \in \mathcal{L}(W;W')$ (for $u, v \in W$) with

$$\begin{aligned}{}_{W'}\langle\, Mu, v\,\rangle_W &= {}_{W'}\langle\, DPu, Pv\,\rangle_W - {}_{W'}\langle\, DQu, Qv\,\rangle_W\\ &\quad + {}_{W'}\langle\, D(P+Q-PQ)u, v\,\rangle_W - {}_{W'}\langle\, Du, (P+Q-PQ)v\,\rangle_W\,.\end{aligned}$$

Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and M satisfies (M1)–(M2). ∎

Gramm operator

Graph space $(W, \langle \cdot \mid \cdot \rangle_T)$ is a Hilbert space, where another (indefinite) inner product is defined:

$$[\, u \mid v \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,, \qquad u, v \in W \;.$$

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This inner product is dominated by the graph norm, which insures the existence of a linear operator $G \in \mathcal{L}(W; W)$ such that

$$[\, u \mid v \,] = \langle\, Gu \mid v \,\rangle_T \qquad \text{and} \qquad \langle\, Gu \mid v \,\rangle_T = \langle\, u \mid Gv \,\rangle_T \,.$$

Such an operator is called the *Gramm operator*.

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Inner product space is a *Kreĭn space* if it admits an orthogonal decomposition to its nonnegative and nonpositive parts, which are complete.

Gramm operator

Graph space $(W, \langle \cdot \mid \cdot \rangle_T)$ is a Hilbert space, where another (indefinite) inner product is defined:

$$[\, u \mid v \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,, \qquad u, v \in W \;.$$

This inner product is dominated by the graph norm, which insures the existence of a linear operator $G \in \mathcal{L}(W; W)$ such that

$$[\, u \mid v \,] = \langle\, Gu \mid v \,\rangle_T \qquad \text{and} \qquad \langle\, Gu \mid v \,\rangle_T = \langle\, u \mid Gv \,\rangle_T \;.$$

Such an operator is called the *Gramm operator*.

Inner product space is a *Kreĭn space* if it admits an orthogonal decomposition to its nonnegative and nonpositive parts, which are complete.

Equivalently, a Hilbert space is a Kreĭn space if its Gramm operator is invertible.

Kreĭn spaces

$(W, [\cdot \mid \cdot])$ is not a Kreĭn space – it is a degenerate space, because its Gramm operator $G := j \circ D \quad (j : W' \longrightarrow W$ is the canonical isomorphism$)$ has large kernel:

$$\ker G = W_0 \, .$$

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Theorem. *If G is the Gramm operator of the space W, then the quotient space $\hat{W} := W/\ker G$ is a Kreĭn space if and only if $\operatorname{im} G$ is closed.* ∎

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$\hat{W} := W/W_0$ is the Kreĭn space, with

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Important: $\operatorname{im} D$ is closed and $\ker D = W_0$.

Quotient Kreĭn space

Lemma. *Let $U \supseteq W_0$ and Y be subspaces of W. Then*

a) U is closed if and only if $\hat{U} := \{\hat{v} : v \in U\}$ is closed in $\hat{W}$;

b) $(\widehat{U+Y}) = \{u + v + W_0 : u \in U, v \in Y\} = \hat{U} + \hat{Y}$;

c) $U + Y$ is closed if and only if $\hat{U} + \hat{Y}$ is closed;

d) $(\hat{Y})^{[\perp]^{\hat{}}} = \widehat{Y^{[\perp]}}$.

e) if Y is maximal non-negative (non-positive) in W, than $\hat{Y}$ is maximal non-negative (non-positive) in $\hat{W}$;

f) if $\hat{U}$ is maximal non-negative (non-positive) in $\hat{W}$, then U is maximal non-negative (non-positive) in W. ∎

(V1)–(V2) $\iff$ (X1)–(X2)

Theorem. *a) If subspaces V and $\tilde{V}$ satisfy (V1)–(V2), then V is maximal non-negative in W (satisfies (X1)–(X2)) and $\tilde{V}$ is maximal non-positive in W.*

b) If V is maximal non-negative in W, then V and $\tilde{V} := V^{[\perp]}$ satisfy (V1)–(V2). ∎

(M1)–(M2) $\Longrightarrow$ (V1)–(V2) (recall)

Theorem. *[EGC] (T1)–(T3) and $M \in \mathcal{L}(W; W')$ satisfy (M) imply*

$$V := \ker(D - M) \quad \textit{and} \quad \tilde{V} := \ker(D + M^*) \quad \textit{satisfy (V).}$$

∎

Corollary. *Under above assumptions*

$$T_{|_{\ker(D-M)}} : \ker(D - M) \longrightarrow L \qquad i \qquad \tilde{T}_{|_{\ker(D+M^*)}} : \ker(D + M^*) \longrightarrow L$$

are isomorphisms.

∎

(M1)–(M2) $\longleftarrow$ (V1)–(V2) (recall)

Theorem. *Let V and $\tilde V$ satisfy (V1)–(V2), and suppose that there exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde V)$ such that*

$$\begin{aligned} (\forall\, v \in V) \quad & D(v - Pv) = 0\,, \\ (\forall\, v \in \tilde V) \quad & D(v - Qv) = 0\,, \\ & DPQ = DQP\,. \end{aligned}$$

Let us define $M \in \mathcal{L}(W;W')$ (for $u, v \in W$) with

$$\begin{aligned} {}_{W'}\langle\, Mu, v\,\rangle_W = {}& {}_{W'}\langle\, DPu, Pv\,\rangle_W - {}_{W'}\langle\, DQu, Qv\,\rangle_W \\ & + {}_{W'}\langle\, D(P+Q-PQ)u, v\,\rangle_W - {}_{W'}\langle\, Du, (P+Q-PQ)v\,\rangle_W\,. \end{aligned}$$

Then $V := \ker(D - M)$, $\tilde V := \ker(D + M^*)$, and M satisfies (M1)–(M2). ∎

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (direct proof)

Theorem. *If $V, \tilde{V}$ are two closed subspaces of W that satisfy $W_0 \subseteq V \cap \tilde{V}$, then the following statements are equivalent:*
a) There exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$, such that

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b) There exist projectors $P', Q' \in \mathcal{L}(W;W)$*, such that*

$$\begin{gathered} {P'}^2 = P' \quad \text{and} \quad {Q'}^2 = Q'\,, \\ \operatorname{im} P' = V \quad \text{and} \quad \operatorname{im} Q' = \tilde{V}\,, \\ P'Q' = Q'P'\,. \end{gathered}$$

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∎

(b) is equivalent to closedness of $V + \tilde{V}$.

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

Theorem.
a) $V, \tilde{V} \leqslant W$ satisfy (V), and exists a closed subspace $W_2 \subseteq C^-$ of W, $V \dot{+} W_2 = W$, then there exist an operator $M \in \mathcal{L}(W; W')$ satisfying (M) and $V = \ker(D - M)$.

If we define W_1 as orthogonal complement of W_0 in V, so that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and denote by R_1, R_0, R_2 projectors that correspond to above direct sum, then one such operator is given with $M = D(R_1 - R_2)$.

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b) $M \in \mathcal{L}(W; W')$ *an operator satisfying (M1)–(M2),* $V := \ker(D - M)$*.*

For W_2*, the orthogonal complement of* W_0 *in* $\ker(D + M)$*,* $W_2 \subseteq C^-$ *is closed,* $V \dot{+} W_2 = W$*, and* M *coincide with the operator in (a).* ∎

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

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Lemma. *Let* $W_2'' \leqslant W$ *satisfies* $W_2'' \subseteq C^-$ *and* $W_2'' + V = W$*.*
Then there is a closed subspace W_2 *of* W*, such that* $W_2 \subseteq C^-$ *and* $W_2 \dot{+} V = W$*.* ∎

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

Lemma. *If* $U_1 + U_2 = W$ *for some subspaces* $U_1 \subseteq C^+$ *and* $U_2 \subseteq C^-$ *of* W*, then* $U_1 \cap U_2 \subseteq W_0$*.*
If additionally U_1 *is maximal nonnegative and* U_2 *maximal nonpositive, then* $U_1 \cap U_2 = W_0$*.* ∎

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Theorem. *For a maximal nonnegative subspace V of W, it is equivalent:*
a) There is a maximal nonpositive subspace W_2 of W, such that $W_2 + V = W$;
b) There is a nonpositive subspace $W_{\hat{2}}$ of $\hat{W}$, such that $W_{\hat{2}} + \hat{V} = \hat{W}$. ∎

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Corollary. *The conditions (V) and (M) are equivalent.* ∎

Scalar elliptic equation

Consider

$$-\mathrm{div}\,(\mathbf{A}\nabla u) + cu = f$$

in $\Omega \subseteq \mathbf{R}^d$,

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$$\mathbf{A} \in \mathcal{M}_d(\alpha', \beta'; \Omega) := \Big\{ \mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R})) :$$
$$(\forall \boldsymbol{\xi} \in \mathbf{R}^d)\ \mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \alpha' |\boldsymbol{\xi}|^2 \ \&\ \mathbf{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \frac{1}{\beta'} |\mathbf{A}\boldsymbol{\xi}|^2 \Big\}$$

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New unknown vector function taking values in $\mathbf{R}^{d+1}$:

$$\mathsf{u} = \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}\nabla_{\mathbf{x}} u \\ u \end{bmatrix} .$$

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Then the starting equation can be written as a first-order system

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1} \mathsf{u}_d = 0 \\ \mathrm{div}\,\mathsf{u}_d + c u_{d+1} = f \end{cases} ,$$

Scalar elliptic equation (cont.)

which is a Friedrichs system with the choice of

$$\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R}), \qquad \mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0 & c \end{bmatrix}.$$

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The graph space: $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$.

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The graph space: $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$.
Dirichlet, Neumann and Robin boundary conditions are imposed by the following choice of V and $\widetilde{V}$:

$$\begin{aligned} V_D = \widetilde{V}_D &:= \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega) , \\ V_N = \widetilde{V}_N &:= \{ (\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = 0 \} , \\ V_R &:= \{ (\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = a {u_{d+1}}_{|\Gamma} \} , \\ \widetilde{V}_R &:= \{ (\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = -a {u_{d+1}}_{|\Gamma} \} . \end{aligned}$$

Heat equation

. . . with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \operatorname{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ u = 0 \text{ on } \langle 0, T\rangle \times \Gamma \\ u(0,\cdot) = 0 \text{ on } \Omega \end{cases}$$

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...as a Friedrichs system:

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}}\mathsf{u}_d + cu_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d = f \end{cases},$$

(note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$).

Friedrichs operator and the graph space

The operator T is given by

$$T \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1} \mathsf{u}_d \\ \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}} \mathsf{u}_d + c u_{d+1} - \mathbf{A}^{-1} \mathsf{b} \cdot \mathsf{u}_d \end{bmatrix},$$

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while the corresponding graph space is

$$\begin{aligned} W &= \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1}) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \\ &\qquad\qquad \& \quad \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(\Omega_T) \Big\} \\ &= \Big\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \Big\} \\ &= \Big\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \Big\}. \end{aligned}$$

Lemma. *The projection* $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top \mapsto u_{d+1}$ *is a continuous linear operator from* W *to* $W(0,T)$*, which is continuously embedded to* $\mathrm{C}([0,T];\mathrm{L}^2(\Omega))$.

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Properties of the last component

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The space

$$W(0,T) = \left\{u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) : \partial_t u \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))\right\},$$

is a Banach space when equipped by norm

$$\|\mathsf{u}\|_{W(0,T)} = \sqrt{\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\partial_t u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))}}.$$

Finally

Let

$$\begin{aligned} V &= \left\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. on } \Omega \right\}, \\ \widetilde{V} &= \left\{ \mathsf{v} \in W : v_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v_{d+1}(\cdot,T) = 0 \text{ a.e. on } \Omega \right\}. \end{aligned}$$

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\end{aligned}
$$

Do they satisfy (V1)–(V2)? Technical...

Theorem

The above V and $\widetilde{V}$ satisfy (V1)–(V2), and therefore the operator $T_{|_V} : V \longrightarrow L$ is an isomorphism.

Two-field theory...

Heat equation with b = 0 and $c = 0$:

$$\begin{cases} \partial_t u - \mathrm{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) = f \text{ in } \Omega_T \\ u = 0 \text{ on } \Gamma \times \langle 0, T\rangle \\ u(\cdot, 0) = 0 \text{ on } \Omega \end{cases}$$

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Two field theory:

developed by Ern and Guermond for elliptic problems

Two-field theory...

Heat equation with $\mathsf{b} = 0$ and $c = 0$:

$$\begin{cases} \partial_t u - \operatorname{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) = f \text{ in } \Omega_T \\ u = 0 \text{ on } \Gamma \times \langle 0, T\rangle \\ u(\cdot, 0) = 0 \text{ on } \Omega \end{cases}$$

Two field theory:

developed by Ern and Guermond for elliptic problems

matrices need to be of the form

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{0} & \mathsf{B}^k \\ (\mathsf{B}^k)^\top & a^k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix},$$

where $\mathsf{B}^k \in \mathbf{R}^d$ are constant vectors, $a^k \in \mathrm{W}^{1,\infty}(\Omega_T)$, $\mathbf{C}^d \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$ and $c^{d+1} \in \mathrm{L}^\infty(\Omega_T)$, $k \in 1..(d+1)$.

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For the heat equation matrices have this form!

. . . with partial coercivity

Instead of coercivity (positivity) condition (F2), the following is required:

$$(\exists\, \mu_1 > 0)(\forall\, \boldsymbol{\xi} = (\boldsymbol{\xi}_d, \xi_{d+1}) \in \mathbf{R}^{d+1})$$

$$\Big(\mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d+1} \partial_k \mathbf{A}_k\Big)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 2\mu_1 |\boldsymbol{\xi}_d|^2 \qquad \text{(a.e. on } \Omega\text{)}\,,$$

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$$\sqrt{\langle\, \mathcal{L}\mathsf{u} \mid \mathsf{u}\,\rangle_{\mathrm{L}^2(\Omega_T;\mathbf{R}^{d+1})}} + \|\mathsf{B}u_{d+1}\|_{\mathrm{L}^2(\Omega_T;\mathbf{R}^d)} \geqslant \mu_2 \|u_{d+1}\|_{\mathrm{L}^2(\Omega_T)}\,,$$

where $\mathsf{B}u_{d+1} := \sum_{k=1}^{d+1} \mathsf{B}^k \partial_k u_{d+1} = \nabla_{\mathbf{x}} u_{d+1}$.

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For our system both conditions are trivially fulfilled.

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For our system both conditions are trivially fulfilled.

Therefore, we have the well-posedness result.

Some further applications ...

Dirac system
Maxwell system

Open problems . . .

- Find all representations of a particular equation in the form of a Friedrichs system.
- Application to other equations of practical importance (mixed-type problems).
- Compare the results to those already known in the classical setting.

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Some used properties

Theorem. *a) $[\cdot \mid \cdot]$-orthogonal complement of a maximal non-negative (non-positive) subspace is non-positive (non-negative).*

b) Each maximal semi-definite subspace contains all isotropic vectors in W.

c) If L is a non-negative (non-positive) subspace of a Krein space, such that $L^{[\perp]}$ is non-positive (non-negative), then $\mathrm{Cl}\, L$ is maximal non-negative (non-positive).

d) Each maximal semi-definite subspace of a Krein space is closed.

e) A subspace L of a Krein space is closed if and only if $L = L^{[\perp][\perp]}$.

f) For a subspace L of a Krein space W it holds

$$L \cap L^{[\perp]} = \{0\} \qquad \Longleftrightarrow \qquad \mathrm{Cl}\,(L + L^{[\perp]}) = W\,.$$

∎