H-distributions in various settings

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Joint work with Marko Erceg, Ivan Ivec, Marin Mišur and Darko Mitrović

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Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\boldsymbol{\xi}) \longrightarrow 0$ pointwise.
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If $\varphi u_n \not\rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then $\widehat{\varphi u_n} \not\rightharpoonup 0$; some information must go to infinity.

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How does it go to infinity in various directions? Take $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, and consider:

$$\lim_n \int_{\mathbf{R}^d} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|) |\widehat{\varphi u_n}|^2 d\boldsymbol{\xi} = \int_{\mathrm{S}^{d-1}} \psi(\boldsymbol{\xi}) d\nu_\varphi(\boldsymbol{\xi}) \,.$$

The limit is a linear functional in ψ, thus an integral over the sphere of some nonegativne Radon measure (a bounded sequence of Radon measures has an accumulation point), which depends on φ. How does it depent on φ?

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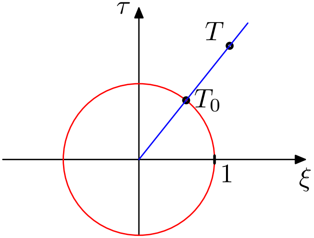

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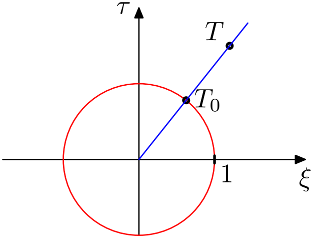

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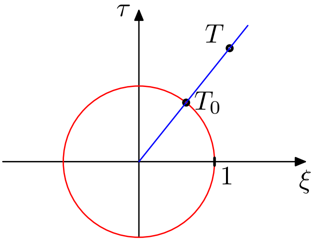

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$$S^1 \ldots \ r^2(\tau, \xi) := \tau^2 + \xi^2 = 1$$

and projection $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ onto the curve (surface):

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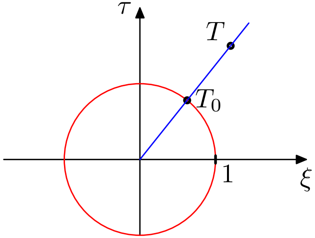

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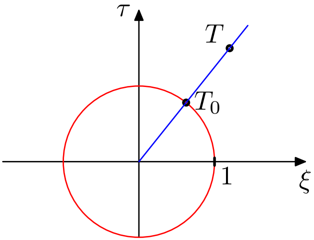

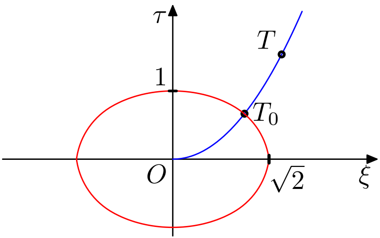

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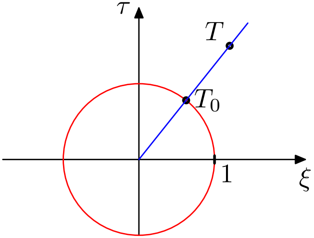

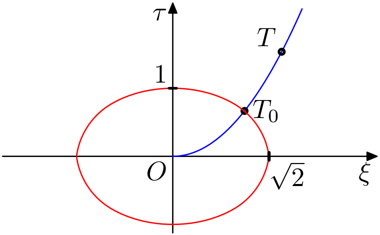

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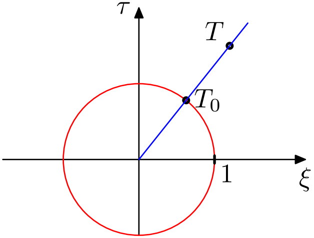

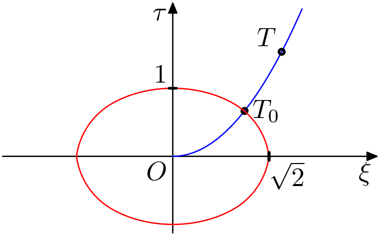

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Now we can state the main theorem, where we use the notation

$$\mathsf{v}\cdot\mathsf{u} := \sum v_i \bar{u}_i \ , \quad (\mathsf{v}\otimes\mathsf{u})\mathsf{a} := (\mathsf{a}\cdot\mathsf{u})\mathsf{v} \text{ , while } \quad (f \boxtimes g)(\mathbf{x},\boldsymbol{\xi}) := f(\mathbf{x})g(\boldsymbol{\xi}) \ .$$

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there exists its subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times \mathrm{S}^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and

$$\psi \in \mathrm{C}(\mathrm{S}^{d-1})$$

one has

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There are some other variants (E. Ju. Panov, ...).

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Lemma. (general form of the first commutation lemma — Luc Tartar)
If $b \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfy the condition*

$$(\forall\, \rho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\text{a.e. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \rho)) \,,$$

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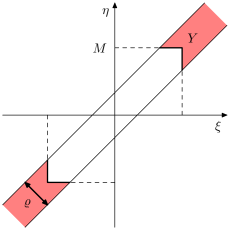
For given $M, \rho \in \mathbf{R}^+$ denote the set

$$Y = Y(M, \rho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M \ \&\ |\boldsymbol{\xi} - \boldsymbol{\eta}| \leqslant \rho\} \ .$$



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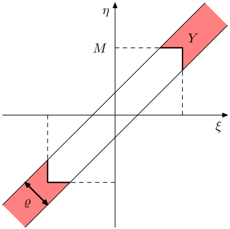
then $C := [\mathcal{A}_a, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$*.* ∎

For given $M, \rho \in \mathbf{R}^+$ denote the set

$$Y = Y(M, \rho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M \;\&\; |\boldsymbol{\xi} - \boldsymbol{\eta}| \leqslant \rho\} \,.$$



In both cases discussed above, this lemma can also be proven directly, based on elementary inequalities.

The importance of First commutation lemma

If we take $\mathsf{u}_n = (u_n, v_n)$, and consider $\mu = \mu_{12}$, we have

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\begin{aligned}
\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} \psi \, d\boldsymbol{\xi} &= \lim_{n'} \langle \mathcal{A}_\psi(\varphi_1 u_{n'}) | \varphi_2 v_{n'} \rangle \\
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Thus the limit is a bilinear functional in $\varphi_1 \bar{\varphi}_2$ and ψ, and we have the bound:

$$\Big| \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} d\mathbf{x} \Big| \leq C \|\psi\|_{C(S^{d-1})} \|\varphi_1 \overline{\varphi_2}\|_{C_0(\mathbf{R}^d)} \, .$$

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This form makes sense even for $p < 2$ (for $p > 2$ we use the fact that $u_n \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$).

Localisation principle for classical H-measures

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}\ ) + \mathbf{Bu} \ = \mathsf{f} \quad , \qquad \mathbf{A}^k \in \mathrm{C}_b(\mathbf{R}^d; \mathrm{M}_{l\times r})$$

Assume:

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0 \ , \qquad \text{and defines } \boldsymbol{\mu}$$

$$\mathsf{f}_n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \ .$$

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Theorem. (localisation principle) If u_n satisfies:

$$\sum_{k=1}^{d} \partial_k\big(\mathbf{A}^k \mathsf{u}_n\big) \longrightarrow 0 \qquad \text{in } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{C}^r) \ ,$$

then for $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{k=1}^{d} \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}^\top = \mathbf{0} \ .$$

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Thus, if $l = r$, the support of H-measure $\boldsymbol{\mu}$ is contaned in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{p}$ is a singular matrix.

The localisation principle is behind the applications to the small-amplitude homogenisation, which can be used in optimal design.

It contains a generalisation of compactness by compensation to variable coefficients.

Localisation principle for parabolic H-measures

In the parabolic case the details become more involved.

Anisotropic Sobolev spaces ($s \in \mathbf{R}$; $k_p(\tau, \boldsymbol{\xi}) := \sqrt[4]{1 + (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}$)

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{S}' : k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d}) \right\} .$$

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Theorem. **(localisation principle)** Let $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$, uniformly compactly supported in t, satisfy ($s \in \mathbf{N}$)

$$\sqrt{\partial_t}^{\,s}(\mathbf{A}^0 \mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow 0 \quad \text{strongly in} \quad \mathrm{H}_{\mathrm{loc}}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d}) ,$$

where $\mathbf{A}^0, \mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}_b(\mathbf{R}^{1+d}; \mathrm{M}_{l\times r}(\mathbf{C}))$, for some $l \in \mathbf{N}$, while $\sqrt{\partial_t}$ is a pseudodifferential operator with symbol $\sqrt{2\pi i\tau}$, i.e.

$$\sqrt{\partial_t} u = \overline{\mathcal{F}}\left(\sqrt{2\pi i\tau}\, \hat{u}(\tau) \right) .$$

Then for a parabolic H-measure $\boldsymbol{\mu}$ associated to (a sub)sequence (of) (u_n) one has

$$\left((\sqrt{2\pi i\tau})^s \mathbf{A}^0 + \sum_{|\boldsymbol{\alpha}|=s} (2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}} \right) \boldsymbol{\mu}^\top = \mathbf{0}.$$

Good bounds: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)\ , \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

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Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|}\ ,$$

then for any $p \in \langle 1, \infty\rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty)\ .$$

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For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

The main theorem

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The H-distribution would correspond to a non-diagonal block for an H-measure.

The proof is based on First commutation lemma

$\psi \in \mathrm{C}^{\kappa}(\mathrm{S}^{d-1})$ satisfies the conditions of the Hörmander-Mihlin theorem.
Therefore, $\mathcal{A}_\psi$ and M_φ are bounded operators on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty\rangle$.
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Lemma. *Let (v_n) be bounded in both $\mathrm{L}^2(\mathbf{R}^d)$ and $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 2, \infty]$, and let $v_n \rightharpoonup 0$ in $\mathcal{D}'$. Then the sequence (Cv_n) strongly converges to zero in $\mathrm{L}^q(\mathbf{R}^d)$, for any $q \in [2, r] \setminus \{\infty\}$.* ∎

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$\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ satisfies the conditions of the Hörmander-Mihlin theorem. Therefore, $\mathcal{A}_\psi$ and M_φ are bounded operators on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty \rangle$. We are interested in the properties of their commutator, $C = \mathcal{A}_\psi M_\varphi - M_\varphi \mathcal{A}_\psi$.

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If $q < r$, we can apply the classical interpolation inequality:

$$\|Cv_n\|_q \leqslant \|Cv_n\|_2^\alpha \|Cv_n\|_r^{1-\alpha} \ ,$$

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We still need a lemma on *compactness* of uniformly bounded bilinear forms, and an application of the Schwartz kernel theorem.

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d \rangle$, *such that*

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Take an arbitrary (v_n) *bounded in* $\mathrm{L}^\infty(\mathbf{R}^d)$, *and by* μ *denote the* H*-distribution corresponding to a subsequence of* (u_n) *and* (v_n). *Then*

$$(\mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}) \mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}$ *being the symbol of the linear PDO with* C^κ_0 *coefficients.* ∎

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In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\boldsymbol{\xi}|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\frac{\xi_j}{i|\boldsymbol{\xi}|}}$.

Note that

$$\int I_1(\phi)\partial_j g = \int (R_j\phi)g, \quad g \in \mathcal{S}(\mathbf{R}^d).$$

Using the density argument and that R_j is bounded from $\mathrm{L}^p(\mathbf{R}^d)$ to itself, we conclude $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in \mathrm{L}^p(\mathbf{R}^d)$.

Compactness by compensation: L^2 case

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The prototype of this compensation effect is Murat-Tartar's div-rot lemma.

For simplicity consider 2D case, (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^2(\mathbf{R}^2)$, such that $(\partial_x u_n^1 + \partial_y u_n^2)$ and $(\partial_y v_n^1 - \partial_x v_n^2)$ are both contained in a compact set of $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$ (which then implies that they converge to zero strongly in $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$).

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We can define $\mathsf{U}_n := \begin{bmatrix} \mathsf{u}_n \\ \mathsf{v}_n \end{bmatrix}$, which (on a subsequence) defines a 4×4 H-measure $\boldsymbol{\mu}$. By the localisation principle, as the above relations can be written in the form ($\mathbf{A}^1, \mathbf{A}^2$ are 4×4 constant matrices with all entries zero except $A^1_{11} = A^2_{12} = A^2_{33} = 1$ and $A^1_{34} = -1$)

$$\mathbf{A}^1 \partial_1 \mathsf{U}_n + \mathbf{A}^2 \partial_2 \mathsf{U}_n \to 0 \text{ strongly in } \mathrm{H}^{-1}_{loc}(\mathbf{R}^2)^4 ,$$

the corresponding H-measure satisfies $(\xi_1 \mathbf{A}^1 + \xi_2 \mathbf{A}^2)\boldsymbol{\mu} = \mathbf{0}$. After straightforward calculations this shows that $u_n^1 v_n^1 + u_n^2 v_n^2 \longrightarrow 0$ weak $*$ in the sense of Radon measures (and therefore in the sense of distributions as well).

What for sequences in L^p?

For the above we have used only the non-diagonal blocks $\boldsymbol{\mu}_{12} = \boldsymbol{\mu}_{21}^*$ of

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix} ,$$

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Assume now (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^p(\mathbf{R}^2)$ and $\mathrm{L}^{p'}(\mathbf{R}^2)$, and $(\partial_1 u_n^1 + \partial_2 u_n^2)$ bounded in $\mathrm{L}^p(\mathbf{R}^2)$, while $(\partial_2 v_n^1 - \partial_1 v_n^2)$ in $\mathrm{L}^{p'}(\mathbf{R}^2)$ (thus precompact in $\mathrm{W}^{-1,p}_{loc}(\mathbf{R}^2)$, and $\mathrm{W}^{-1,p'}_{loc}(\mathbf{R}^2)$).

Then $(u_n^1 v_n^1 + u_n^2 v_n^2)$ is bounded in $\mathrm{L}^1(\mathbf{R}^2)$, so also in $\mathcal{M}_b$ (Radon measures), and by weak $*$ compactness it has a weakly converging subsequence. However, we can say more—the whole sequence converges to zero.

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Denote by μ^{ij} the H-distribution corresponding to (some sub)sequences (of) (u_n^1, u_n^2) and (v_n^1, v_n^2).
Since $(\partial_1 u_n^1 + \partial_2 u_n^2)$ is bounded in $\mathrm{L}^p(\mathbf{R}^2)$, and $(\partial_2 v_n^1 - \partial_1 v_n^2)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^2)$, they are weakly precompact, while the only possible limit is zero, so

$$\partial_1 u_n^1 + \partial_2 u_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^p , \qquad \text{and}$$

$$\partial_2 v_n^1 - \partial_1 v_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^{p'} .$$

From the compactness of the Riesz potential I_1 mentioned above, we conclude that for $\varphi \in \mathrm{C}_c(\mathbf{R}^2)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ the following limit holds in $\mathrm{L}^p(\mathbf{R}^2)$:

$$\mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_1}{|\boldsymbol{\xi}|}}(\varphi u_n^1) + \mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_2}{|\boldsymbol{\xi}|}}(\varphi u_n^2) = \mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\partial_1(\varphi u_n^1) + \partial_2(\varphi u_n^2)) \to 0 \ .$$

Multiplying it first by φv_n^1 and then by φv_n^2, integrating over $\mathbf{R}^2$ and passing to the limit, we conclude from the existence theorem that:

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$$w_n^j = \varphi \mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\varphi u_n^j) \in \mathrm{W}^{1,p'}(\mathbf{R}^d), \quad j = 1, 2.$$

From the last limits on the preceeding slide we get

$$\langle (\varphi v_n^1, -\varphi v_n^2), \nabla w_n^j \rangle = -\langle \mathsf{rot}\,(\varphi v_n^1, \varphi v_n^2), w_n^j \rangle \to 0 \ \text{ as } \ n \to \infty,$$

for $j = 1, 2$. Rewriting it in the integral formulation, we obtain again from the existence theorem:

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From the algebraic relations above, we can easily conclude

$$\xi_1\left(\mu^{11} + \mu^{22}\right) = 0 \;\text{ and }\; \xi_2\left(\mu^{11} + \mu^{22}\right) = 0,$$

implying that the distribution $\mu^{11} + \mu^{22}$ is supported on the set $\{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap P = \emptyset$, which implies $\mu^{11} + \mu^{22} \equiv 0$.
After inserting $\psi \equiv 1$ in the definition of H-distribution, we immediately reach the conclusion.

This proof is similar to the L^2 case, but it should be noted that we had used only a non-diagonal block of 4×4 H-measure, which corresponds to the only available 2×2 H-distribution.

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There is no reason to limit oneself to two dimensions; take (u_n) and (v_n) converging weakly to zero in $\mathrm{L}^p(\mathbf{R}^d)^d$ and $\mathrm{L}^{p'}(\mathbf{R}^d)^d$, and by $\boldsymbol{\mu}$ denote $d \times d$ matrix H-distribution corresponding to some chosen subsequences of (u_n) and (v_n).

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Theorem. *Let* (u_n) *and* (v_n) *be vector valued sequences converging to zero weakly in* $\mathrm{L}^p(\mathbf{R}^d)^d$ *and* $\mathrm{L}^{p'}(\mathbf{R}^d)^d$*, respectively. Assume the sequence* $(\operatorname{div} \mathsf{u}_n)$ *is bounded in* $\mathrm{L}^p(\mathbf{R}^d)$*, and the sequence* $(\operatorname{rot} \mathsf{v}_n)$ *is bounded in* $\mathrm{L}^{p'}(\mathbf{R}^d)^{d \times d}$*. Then the sequence* $(\mathsf{u}_n \cdot \mathsf{v}_n)$ *converges to zero in the sense of distributions (or vaguely in the sense of Radon measures).* ∎

Lebesgue spaces with mixed norm

For $\mathbf{p} \in [1, \infty\rangle^d$, by $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ denote the space of f on $\mathbf{R}^d$ with finite norm

$$\|f\|_{\mathbf{p}} = \left(\int_{\mathbf{R}} \cdots \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(x_1, \ldots, x_d)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} \cdots dx_d \right)^{1/p_d} .$$

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These spaces can be seen as vector-valued Lebesgue spaces in the sense

$$\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) = \mathrm{L}^{p_d}_{x_d}(\mathbf{R}; \mathrm{L}^{(p_1, \ldots, p_{d-1})}_{x_1, \ldots, x_{d-1}}(\mathbf{R}^{d-1})) .$$

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Theorem. *Let $m \in \mathrm{L}^\infty(\mathbf{R}^d \setminus \{0\})$ for some $A > 0$ and any $|\boldsymbol{\alpha}| \leqslant [\frac{d}{2}] + 1$*
(a) either Mihlin's condition $\quad |\partial^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}} m(\boldsymbol{\xi})| \leqslant A|\boldsymbol{\xi}|^{-|\boldsymbol{\alpha}|}$ $\quad$ *or*
(b) Hörmander's condition

$$\sup_{R>0} R^{-d+2|\boldsymbol{\alpha}|} \int_{R<|\boldsymbol{\xi}|<2R} |\partial^{\boldsymbol{\alpha}}_{\boldsymbol{\xi}} m(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leqslant A^2 < \infty .$$

Then m lies in $\mathcal{M}_{\mathbf{p}}$, for any $\mathbf{p} \in \langle 1,\infty\rangle^d$, and we have the estimate

$$\begin{aligned} \|m\|_{\mathcal{M}_{\mathbf{p}}} &\leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max\{p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}}\}(A + \|m\|_{\mathrm{L}^\infty}) \\ &\leqslant c' \prod_{j=0}^{d-1} \max\{p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}}\}(A + \|m\|_{\mathrm{L}^\infty}) , \end{aligned}$$

where c and c' are constants that depend only on d. ∎

Lemma. *Let* (v_n) *be bounded both in* $\mathrm{L}^2(\mathbf{R}^d)$ *and in* $\mathrm{L}^{\mathbf{r}}(\mathbf{R}^d)$*, for some* $\mathbf{r} \in [2, \infty]^d$*, and such that* $v_n \longrightarrow 0$ *in* $\mathcal{D}'$*. Then* (Cv_n)*, where the commutator is defined by* $C := \mathcal{A}_\psi M_\varphi - M_\varphi \mathcal{A}_\psi$*, strongly converges to zero in* $\mathrm{L}^{\mathbf{q}}(\mathbf{R}^d)$*, for any* $\mathbf{q} \in [2, \infty\rangle^d$ *such that there exists* $\lambda \in \langle 0, 1\rangle$ *for which it holds*

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∎

Theorem. *Let* $\kappa = [d/2] + 1$ *and* $\mathbf{p} \in \langle 1, \infty\rangle^d$*. If* $u_n \longrightarrow 0$ *weakly in* $\mathrm{L}^{\mathbf{p}}_{\mathrm{loc}}(\mathbf{R}^d)$*,* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^{\mathbf{q}}_{\mathrm{loc}}(\mathbf{R}^d)$*, for some* $\mathbf{q} \in [2, \infty]^d$ *such that* $\mathbf{q} > \mathbf{p}'$*, then there exist subsequences* $(u_{n'})$ *and* $(v_{n'})$ *and a complex distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times S^{d-1})$*, such that for* $\phi_1, \phi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(S^{d-1})$ *one has*

$$\begin{aligned}\lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\Big\langle \mathcal{A}_\psi(\phi_1 u_{n'}), \phi_2 v_{n'} \Big\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)} &= \lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\Big\langle \phi_1 u_{n'}, \mathcal{A}_{\overline{\psi}}(\phi_2 v_{n'}) \Big\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)} \\ &= \langle \mu, \overline{\phi}_1 \phi_2 \boxtimes \overline{\psi}\rangle\,,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \longrightarrow \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ *is the Fourier multiplier operator.* ∎

Velocity averaging

A sequence of solutions to some (fractional order) PDE

$$\sum_{k=1}^{d} \partial_{x_k}^{\alpha_k}(a_k(\mathbf{x},\mathbf{v})u_n(\mathbf{x},\mathbf{v})) = g_n(\mathbf{x},\mathbf{v}) \ ,$$

is often only weakly convergent in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^{d+m})$. Sometimes it is sufficient to have strong precompactness only of the averaged sequence; for some $\rho \in \mathrm{C}_c(\mathbf{R}^m)$:

$$\int_{\mathbf{R}^m} \rho(\mathbf{v})u_n(\mathbf{x},\mathbf{v})\,d\mathbf{v} \ .$$

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In abstract terms, take E a separable Banach space, and $p \in \langle 1, \infty\rangle$.

Theorem. [M. Lazar & D. Mitrović (CRAS, 2013)]
A continuous bilinear functional $B : \mathrm{L}^p(\mathbf{R}^d) \boxtimes E \longrightarrow \mathbf{C}$ *can be extended to a continuous functional on* $\mathrm{L}^p(\mathbf{R}^d; E)$ *if and only if there is a* $b \in \mathrm{L}^{p'}(\mathbf{R}^d; \mathbf{R}^+_0)$ *such that*

$$(\forall\, \psi \in E) \quad |\tilde{B}\psi(\mathbf{x})| \leqslant b(\mathbf{x}) \|\psi\|_E \ ,$$

where $\tilde{B}$ *is defined by* $\langle \tilde{B}\psi, \varphi\rangle = B(\varphi, \psi)$*.* ∎

An H-distribution

Instead of the sphere (or ellipsoid) take a manifold

$$\mathrm{P} := \left\{ \boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{l\alpha_k} = 1 \right\} ,$$

where l is such that $l\alpha_k > d$, $k \in 1..d$.

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A function ψ from P can be extended to ψ_P on $\mathbf{R}^d \setminus \{0\}$ by projections

$$(\pi_P(\boldsymbol{\xi}))_i = \xi_i \left(\xi_1^{l\alpha_1} + \cdots + \xi_d^{l\alpha_d} \right)^{-1/l\alpha_i}.$$

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Theorem. [M. Lazar & D. Mitrović (CRAS, 2013)] (u_n) *bounded in* $\mathrm{L}^s(\mathbf{R}^{d+m})$*, supported in a compact (*$s \in \langle 1, 2 \rangle$*), and* (v_n) *bounded in* $\mathrm{L}^\infty_c(\mathbf{R}^m)$*.*
Then for any $\bar{s} \in \langle 1, s \rangle$*, on a subsequence, there is a continuous bilinear functional* B *on* $\mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m}) \boxtimes \mathrm{C}^d(P)$ *such that for* $\varphi \in \mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m})$ *and* $\psi \in \mathrm{C}^d(P)$

$$B(\varphi, \psi) = \lim_n \int_{\mathbf{R}^{d+m}} \varphi(\mathbf{x}, \mathbf{v}) u_n(\mathbf{x}, \mathbf{v}) (\mathcal{A}_{\psi_P} v_n)(\mathbf{x}) \, d\mathbf{x} d\mathbf{v} .$$

Furthermore, B *can be extended to a continuous bilinear functional on* $\mathrm{L}^{\bar{s}'}(\mathbf{R}^{d+m}; \mathrm{C}^d(P))$*.*

∎

What for $q < \infty$?

An analogous construction led M. Mišur and D. Mitrović to a construction of another variant, with an application to compactness by compensation.

Theorem. [M. Mišur & D. Mitrović (JFA, 2015)] *(u_n) bounded in $\mathrm{L}^p(\mathbf{R}^d)$, $p > 1$, and (v_n) bounded in $\mathrm{L}^q(\mathbf{R}^d)$, where $1/r := 1/p + 1/q < 1$, and v_n are supported in a compact.*
Then for any $\bar{s} \in \langle 1, r\rangle$, on a subsequence, there is a continuous bilinear functional B on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d) \boxtimes \mathrm{C}^d(P)$ such that for $\varphi \in \mathrm{L}^{\bar{s}'}(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^d(P)$

$$B(\varphi, \psi) = \lim_n \int_{\mathbf{R}^d} \varphi(\mathbf{x}) u_n(\mathbf{x}) (\mathcal{A}_{\psi_P} v_n)(\mathbf{x})\, d\mathbf{x}\ .$$

Furthermore, B can be extended to a continuous bilinear functional on $\mathrm{L}^{\bar{s}'}(\mathbf{R}^d; \mathrm{C}^d(P))$. ∎

Strong consistency condition

Introduce the set

$$\Lambda_{\mathcal{D}} = \Big\{ \boldsymbol{\mu} \in \ L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}^d))')^r : \Big(\sum_{k=1}^{n} (2\pi i \xi_k)^{\alpha_k} \mathbf{A}^k \Big) \boldsymbol{\mu} = \mathbf{0}_m \Big\},$$

where the given equality is understood in the sense of $L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}^d))')^m$.

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where the given equality is understood in the sense of $L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}^d))')^m$.

Let us assume that coefficients of the bilinear form $q(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\eta}) = \mathbf{Q}(\mathbf{x})\boldsymbol{\lambda} \cdot \boldsymbol{\eta}$, belong to space $L^t_{loc}(\mathbf{R}^d)$, where $1/t + 1/p + 1/q < 1$.

We say that set $\Lambda_{\mathcal{D}}$, bilinear form q and matrix $\boldsymbol{\mu} = [\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r], \boldsymbol{\mu}_j \in L^{\bar{s}}(\mathbf{R}^d; (C^d(\mathrm{P}^d))')^r$ satisfy the strong consistency condition if $(\forall j \in \{1, \dots, r\})\ \boldsymbol{\mu}_j \in \Lambda_{\mathcal{D}}$, and it holds

$$\langle \phi \mathbf{Q} \otimes 1, \boldsymbol{\mu} \rangle \geq \mathbf{0}, \quad \phi \in L^{\bar{s}}(\mathbf{R}^d; \mathbf{R}^+_0).$$

Compactness by compensation

Theorem. *Assume that sequences* (u_n) *and* (v_n) *are bounded in* $L^p(\mathbf{R}^d;\mathbf{R}^r)$ *and* $L^q(\mathbf{R}^d;\mathbf{R}^r)$*, respectively, and converge toward* u *and* v *in the sense of distributions.*
Assume that

$$\mathbf{G}_n := \sum_{k=1}^{d} \partial_k^{\alpha_k}(\mathbf{A}^k \mathsf{u}_n) \to \mathbf{0} \textit{ in } W^{-1,p}(\Omega;\mathbf{R}^m),$$

holds and that

$$q(\mathbf{x};\mathsf{u}_n,\mathsf{v}_n) \rightharpoonup \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

If the set $\Lambda_{\mathcal{D}}$*, the bilinear form* q*, and matrix* H*-distribution* $\boldsymbol{\mu}$*, corresponding to subsequences of* $(\mathsf{u}_n - \mathsf{u})$ *and* $(\mathsf{v}_n - \mathsf{v})$*, satisfy the strong consistency condition, then*

$$q(\mathbf{x};\mathsf{u},\mathsf{v}) \leq \omega \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d).$$

∎

Further variants and open questions

J. Aleksić, S. Pilipović, I. Vojnović (preprint)

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J. Aleksić, S. Pilipović, I. Vojnović (preprint)

F. Rindler (ARMA, 2015): microlocal compactness forms

Thank you for your attention.