Friedrichs systems

Nenad Antonić

Department of Mathematics
Faculty of Science
University of Zagreb

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Joint work with Krešimir Burazin, Marko Vrdoljak and Marko Erceg

Friedrichs' system (KOF1958)

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$d, r \in \mathbf{N}$, $\Omega \subseteq \mathbf{R}^d$ open and bounded with Lipschitz boundary Γ;

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$$\text{(F2)} \qquad (\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad (\text{ae on } \Omega)\,.$$

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The operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{R}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{R}^r)$

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

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Symmetric hyperbolic systems (KOF1954)

Summing over repeated indices:

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad .$$

In divergence form:

$$\partial_k(\mathbf{A}^k \mathsf{u}) + (\mathbf{B} - \partial_k \mathbf{A}^k)\mathsf{u} = \mathsf{f} \ .$$

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It is symmetric if all matrices $\mathbf{A}^k$ are symmetric; and hyperbolic (Friedrichs) if one of the matrices is even positive definite.

The wave equation

In d-dimensional space:

$$(\rho u')' - \operatorname{div}(\mathbf{A}\nabla u) = g \ .$$

Time $t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$:

$$(*) \qquad \partial_0(\rho \partial_0 u) - \sum_{i,j=1}^{d} \partial_i (a^{ij} \partial_j u) = g \ .$$

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This transformation gives us only one equation. For a system with $d+2$ unknowns to be formally deterministic, we need $d+1$ more equations. Clearly, defining equations for v^i would lead to a formally deterministic system, which is not symmetric.

The wave equation (cont.)

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Take the derivatives of the products in $(*)$:

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This will be the second equation of the system.

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The remaining d equations will be the Schwarz symmetry relations, with one index being 0, but multiplied by $\mathbf{A}^\top$:

$$\begin{aligned} \partial_0 u - v_0 &= 0 \\ \rho \partial_0 v_0 - a^{ij} \partial_i v_j + b^j v_j &= g \\ a^{ij} \partial_0 v_i - a^{ij} \partial_i v_0 &= 0 \ , \end{aligned}$$

where $b^0 := \partial_0 \rho, b^j := -\partial_i a^{ij} = [-\mathrm{div}\, \mathbf{A}^\top]^j$, for $j \in 1..d$.

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where $b^0 := \partial_0 \rho, b^j := -\partial_i a^{ij} = [-\mathrm{div}\, \mathbf{A}^\top]^j$, for $j \in 1..d$.
Actually, we can take $v_0 = \partial_0 u$ as a definition of u, and solve first for the remaining unknowns.

The wave equation in the required form

$$\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A}^\top & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{u} + \sum_{i=1}^{d} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{in} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{in} & & & \end{bmatrix} \partial_i \mathsf{u} + \begin{bmatrix} b^0 & b^1 & \cdots & b^n \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{u} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

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$$+ \begin{bmatrix} b^0 & b^1 & \cdots & b^n \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{u} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

$\mathbf{A}^i$ are symmetric, $\mathbf{A}^0$ is even positive definite ($\rho > 0$ and $\mathbf{A}$ is p.d.).
In particular, the system to which we reduced the wave equation is *hyperbolic in the sense of Petrovski.*

The wave equation (cont.)

For initial data $u(0,.) = u_0$ and $u'(0,.) = u_1$, take:

$$\begin{pmatrix} u(0,.) = u_0 \\ \partial_0 u(0,.) = u_1 \\ \partial_i u(0,.) = \partial_i u_0 \end{pmatrix} , \text{for } i \in 1..d$$

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as the initial data for the system.
u_0 is defined on $\mathbf{R}^d$, so we can compute its derivatives in the spatial directions.
To check:
the identities defining v_i (and therefore the symmetry relations).
For $i \in 1..d$:

$$\partial_0 v_i = \partial_i v_0 = \partial_i \partial_0 u = \partial_0 \partial_i u \ .$$

(The first equality follows from the regularity of $\mathbf{A}^\top$, because $\mathbf{A}^\top(\partial_0 \mathsf{v} - \nabla v_0) = 0$ implies $\partial_0 v_i = \partial_i v_0$.)
Now, we have that $\partial_0(v_i - \partial_i u) = 0$, and $v_i - \partial_i u = 0$ at $t = 0$, and we conclude that the last identity holds for any $t > 0$.

Maxwell's systems

In a material with electric permeability ϵ, conductivity $\boldsymbol{\sigma}$ and magnetic susceptibility $\boldsymbol{\mu}$

$$\mathsf{D}' = \operatorname{rot}\mathsf{H} - \mathsf{J} + \mathsf{F}$$
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together with $\operatorname{div}\mathsf{D} = \rho$ and $\operatorname{div}\mathsf{B} = 0$, and with the constitutive laws:

$$\mathsf{D}(.,t) = \boldsymbol{\epsilon}\mathsf{E}(.,t)$$
$$\mathsf{J}(.,t) = \boldsymbol{\sigma}\mathsf{E}(.,t)$$
$$\mathsf{B}(.,t) = \boldsymbol{\mu}\mathsf{H}(.,t)\ .$$

Maxwell's systems (cont.)

E and H as variables, $\mathsf{u} := \begin{bmatrix} \mathsf{E} \\ \mathsf{H} \end{bmatrix}$, the system can be written in the form of a symmetric system:

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The constant antisymmetric matrices $\mathbf{Q}_k$ are given by:

$$\mathbf{Q}_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{Q}_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \mathbf{Q}_3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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In the above we have used the fact that the rotator (curl) of a vector field E can be written as:

$$\operatorname{rot} \mathsf{E} = \begin{bmatrix} \partial_2 E^3 - \partial_3 E^2 \\ \partial_3 E^1 - \partial_1 E^3 \\ \partial_1 E^2 - \partial_2 E^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \partial_1 \mathsf{E} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \partial_2 \mathsf{E} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial_3 \mathsf{E}\,.$$

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If we assume the uniform boundedness and symmetry of the permeability and susceptibility tensors, the above system is even symmetric hyperbolic.

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$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

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– still it does not cover all of Gårding's theory of general elliptic equations, or Lerray's of general hyperbolic equations.

Example – heat equation, first form

Heat equation with lower order terms ($\Omega \subseteq \mathbf{R}^d$, $T > 0$ and $\Omega_T := \langle 0, T\rangle \times \Omega$):

$$\partial_t u - \mathsf{div}\,(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u + cu = f \qquad \text{in } \Omega_T\ ,$$

where $f \in \mathrm{L}^2(\Omega_T)$, $c \in \mathrm{L}^\infty(\Omega_T)$, $\mathbf{b} \in \mathrm{L}^\infty(\Omega_T; \mathbf{R}^d)$ and $\mathbf{A} \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$ is symmetric with eigenvalues between $\alpha > 0$ and $\beta \geqslant \alpha$ a.e. on Ω_T.

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Similarly as for the wave equation: $\mathbf{A} = [\mathsf{a}^1 \cdots \mathsf{a}^d]$ and $\mathsf{w} = \nabla u$

$$\begin{bmatrix} 1 & \mathsf{0}^\top \\ 0 & \mathbf{0} \end{bmatrix} \partial_t \begin{bmatrix} u \\ \mathsf{w} \end{bmatrix} - \sum_{i=1}^d \begin{bmatrix} \mathsf{div}\,\mathsf{a}^i & (\mathsf{a}^i)^\top \\ \mathsf{a}^i & \mathbf{0} \end{bmatrix} \partial_{x^i} \begin{bmatrix} u \\ \mathsf{w} \end{bmatrix} + \begin{bmatrix} c & \mathsf{b}^\top \\ 0 & \mathbf{A} \end{bmatrix} \begin{bmatrix} u \\ \mathsf{w} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

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It is clearly symmetric; positivity should be checked.

Example – heat equation, second form

New unknown vector function taking values in $\mathbf{R}^{d+1}$:

$$\mathsf{u} = \begin{bmatrix} u \\ \mathsf{v} \end{bmatrix} = \begin{bmatrix} u \\ -\mathbf{A}\nabla u \end{bmatrix} .$$

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Then the heat equation can be written as a first-order system

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$$\begin{bmatrix} 1 & 0^\top \\ 0 & \mathbf{0} \end{bmatrix} \partial_t \begin{bmatrix} u \\ \mathsf{v} \end{bmatrix} + \sum_{i=1}^{d} \begin{bmatrix} 1 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \partial_{x^i} \begin{bmatrix} u \\ \mathsf{v} \end{bmatrix} + \begin{bmatrix} c & -\mathbf{A}^{-1}\mathsf{b} \\ 0 & \mathbf{A}^{-1} \end{bmatrix} \begin{bmatrix} u \\ \mathsf{v} \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} .$$

The condition (F1) holds. The positivity condition $\mathbf{C} + \mathbf{C}^\top \geqslant 2\mu_0 \mathbf{I}$ is fulfilled if and only if $c - \frac{1}{4}\mathbf{A}^{-1}\mathbf{b} \cdot \mathbf{b}$ is uniformly positive.

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Two unknown functions:

$$v := \partial_x u$$

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The Schwarz symmetries give us more equations, and the following choice leads to a symmetric system:

$$\begin{aligned} \partial_x u - v &= 0 \\ -y\partial_x v - \partial_y w &= 0 \\ \partial_x w - \partial_y v &= 0 \; . \end{aligned}$$

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Any solution of this equation satisfies the symmetric system:

$$\mathbf{A}^1 \partial_x \mathsf{u} + \mathbf{A}^2 \partial_y \mathsf{u} = 0 \ ,$$

where the matrices are given by:

$$\mathbf{A}^1 := \begin{bmatrix} -y & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{A}^2 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} .$$

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Thus, a symmetric *hyperbolic* system corresponds to the Tricomi's equation in the *lower* half plane.
It is not positive ([KOF1958] — a transformation providing the right form).

Boundary conditions

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allows the treatment of different types of usual boundary conditions.

Assumptions on the boundary matrix $\mathbf{M}$

We assume (for ae $\mathbf{x} \in \Gamma$) [KOF1958]

(FM1) $$(\forall \boldsymbol{\xi} \in \mathbf{R}^r) \quad \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

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Such $\mathbf{M}$ is called *the admissible boundary condition*.

The boundary problem: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{R}^r)$ find u such that

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases} .$$

Different ways to enforce boundary conditions

Instead of

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u} = 0 \quad \text{on } \Gamma\,,$$

Lax proposed boundary conditions with

$$\mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma\,,$$

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Assumptions on N

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$) [PDL]

(FX1) $N(\mathbf{x})$ is non-negative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$:

$$(\forall \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,;$$

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or [RSP&LS1966]

Let $N(\mathbf{x})$ and $\tilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ satisfy (for ae $\mathbf{x} \in \Gamma$)

(FV1)

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0$$

$$(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0$$

(FV2)

$$\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}\,.$$

Equivalence of different descriptions of boundary conditions

Theorem. *It holds*

$$\textit{(FM1)–(FM2)} \qquad \Longleftrightarrow \qquad \textit{(FX1)–(FX2)} \qquad \Longleftrightarrow \qquad \textit{(FV1)–(FV2)},$$

with

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In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].

Classical results on well-posedness

Friedrichs:

– uniqueness of the classical solution

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Contributions:
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
– the meaning of traces for functions in the graph space
– weak well-posedness results under additional assumptions (on $\mathbf{A}_\nu$)
– regularity of solution
– numerical treatment

New approach...

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Diff. Eq.* **32** (2007) 317–341.

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. . . and new open questions.

Assumptions

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The Friedrichs operator

Let $\mathcal{D} := \mathrm{C}_c^\infty(\Omega;\mathbf{R}^r)$, $L = \mathrm{L}^2(\Omega;\mathbf{R}^r)$ and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ be defined by

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...fits in this framework.

Prolongations

$(\mathcal{D}, \langle \cdot \mid \cdot \rangle_T)$ is an inner product space, where

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Therefore $T = \tilde{T}^*_{|W_0}$, and analogously $\tilde{T} = T^*_{|W_0}$.
Abusing notation: $T, \tilde{T} \in \mathcal{L}(L; W_0')$ ... (T1)–(T3)

Formulation of the problem

Lemma. *The graph space*

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\},$$

is a Hilbert space with respect to $\langle \cdot \mid \cdot \rangle_T$.

∎

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Lemma. *The graph space*

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\}\,,$$

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Problem: for given $f \in L$ find $u \in W$ such that $Tu = f$.

Find sufficient conditions on $V \leqslant W$ such that $T_{|_V} : V \longrightarrow L$ is an isomorphism.

Boundary operator

Boundary operator $D \in \mathcal{L}(W; W')$:

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{}_{W'}\langle\, Du, v \,\rangle_W := \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L\,, \qquad u, v \in W\,.
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Lemma. *D is symmetric and satisfies*

$$ \ker D = W_0 $$
$$ \operatorname{im} D = W_0^0 := \{g \in W' : (\forall\, u \in W_0) \quad {}_{W'}\langle\, g, u \,\rangle_W = 0\} \,. $$

In particular, $\operatorname{im} D$ is closed in W'. ∎

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If T is the Friedrichs operator $\mathcal{L}$, then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ we have

$$_{W'}\langle D\mathsf{u}, \mathsf{v} \rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) .$$

Well-posedness theorem

Let V and $\tilde{V}$ be subspaces of W that satisfy

(V1)
$$(\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u\,\rangle_W \geqslant 0$$
$$(\forall\, v \in \tilde{V}) \qquad {}_{W'}\langle\, Dv, v\,\rangle_W \leqslant 0$$

(V2)
$$V = D(\tilde{V})^0\,, \qquad \tilde{V} = D(V)^0\,.$$

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Theorem. *Under assumptions* $(T1) - (T3)$ *and* $(V1) - (V2)$, *the operators* $T_{|_V} : V \longrightarrow L$ *and* $\tilde{T}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$ *are isomorphisms.* ∎

[AE&JLG&GC2007]

Correspondence with *classical* assumptions

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$$(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \leqslant 0\,,$$

(FV2)
$$\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}\,,$$

(for ae $\mathbf{x} \in \Gamma$)

Other sets of conditions in the classical setting (recall)

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$)

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(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which contains $N(\mathbf{x})$,

admissible boundary conditions: there exists a matrix function $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{R})$ such that (for ae $\mathbf{x} \in \Gamma$)

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(M2) $$W = \ker(D - M) + \ker(D + M)\,.$$

Equivalence of different descriptions of b.c.

Theorem. *(**classical**)* *It holds*

$$\textit{(FM1)–(FM2)} \quad \Longleftrightarrow \quad \textit{(FV1)–(FV2)} \quad \Longleftrightarrow \quad \textit{(FX1)–(FX2)},$$

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$$N(\mathbf{x}) := \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big).$$

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Theorem. *(***A. Ern, J.-L. Guermond, G. Caplain***)* *It holds*

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■

(M1)–(M2) $\longleftarrow$ (V1)–(V2)

Theorem. *Let V and $\tilde{V}$ satisfy (V1)–(V2), and suppose that there exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$ such that*

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Let us define $M \in \mathcal{L}(W;W')$ (for $u, v \in W$) with

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Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and M satisfies (M1)–(M2). ∎

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Lemma. *Suppose additionally that $V + \tilde{V}$ is closed. Then the operators P and Q from previous theorem do exist.* ∎

When this is satisfied?

Lemma. *(**K. Burazin, N.A.**)*
If $\operatorname{codim} W_0 (= \dim W/W_0)$ is finite, then the set $V + \tilde{V}$ is closed whenever V and $\tilde{V}$ satisfy (V1)–(V2). ∎

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Moreover, there do not exist operators P *and* Q *with desired properties.*

∎

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$$V := \{(\mathsf{p}, u)^\top \in W : \mathcal{T}_{\mathrm{div}}\mathsf{p} = \alpha \mathcal{T}_{\mathrm{H}^1} u\} \,,$$
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Lemma.
The above V and $\tilde{V}$ satisfy (V1)-(V2), $V \cap \tilde{V} = W_0$ and $V + \tilde{V} \neq W$. There exists an operator $M \in \mathcal{L}(W; W')$, that satisfies (M1)–(M2) and $V = \ker(D - M)$. ∎

New notation

$$[\, u \mid v \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,, \qquad u, v \in W$$

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subspace V is maximal non-negative in $(W, [\,\cdot \mid \cdot\,])$:

(X1) V is non-negative in $(W, [\,\cdot \mid \cdot\,])$: $\quad (\forall\, v \in V) \quad [\, v \mid v \,] \geqslant 0\,,$

(X2) there is no non-negative subspace in $(W, [\,\cdot \mid \cdot\,])$ containing V.

Kreĭn spaces

$(W, [\,\cdot \mid \cdot\,])$ is not a Kreĭn space – it is a degenerate space, because its Gramm operator $G := j \circ D$ $\quad (j : W' \longrightarrow W$ is the canonical isomorphism$)$ has large kernel:

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Important: $\operatorname{im} D$ is closed and $\ker D = W_0$.

Quotient Kreĭn space

Lemma. *Let* $U \supseteq W_0$ *and* Y *be subspaces of* W*. Then*

a) U *is closed if and only if* $\hat{U} := \{\hat{v} : v \in U\}$ *is closed in* $\hat{W}$*;*

b) $(\widehat{U+Y}) = \{u + v + W_0 : u \in U, v \in Y\} = \hat{U} + \hat{Y}$;

c) $U + Y$ *is closed if and only if* $\hat{U} + \hat{Y}$ *is closed;*

d) $(\hat{Y})^{[\perp]\hat{}} = \widehat{Y^{[\perp]}}$.

e) if Y *is maximal non-negative (non-positive) in* W*, than* $\hat{Y}$ *is maximal non-negative (non-positive) in* $\hat{W}$*;*

f) if $\hat{U}$ *is maximal non-negative (non-positive) in* $\hat{W}$*, then* U *is maximal non-negative (non-positive) in* W*.*

∎

(V1)–(V2) $\iff$ (X1)–(X2)

Theorem. *a) If subspaces V and $\tilde{V}$ satisfy (V1)–(V2), then V is maximal non-negative in W (satisfies (X1)–(X2)) and $\tilde{V}$ is maximal non-positive in W.*

b) If V is maximal non-negative in W, then V and $\tilde{V} := V^{[\perp]}$ satisfy (V1)–(V2). ∎

(M1)–(M2) $\Longrightarrow$ (V1)–(V2) (recall)

Theorem. *[EGC] (T1)–(T3) and $M \in \mathcal{L}(W;W')$ satisfy (M) imply*

$$V := \ker(D - M) \quad \textit{and} \quad \tilde{V} := \ker(D + M^*) \quad \textit{satisfy (V).}$$

∎

Corollary. *Under above assumptions*

$$T_{|_{\ker(D-M)}} : \ker(D - M) \longrightarrow L \qquad i \qquad \tilde{T}_{|_{\ker(D+M^*)}} : \ker(D + M^*) \longrightarrow L$$

are isomorphisms.

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{}_{W'}\langle\, Mu, v\,\rangle_W = {}& {}_{W'}\langle\, DPu, Pv\,\rangle_W - {}_{W'}\langle\, DQu, Qv\,\rangle_W \\
& + {}_{W'}\langle\, D(P+Q-PQ)u, v\,\rangle_W - {}_{W'}\langle\, Du, (P+Q-PQ)v\,\rangle_W\,.
\end{aligned}
$$

Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and M satisfies (M1)–(M2). ∎

Lemma. *Suppose additionally that $V + \tilde{V}$ is closed. Then the operators P and Q from previous theorem do exist.* ∎

(M1)–(M2) $\longleftarrow$ (V1)–(V2) (recall)

Theorem. *Let V and $\tilde{V}$ satisfy (V1)–(V2), and suppose that there exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$ such that*

$$\begin{aligned} (\forall\, v \in V) \quad & D(v - Pv) = 0\,, \\ (\forall\, v \in \tilde{V}) \quad & D(v - Qv) = 0\,, \\ & DPQ = DQP\,. \end{aligned}$$

Let us define $M \in \mathcal{L}(W;W')$ (for $u, v \in W$) with

$$\begin{aligned} {}_{W'}\langle\, Mu, v\,\rangle_W &= {}_{W'}\langle\, DPu, Pv\,\rangle_W - {}_{W'}\langle\, DQu, Qv\,\rangle_W \\ &\quad + {}_{W'}\langle\, D(P+Q-PQ)u, v\,\rangle_W - {}_{W'}\langle\, Du, (P+Q-PQ)v\,\rangle_W\,. \end{aligned}$$

Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and M satisfies (M1)–(M2). ∎

Lemma. *Suppose additionally that $V + \tilde{V}$ is closed. Then the operators P and Q from previous theorem do exist.* ∎

Closedness of $V + \tilde{V}$ is actually equivalent to the existence of operators P and Q.

On existence of P and Q

Our original approach was indirect:
Firstly, the existence of P and Q implies the existence of certain projectors in the quotient Kreĭn space; more precisely:

$$\hat{P}\hat{w} := \widehat{Pw}\,, \quad \hat{Q}\hat{w} := \widehat{Qw}\,, \quad w \in W$$

the projectors $\hat{P}, \hat{Q} : \hat{W} \longrightarrow \hat{W}$ are defined, satisfying

$$\hat{P}^2 = \hat{P} \quad \text{and} \quad \hat{Q}^2 = \hat{Q}\,,$$
$$\operatorname{im}\hat{P} = \hat{V} \quad \text{and} \quad \operatorname{im}\hat{Q} = \hat{\tilde{V}}\,,$$
$$\hat{P}\hat{Q} = \hat{Q}\hat{P}\,.$$

Secondly, this allowed us to prove the existence of corresponding projectors on W.

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (direct proof)

Theorem. *If $V, \tilde{V}$ are two closed subspaces of W that satisfy $W_0 \subseteq V \cap \tilde{V}$, then the following statements are equivalent:*
a) There exist operators $P \in \mathcal{L}(W; V)$ and $Q \in \mathcal{L}(W; \tilde{V})$, such that

$$\begin{aligned} (\forall\, v \in V) \quad & D(v - Pv) = 0\,, \\ (\forall\, v \in \tilde{V}) \quad & D(v - Qv) = 0\,, \\ & DPQ = DQP\,. \end{aligned}$$

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b) There exist projectors $P', Q' \in \mathcal{L}(W;W)$*, such that*

$$
\begin{gathered}
{P'}^2 = P' \quad \text{and} \quad {Q'}^2 = Q'\,, \\
\operatorname{im} P' = V \quad \text{and} \quad \operatorname{im} Q' = \tilde{V}\,, \\
P'Q' = Q'P'\,.
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(b) is equivalent to closedness of $V + \tilde{V}$.

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

Theorem.
a) $V, \tilde{V} \leqslant W$ satisfy (V), and exists a closed subspace $W_2 \subseteq C^-$ of W, $V \dot{+} W_2 = W$, then there exist an operator $M \in \mathcal{L}(W; W')$ satisfying (M) and $V = \ker(D - M)$.

If we define W_1 as orthogonal complement of W_0 in V, so that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and denote by R_1, R_0, R_2 projectors that correspond to above direct sum, then one such operator is given with $M = D(R_1 - R_2)$.

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

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b) $M \in \mathcal{L}(W; W')$ *an operator satisfying (M1)–(M2),* $V := \ker(D - M)$*.*

For W_2*, the orthogonal complement of* W_0 *in* $\ker(D + M)$*,* $W_2 \subseteq C^-$ *is closed,* $V \dot{+} W_2 = W$*, and* M *coincide with the operator in (a).* ∎

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Lemma. *Let* $W_2'' \leqslant W$ *satisfies* $W_2'' \subseteq C^-$ *and* $W_2'' + V = W$.
Then there is a closed subspace W_2 *of* W, *such that* $W_2 \subseteq C^-$ *and* $W_2 \dot{+} V = W$. ∎

Lemma. *If* $U_1 + U_2 = W$ *for some subspaces* $U_1 \subseteq C^+$ *and* $U_2 \subseteq C^-$ *of* W*, then* $U_1 \cap U_2 \subseteq W_0$*.*
If additionally U_1 *is maximal nonnegative and* U_2 *maximal nonpositive, then* $U_1 \cap U_2 = W_0$*.*

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Theorem. *For a maximal nonnegative subspace* V *of* W*, it is equivalent:*
a) There is a maximal nonpositive subspace W_2 *of* W*, such that* $W_2 + V = W$*;*
b) There is a nonpositive subspace $W_{\hat{2}}$ *of* $\hat{W}$*, such that* $W_{\hat{2}} + \hat{V} = \hat{W}$*.* ∎

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Lemma. *If $U_1 + U_2 = W$ for some subspaces $U_1 \subseteq C^+$ and $U_2 \subseteq C^-$ of W, then $U_1 \cap U_2 \subseteq W_0$.*
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Theorem. *For a maximal nonnegative subspace V of W, it is equivalent:*
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b) There is a nonpositive subspace $W_{\hat{2}}$ of $\hat{W}$, such that $W_{\hat{2}} + \hat{V} = \hat{W}$. ∎

Corollary. *The conditions (V) and (M) are equivalent.* ∎

Some used properties

Theorem. *a) $[\cdot \mid \cdot]$-orthogonal complement of a maximal non-negative (non-positive) subspace is non-positive (non-negative).*

b) Each maximal semi-definite subspace contains all isotropic vectors in W.

c) If L is a non-negative (non-positive) subspace of a Krein space, such that $L^{[\perp]}$ is non-positive (non-negative), then $\mathsf{Cl}\, L$ is maximal non-negative (non-positive).

d) Each maximal semi-definite subspace of a Krein space is closed.

e) A subspace L of a Krein space is closed if and only if $L = L^{[\perp][\perp]}$.

f) For a subspace L of a Krein space W it holds

$$L \cap L^{[\perp]} = \{0\} \qquad \Longleftrightarrow \qquad \mathsf{Cl}\,(L + L^{[\perp]}) = W\,.$$

∎

Posing and solving the problem

Problem: for given $f \in L$ find $u \in W$ such that $Tu = f$.

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Theorem. *Assume* $(T1) - (T3)$ *and the existence of* $M \in \mathcal{L}(W; W')$ *satisfying*

(M1) $$(\forall\, u \in W) \quad {}_{W'}\langle Mu, u \rangle_W \geqslant 0 \,,$$

(M2) $$W = \ker(D - M) + \ker(D + M) \,.$$

Then the operator $T_{|_{\ker(D-M)}} : \ker(D - M) \longrightarrow L$ *is an isomorphism.* ∎

Application to the classical theory

Let $\mathcal{D} := \mathrm{C}_c^\infty(\Omega;\mathbf{R}^r)$, $L = \mathrm{L}^2(\Omega;\mathbf{R}^r)$ and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ be defined by

$$T\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{C}\mathsf{u}\,,$$

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$$W = \Big\{\mathsf{u} \in \mathrm{L}^2(\Omega;\mathbf{R}^r) : \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} \ \in \mathrm{L}^2(\Omega;\mathbf{R}^r)\Big\}\,.$$

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(FM1) $$(\forall \boldsymbol{\xi} \in \mathbf{R}^r) \quad \mathbf{M}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

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For given matrix field $\mathbf{M}$ is there an operator M determined by $\mathbf{M}$ in some *natural way*?

What is *a natural way*?

Abstract well-posedness result:

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Weak well-posedness result for the original problem:

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i.e. smooth $\mathsf{u} \in \ker(A-M)$ should satisfy $(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|\partial\Omega} = 0$

Representation of D and M via matrix fields

For $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$ we have

$$ {}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,. $$

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For a given field $\mathbf{M}$, it is reasonable to seek an operator M of the form

(m) $$ {}_{W'}\langle\, M\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{M}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) \,. $$

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Question: Do (FM) and (m) define $M \in \mathcal{L}(W; W')$ satisfying (M)?

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Question: ...perhaps under some additional assumptions...?

Idea: represent $\mathbf{M}$ by $\mathbf{A}_{\boldsymbol{\nu}}$

Lemma. *If $\mathbf{M}$ satisfies (FM), then (for ae $\mathbf{x} \in \Gamma$) there is a pair of projectors $\mathbf{S}_+(\mathbf{x}), \mathbf{S}_-(\mathbf{x})$*
(i.e. $\mathbf{S}_+(\mathbf{x}) + \mathbf{S}_-(\mathbf{x}) = \mathbf{I}$ and $\mathbf{S}_+(\mathbf{x})\mathbf{S}_-(\mathbf{x}) = \mathbf{S}_-(\mathbf{x})\mathbf{S}_+(\mathbf{x}) = \mathbf{0}$), s.t.

$$(\mathbf{A}_{\boldsymbol{\nu}} + \mathbf{M})(\mathbf{x}) = 2\mathbf{S}_+^\top(\mathbf{x})\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \qquad \& \qquad (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})(\mathbf{x}) = 2\mathbf{S}_-^\top(\mathbf{x})\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\,.$$

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∎

Therefore

$$\mathbf{M}(\mathbf{x}) = \Big(\mathbf{I} - 2\mathbf{S}_-^\top(\mathbf{x})\Big)\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\,.$$

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Lemma. *If $\mathbf{M}$ satisfies (FM), then (for ae $\mathbf{x} \in \Gamma$) there is a pair of projectors $\mathbf{S}_{+}(\mathbf{x}), \mathbf{S}_{-}(\mathbf{x})$ (i.e. $\mathbf{S}_{+}(\mathbf{x}) + \mathbf{S}_{-}(\mathbf{x}) = \mathbf{I}$ and $\mathbf{S}_{+}(\mathbf{x})\mathbf{S}_{-}(\mathbf{x}) = \mathbf{S}_{-}(\mathbf{x})\mathbf{S}_{+}(\mathbf{x}) = \mathbf{0}$), s.t.*

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$$\mathbf{M}(\mathbf{x}) = \Big(\mathbf{I} - 2\mathbf{S}_{-}^{\top}(\mathbf{x})\Big)\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\,.$$

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N. Antonić, K. Burazin: *Boundary operator from matrix field formulation of boundary conditions for Friedrichs systems*, *Journal of Differential Equations* **250** (2011) 3630–3651.

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. . . not good enough for applications to hyperbolic equations

$\mathbf{P}$ is not necessarily a projector

Lemma

For a matrix field $\mathbf{M}$ *the following statements are equivalent.*

- $\mathbf{M}$ *satisfies (FM2).*
- *There is a matrix field* $\mathbf{P}$ *such that* $\mathbf{M} = \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I} - 2\mathbf{P})$ *and* $\ker(\mathbf{A}_{\boldsymbol{\nu}}\mathbf{P}) + \ker(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{I} - \mathbf{P})) = \mathbf{R}^r$ *ae in* $\partial\Omega$.

Main result for Friedrichs systems

Theorem. *Let matrix field* $\mathbf{M} \in \mathrm{L}^\infty(\Gamma; \mathrm{M}_r(\mathbf{R}))$ *satisfy (FM), and let* $\mathbf{S}_-$ *be extendable to a measurable function on* $\mathsf{Cl}\,\Omega$*, and satisfy:*
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($\mathcal{S}_{-,p}(\mathsf{v}) := \mathbf{S}_{-,p}\mathsf{v}$ *for* $\mathsf{v} \in W$*)*

(S2) $(\forall\, \mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{R}^r))\; \mathbf{S}_{-,p}\mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{R}^r)\ \&\ \mathcal{T}_{\mathrm{H}^1}(\mathbf{S}_{-,p}\mathsf{v}) = \mathbf{S}_-\mathcal{T}_{\mathrm{H}^1}\mathsf{v}$.

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Test on examples . . . assumptions are reasonable . . .

An example – scalar elliptic equation

$\Omega \subseteq \mathbf{R}^2$, $\mu > 0$ and $f \in \mathrm{L}^2(\Omega)$ given.

$$-\triangle u + \mu u = f$$

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which is a Friedrichs system with the choice of

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

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Note

$$\mathbf{A}_{\boldsymbol{\nu}} = \nu_1 \mathbf{A}_1 + \nu_2 \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & \nu_1 \\ 0 & 0 & \nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix}.$$

Elliptic equation – different boundary conditions

$$\mathbf{M} \qquad \mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} \qquad (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix}_{|\Gamma} = 0$$

$$\begin{bmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 2\nu_1 \\ 0 & 0 & 2\nu_2 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_{|\Gamma} = 0$$

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$\mathbf{M}$	$\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}$	$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix}_{\vert\Gamma} = 0$
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All above matrices $\mathbf{M}$ satisfy (FM).

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Constants can easily be extended, but we need $\boldsymbol{\nu} : \Gamma \longrightarrow \mathbf{R}^r$ to be Lipschitz in order to have bounded multiplication for the Robin b.c.

Practical sufficient conditions

Lemma

For constant $\mathbf{A}_k \in \mathrm{M}_r(\mathbf{R})$ *and* $\mathbf{P} \in \mathrm{M}_r(\mathbf{R})$ *the multiplication operator* $\mathsf{u} \mapsto \mathbf{P}\mathsf{u}$ *belongs to* $\mathcal{L}(W)$ *if and only if there exists* $\mathbf{S} \in \mathrm{M}_r(\mathbf{R})$ *such that* $\mathbf{A}_k\mathbf{P} = \mathbf{S}\mathbf{A}_k$ *for* $k \in 1..d$*.*

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Theorem (sufficient conditions)

Let $\mathbf{P} : \mathsf{Cl}\,\Omega \longrightarrow \mathrm{M}_r(\mathbf{R})$ *be a Lipschitz matrix function satisfying:*

– $(\exists\, \mathbf{S} \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R})))(\forall\, k \in 1..d) \qquad \mathbf{A}_k\mathbf{P} = \mathbf{S}\mathbf{A}_k$

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Then formula (m), for $\mathbf{M}(\mathbf{x}) := \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})(\mathbf{I} - 2\mathbf{P}(\mathbf{x}))$ *on* $\partial\Omega$*, defines a bounded operator* $M \in \mathcal{L}(W; W')$ *satisfying (M).*

Tests on examples

Applications on hyperbolic equations (transport and wave equation)

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N. Antonić, K. Burazin, M. Vrdoljak: *Second-order equations as Friedrichs systems*, Nonlin. Analysis B: Real World Appl. **14** (2014) 290–305.

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. . . still unable do get good results for mixed type problems

Heat equation

. . . with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \text{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b} \cdot \nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ u = 0 \text{ on } \partial\Omega \times \langle 0, T \rangle \\ u(\cdot, 0) = 0 \text{ on } \Omega \end{cases}$$

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...as a Friedrichs system:

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}}\mathsf{u}_d + cu_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d = f \end{cases},$$

(note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$).

Friedrichs operator and the graph space

The operator T is given by

$$T\begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d \\ \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}}\mathsf{u}_d + c u_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d \end{bmatrix},$$

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while the corresponding graph space is

$$\begin{aligned} W &= \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1}) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \\ &\qquad\qquad \& \quad \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(\Omega_T) \Big\} \\ &= \Big\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \Big\} \\ &= \Big\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \Big\}. \end{aligned}$$

Properties of the last component

Lemma. *The projection* $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top \mapsto u_{d+1}$ *is a continuous linear operator from* W *to* $W(0,T)$*, which is continuously embedded to* $\mathrm{C}([0,T]; \mathrm{L}^2(\Omega))$.

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The space

$$W(0,T) = \left\{ u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) : \partial_t u \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega)) \right\},$$

is a Banach space when equipped by norm

$$\|\mathsf{u}\|_{W(0,T)} = \sqrt{\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\partial_t u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))}}.$$

Main result

Let

$$V = \left\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. on } \Omega \right\},$$

$$\widetilde{V} = \left\{ \mathsf{v} \in W : v_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v_{d+1}(\cdot,T) = 0 \text{ a.e. on } \Omega \right\}.$$

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Do they satisfy (V1)–(V2)?

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Theorem

The above V and $\widetilde{V}$ satisfy (V1)–(V2), and therefore the operator $T_{|_V} : V \longrightarrow L$ is an isomorphism.

Two-field theory...

Heat equation with $\mathsf{b} = 0$ and $c = 0$:

$$\begin{cases} \partial_t u - \operatorname{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) = f \text{ in } \Omega_T \\ u = 0 \text{ on } \Gamma \times \langle 0, T\rangle \\ u(\cdot, 0) = 0 \text{ on } \Omega \end{cases}$$

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matrices need to be of the form

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{0} & \mathsf{B}^k \\ (\mathsf{B}^k)^\top & a^k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix},$$

where $\mathsf{B}^k \in \mathbf{R}^d$ are constant vectors, $a^k \in \mathrm{W}^{1,\infty}(\Omega_T)$, $\mathbf{C}^d \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$ and $c^{d+1} \in \mathrm{L}^\infty(\Omega_T)$, $k \in 1..(d+1)$.

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For the heat equation matrices have this form!

. . . with partial coercivity

Instead of coercivity (positivity) condition (F2), the following is required:

$$(\exists\, \mu_1 > 0)(\forall\, \boldsymbol{\xi} = (\boldsymbol{\xi}_d, \xi_{d+1}) \in \mathbf{R}^{d+1})$$

$$\Big(\mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d+1} \partial_k \mathbf{A}_k\Big)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 2\mu_1 |\boldsymbol{\xi}_d|^2 \qquad (\text{a.e. on } \Omega)\,,$$

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For our system both conditions are trivially fulfilled.

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For our system both conditions are trivially fulfilled.

Therefore, we have the well-posedness result.

An example – stationary diffusion equation

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$$-\mathrm{div}\,(\mathbf{A}\nabla u) + cu = f$$

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New unknown vector function taking values in $\mathbf{R}^{d+1}$:

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Then the starting equation can be written as a first-order system

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \mathrm{div}\,\mathsf{u}_d + c u_{d+1} = f \end{cases} ,$$

An example – stationary diffusion equation (cont.)

which is a Friedrichs system with the choice of

$$\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R}) \,, \qquad \mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0 & c \end{bmatrix} .$$

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The graph space: $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$.
Dirichlet, Neumann and Robin boundary conditions are imposed by the following choice of V and $\widetilde{V}$:

$$\begin{aligned} V_D = \widetilde{V}_D :=& \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega)\,, \\ V_N = \widetilde{V}_N :=& \{(\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = 0\}\,, \\ V_R :=& \{(\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = a {u_{d+1}}_{|_\Gamma}\}\,, \\ \widetilde{V}_R :=& \{(\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = -a {u_{d+1}}_{|_\Gamma}\}\,. \end{aligned}$$

Non-stationary problem

Marko Erceg, Krešimir Burazin: *Non-stationary abstract Friedrichs systems via semigroup theory*, submitted

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Numerics:
E. Burman, A. Ern, M. A. Fernandez, SIAM JNA, 2010.
D. A. Di Pietro, A. Ern, 2012.

Semigroup setting

A priori estimate:

$$(\forall\, t \in [0,T]) \qquad \|\mathsf{u}(t)\|_L^2 \leqslant e^t \left(\|\mathsf{u}_0\|_L^2 + \int_0^t \|\mathsf{f}(s)\|_L^2 \right).$$

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Let $\mathcal{A} : V \subseteq L \longrightarrow L$, $\mathcal{A} := -T_{|_V}$

Then (P) becomes:

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Theorem. *The operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup on L.*

∎

Corollary. *Let T be an operator that satisfies (T1)–(T2) and (T3)$'$, let V be a subspace of its graph space satisfying (V1)–(V2), and* $\mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; L)$.

Corollary. *Let T be an operator that satisfies (T1)–(T2) and (T3)$'$, let V be a subspace of its graph space satisfying (V1)–(V2), and $\mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; L)$. Then for every $\mathsf{u}_0 \in L$ the problem (P) has the unique mild solution $\mathsf{u} \in \mathrm{C}([0,T]; L)$ given with*

$$\mathsf{u}(t) = \mathcal{T}(t)\mathsf{u}_0 + \int_0^t \mathcal{T}(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0,T]\,,$$

where $(\mathcal{T}(t))_{t\geqslant 0}$ is the semigroup generated by $\mathcal{A}$.

Existence and uniqueness result

Corollary. *Let T be an operator that satisfies (T1)–(T2) and (T3)$'$, let V be a subspace of its graph space satisfying (V1)–(V2), and* $\mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; L)$*. Then for every* $\mathsf{u}_0 \in L$ *the problem (P) has the unique mild solution* $\mathsf{u} \in \mathrm{C}([0,T]; L)$ *given with*

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where $(\mathcal{T}(t))_{t\geqslant 0}$ *is the semigroup generated by* $\mathcal{A}$*.*
If additionally $\mathsf{f} \in \mathrm{C}([0,T]; L) \cap \Big(\mathrm{W}^{1,1}(\langle 0,T\rangle; L) \cup \mathrm{L}^1(\langle 0,T\rangle; V)\Big)$ *with V equipped with the graph norm and* $\mathsf{u}_0 \in V$*, then the above mild solution is the classical solution of (P) on* $[0,T\rangle$*.* ∎

Mild solution

Theorem. *Let* $\mathsf{u}_0 \in L$, $\mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; L)$ *and let*

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be the mild solution of (P).

Then $\mathsf{u}', T\mathsf{u}, \mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; W_0')$ *and*

$$\mathsf{u}' + T\mathsf{u} = \mathsf{f}\,,$$

in $\mathrm{L}^1(\langle 0, T\rangle; W_0')$. ∎

Bound on solution

From

$$\mathsf{u}(t) = \mathcal{T}(t)\mathsf{u}_0 + \int_0^t \mathcal{T}(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0,T]\,,$$

we get:

$$(\forall\, t \in [0,T]) \qquad \|\mathsf{u}(t)\|_L \leqslant \|\mathsf{u}_0\|_L + \int_0^t \|\mathsf{f}(s)\|_L ds\,.$$

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$$(\forall\, t \in [0,T]) \qquad \|\mathsf{u}(t)\|_L^2 \leqslant e^t\Big(\|\mathsf{u}_0\|_L^2 + \int_0^t \|\mathsf{f}(s)\|_L^2\Big)\,.$$

Non-stationary Maxwell system 1/5

Let $\Omega \subseteq \mathbf{R}^3$ be open and bounded with a Lipschitz boundary Γ, $\mu, \varepsilon \in \mathrm{W}^{1,\infty}(\Omega)$ positive and *away from zero*, $\mathbf{\Sigma}_{ij} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_3(\mathbf{R}))$, $i, j \in \{1, 2\}$, and $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{L}^1(\langle 0, T\rangle; \mathrm{L}^2(\Omega; \mathbf{R}^3))$.

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\text{(MS)} \qquad \begin{cases} \mu\partial_t \mathsf{H} + \operatorname{rot} \mathsf{E} + \mathbf{\Sigma}_{11}\mathsf{H} + \mathbf{\Sigma}_{12}\mathsf{E} = \mathsf{f}_1 \\ \varepsilon\partial_t \mathsf{E} - \operatorname{rot} \mathsf{H} + \mathbf{\Sigma}_{21}\mathsf{H} + \mathbf{\Sigma}_{22}\mathsf{E} = \mathsf{f}_2 \end{cases} \quad \text{in } \langle 0, T\rangle \times \Omega \,,
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where $\mathsf{E}, \mathsf{H} : [0, T\rangle \times \Omega \longrightarrow \mathbf{R}^3$ are unknown functions.
Change of variable

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\mathsf{u} := \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\mu}\mathsf{H} \\ \sqrt{\varepsilon}\mathsf{E} \end{bmatrix}, \quad c := \frac{1}{\sqrt{\mu\varepsilon}} \in \mathrm{W}^{1,\infty}(\Omega)\,,
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turns (MS) to the Friedrichs system

$$\partial_t \mathsf{u} + T\mathsf{u} = \mathsf{F}\,,$$

Non-stationary Maxwell system 2/5

with

$$\mathbf{A}_1 := c\left[\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & \mathbf{0} & & 0 & 0 & -1 \\ & & & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & \mathbf{0} & \\ 0 & -1 & 0 & & & \end{array}\right], \qquad \mathbf{A}_2 := c\left[\begin{array}{ccc|ccc} & & & 0 & 0 & 1 \\ & \mathbf{0} & & 0 & 0 & 0 \\ & & & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & & & \\ 0 & 0 & 0 & & \mathbf{0} & \\ 1 & 0 & 0 & & & \end{array}\right],$$

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Non-stationary Maxwell system 2/5

with

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(F1) and (F2) are satisfied (with change v $:= e^{-\lambda t}$u for large $\lambda > 0$, if needed)

Non-stationary Maxwell system 3/5

The spaces involved:

$$
\begin{aligned}
L &= \mathrm{L}^2(\Omega; \mathbf{R}^3) \times \mathrm{L}^2(\Omega; \mathbf{R}^3)\,, \\
W &= \mathrm{L}^2_{\mathrm{rot}}(\Omega; \mathbf{R}^3) \times \mathrm{L}^2_{\mathrm{rot}}(\Omega; \mathbf{R}^3)\,, \\
W_0 &= \mathrm{L}^2_{\mathrm{rot},0}(\Omega; \mathbf{R}^3) \times \mathrm{L}^2_{\mathrm{rot},0}(\Omega; \mathbf{R}^3) = \mathsf{Cl}_W \mathrm{C}^\infty_c(\Omega; \mathbf{R}^6)\,,
\end{aligned}
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where $\mathrm{L}^2_{\mathrm{rot}}(\Omega; \mathbf{R}^3)$ is the graph space of the rot operator.

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The boundary condition

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\boldsymbol{\nu} \times \mathsf{E}_{|_\Gamma} = 0
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V = \tilde{V} &= \{\mathsf{u} \in W : \boldsymbol{\nu} \times \mathsf{u}_2 = 0\}\\
&= \{\mathsf{u} \in W : \boldsymbol{\nu} \times \mathsf{E} = 0\}\\
&= \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3) \times \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3)\,.
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Non-stationary Maxwell system 4/5

Theorem. *Let* $\mathsf{E}_0 \in \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3), \mathsf{H}_0 \in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)$ *and let* $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{C}([0,T];\mathrm{L}^2(\Omega;\mathbf{R}^3))$ *satisfy at least one of the following conditions:*

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Then the abstract initial-boundary value problem

$$\begin{cases} \mu\mathsf{H}' + \operatorname{rot}\mathsf{E} + \mathbf{\Sigma}_{11}H + \mathbf{\Sigma}_{12}E = \mathsf{f}_1 \\ \varepsilon\mathsf{E}' - \operatorname{rot}\mathsf{H} + \mathbf{\Sigma}_{21}H + \mathbf{\Sigma}_{22}E = \mathsf{f}_2 \\ \mathsf{E}(0) = \mathsf{E}_0 \\ \mathsf{H}(0) = \mathsf{H}_0 \\ \boldsymbol{\nu}\times\mathsf{E}_{|_\Gamma} = 0 \end{cases},$$

∎

Theorem. *...has unique classical solution given by*

$$\begin{bmatrix} \mathsf{H} \\ \mathsf{E} \end{bmatrix}(t) = \begin{bmatrix} \frac{1}{\sqrt{\mu}}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\varepsilon}}\mathbf{I} \end{bmatrix} \mathcal{T}(t) \begin{bmatrix} \sqrt{\mu}\mathsf{H}_0 \\ \sqrt{\varepsilon}\mathsf{E}_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{\mu}}\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\varepsilon}}\mathbf{I} \end{bmatrix} \int_0^t \mathcal{T}(t-s) \begin{bmatrix} \frac{1}{\sqrt{\mu}}\mathsf{f}_1(s) \\ \frac{1}{\sqrt{\varepsilon}}\mathsf{f}_2(s) \end{bmatrix} ds\,, \quad t \in [0,T]\,,$$

where $(\mathcal{T}(t))_{t\geqslant 0}$ *is the contraction* C_0*-semigroup generated by* $-T$*.* ∎

Other examples

– Symmetric hyperbolic system

$$\begin{cases} \partial_t \mathsf{u} + \sum_{k=1}^{d} \partial_k (\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} = \mathsf{f} \quad \text{in } \langle 0, T\rangle \times \mathbf{R}^d \\ \mathsf{u}(0, \cdot) = \mathsf{u}_0 \end{cases} ,$$

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- Non-stationary div-grad problem

$$\begin{cases} \partial_t \mathsf{q} + \nabla p = \mathsf{f}_1 \quad \text{in } \langle 0, T\rangle \times \Omega\,, \quad \Omega \subseteq \mathbf{R}^d\,, \\ \frac{1}{c_0^2}\partial_t p + \operatorname{div} \mathsf{q} = f_2 \quad \text{in } \langle 0, T\rangle \times \Omega\,, \\ p_{|\partial\Omega} = 0\,, \quad p(0) = p_0\,, \quad \mathsf{q}(0) = \mathsf{q}_0 \end{cases}$$

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– Wave equation

$$\begin{cases} \partial_{tt} u - c^2 \triangle u = f \quad \text{in } \langle 0, T\rangle \times \mathbf{R}^d \\ u(0,\cdot) = u_0\,, \quad \partial_t u(0,\cdot) = u_0^1 \end{cases}.$$

Friedrichs systems in a complex Hilbert space

Let L be a complex Hilbert space, $L' \equiv L$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : L \longrightarrow L$ linear operators that satisfy (T1)–(T3) (or T3' instead of T3).

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Technical differences with respect to the real case, but results remain the same. . .

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For the classical Friedrichs operator we require

(F1) matrix functions $\mathbf{A}_k$ are selfadjoint: $\mathbf{A}_k = \mathbf{A}_k^*$,

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and again (F1)–(F2) imply (T1)–(T3).

Application to Dirac system 1/2

We consider the Cauchy problem

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where $\mathsf{u} : [0, T\rangle \times \mathbf{R}^3 \longrightarrow \mathbf{C}^4$ is an unknown function, $\mathsf{u}_0 : \mathbf{R}^3 \longrightarrow \mathbf{C}^4$, $\mathsf{f} : [0, T\rangle \times \mathbf{R}^3 \longrightarrow \mathbf{C}^4$ are given, and

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\mathbf{A}_k := \begin{bmatrix} \mathbf{0} & \boldsymbol{\sigma}_k \\ \boldsymbol{\sigma}_k & \mathbf{0} \end{bmatrix}, k \in 1..3\,, \qquad \mathbf{C} := \begin{bmatrix} c_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & c_2\mathbf{I} \end{bmatrix},
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where

$$\boldsymbol{\sigma}_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

are Pauli matrices, and $c_1, c_2 \in \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{C})$. . . . (F1)–(F2)

Theorem. *Let* $\mathsf{u}_0 \in W$ *and let* $\mathsf{f} \in \mathrm{C}([0,T]; \mathrm{L}^2(\mathbf{R}^3; \mathbf{C}^4))$ *satisfies at least one of the following conditions:*

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Then the abstract Cauchy problem

$$\begin{cases} \partial_t \mathsf{u} + \displaystyle\sum_{k=1}^{3} \mathbf{A}_k \partial_k \mathsf{u} + \mathbf{C}\mathsf{u} = \mathsf{f} \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases}$$

has unique classical solution given with

$$\mathsf{u}(t) = \mathcal{T}(t)\mathsf{u}_0 + \int_0^t \mathcal{T}(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0,T]\,,$$

where $(\mathcal{T}(t))_{t\geqslant 0}$ *is the contraction* C_0*-semigroup generated by* $-T$*.* ∎

TODO: Time-dependent coefficients

The operator T depends on t (i.e. the matrix coefficients $\mathbf{A}_k$ and $\mathbf{C}$ depend on t if T is a classical Friedrichs operator):

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– Semigroup theory can treat time-dependent case, but conditions that ensure existence/uniqueness result are rather complicated to verify...

TODO: Semilinear problem

Consider

$$\begin{cases} \mathsf{u}'(t) + T\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

where $\mathsf{f} : [0, T\rangle \times L \longrightarrow L$.

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- semigroup theory gives existence and uniqueness of solution
- it requires (locally) Lipschitz continuity of f in variable u

TODO: Semilinear problem

Consider

$$\begin{cases} \mathsf{u}'(t) + T\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

where $\mathsf{f} : [0, T\rangle \times L \longrightarrow L$.

- semigroup theory gives existence and uniqueness of solution
- it requires (locally) Lipschitz continuity of f in variable u
- if $L = \mathrm{L}^2$ it is not *appropriate* assumption, as power functions do not satisfy it; $L = \mathrm{L}^\infty$ is better. . .

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Problems:

– in the classical case (F1)–(F2) need not to imply (T2) and (T3): instead of (T3) we get

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– for semigroup treatment of non-stationary case we need to have $T : \mathcal{D} \subseteq L \longrightarrow L$

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$$W := \{\mathsf{u} \in L : T\mathsf{u} \in L\} = \{\mathsf{u} \in L : T_0 \mathsf{u} \in L\}\,.$$

Moreover, we have equivalence of norms ($\gamma = \sqrt{\max\{2, 1 + 2\beta^2\}}$):

$$\|\mathsf{u}\|_T \leqslant \gamma \|\mathsf{u}\|_{T_0} \leqslant \gamma^2 \|\mathsf{u}\|_T\,, \quad \text{for any } \mathbf{C}\,.$$

Boundary operator and a priori bound

The boundary operator D corresponding to the operator T does not depend on particular $\mathbf{C}$ from $\mathcal{M}_r(\alpha, \beta; \Omega)$.

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If V is a subspace of W that satisfies (V), well-posedness result implies that $T_{|_V} : V \longrightarrow L$ is an isomorphism, with

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Therefore, for fixed T_0 and V satisfying (V), we have a priori bound

$$(\exists\, c > 0)(\forall\, \mathbf{C} \in \mathcal{M}_r(\alpha, \beta; \Omega))(\forall\, \mathsf{u} \in V) \quad \|\mathsf{u}\|_{T_0} \leq c \|(\mathcal{L}_0 + \mathbf{C})\mathsf{u}\|_L\,.$$

Note that constant c depends only on T_0, α and β.

In the sequel $\mathcal{L}_0 = \sum_{k=1}^{d} \mathbf{A}_k \partial_k$ and V are fixed.

Definition (H-convergence for Friedrichs systems)

We say that a sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha, \beta; \Omega)$ H-converges to $\mathbf{C} \in \mathcal{M}_r(\alpha', \beta'; \Omega)$ with respect to T_0 and V if, for any $\mathsf{f} \in L$, the sequence (u_n) in V defined by $\mathsf{u}_n := T_n^{-1}\mathsf{f} \in V$, with $T_n = \mathcal{L}_0 + \mathbf{C}_n$, satisfies

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Theorem

Let $F = \{\mathsf{f}_n : n \in \mathbf{N}\}$ be a dense countable family in $\mathrm{L}^2(\Omega; \mathbf{R}^r)$, $\mathbf{C}, \mathbf{D} \in \mathcal{M}_r(\alpha, \beta; \Omega)$, and $\mathsf{u}_n, \mathsf{v}_n \in V$ solutions of $(T_0 + \mathbf{C})\mathsf{u}_n = \mathsf{f}_n$ and $(T_0 + \mathbf{D})\mathsf{v}_n = \mathsf{f}_n$, respectively. Furthermore, let

$$d(\mathbf{C}, \mathbf{D}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|\mathsf{u}_n - \mathsf{v}_n\|_{\mathrm{H}^{-1}(\Omega;\mathbf{R}^r)} + \|\mathbf{C}\mathsf{u}_n - \mathbf{D}\mathsf{v}_n\|_{\mathrm{H}^{-1}(\Omega;\mathbf{R}^r)}}{\|\mathsf{f}_n\|_{\mathrm{L}^2(\Omega;\mathbf{R}^r)}} .$$

Then the function $d : \mathcal{M}_r(\alpha, \beta; \Omega) \times \mathcal{M}_r(\alpha, \beta; \Omega) \longrightarrow \mathbf{R}$ forms a metric on the set $\mathcal{M}_r(\alpha, \beta; \Omega)$, and the H-convergence is equivalent to the sequential convergence in this metric space.

Compactness assumptions

Additional assumptions: for every sequence $\mathbf{C}_n \in \mathcal{M}_r(\alpha, \beta; \Omega)$ and every $\mathsf{f} \in L$, the sequence $\mathsf{u}_n \in V$ defined by $\mathsf{u}_n := (T_0 + \mathbf{C}_n)^{-1}\mathsf{f}$ satisfies the following: if (u_n) weakly converges to u in W, then also

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(K2) $$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \quad \langle\, T_0\mathsf{u}_n \mid \varphi\mathsf{u}_n \,\rangle_L \longrightarrow \langle\, T_0\mathsf{u} \mid \varphi\mathsf{u} \,\rangle_L \,.$$

Compactness theorems

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For fixed T_0 and V, if family $\mathcal{M}_r(\alpha, \beta; \Omega)$ satisfies (K1) and (K2), then it is compact with respect to H-convergence, i.e. from any sequence $(\mathbf{C}_n)$ in $\mathcal{M}_r(\alpha, \beta; \Omega)$ one can extract a H-converging subsequence whose limit belongs to $\mathcal{M}_r(\alpha, \beta; \Omega)$.

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The proof follows the original proof of Spagnolo in the case of parabolic G-convergence.

Stationary diffusion equation as Friedrichs system

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in $\Omega \subseteq \mathbf{R}^d$,

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Graph space $\ldots W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$

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(K1) For any sequence (u_n) in V

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$$\begin{aligned} {}_{W'}\langle\, D\mathsf{u}, \mathsf{u} \,\rangle_W &= 2\, {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \boldsymbol{\nu} \cdot \mathsf{u}_d, u_{d+1} \,\rangle_{\mathrm{H}^{\frac{1}{2}}} \\ &= \begin{cases} 0 & \ldots \text{ Dirichlet or Neumann} \\ 2a\|u_{d+1}\|^2_{\mathrm{L}^2(\Gamma)} & \ldots \text{ Robin } \ldots W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega) \end{cases} \end{aligned}$$

Properties (K1) and (K2)

(K1) For any sequence (u_n) in V

$$\mathsf{u}_n \rightharpoonup \mathsf{u} \quad \Longrightarrow \quad {}_{W'}\langle D\mathsf{u}_n, \mathsf{u}_n \rangle_W \longrightarrow {}_{W'}\langle D\mathsf{u}, \mathsf{u} \rangle_W$$

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(K2) For any sequence (u_n) in V and any $\varphi \in \mathrm{C}_c^\infty(\Omega)$

$$\mathsf{u}_n \rightharpoonup \mathsf{u} \quad \Longrightarrow \quad \langle\, T_0\mathsf{u}_n \mid \varphi\mathsf{u}_n \,\rangle_L \longrightarrow \langle\, T_0\mathsf{u} \mid \varphi\mathsf{u} \,\rangle_L$$

Compensated compactness

$$\begin{aligned} \langle\, T_0 \mathsf{u}_n \mid \varphi \mathsf{u}_n \,\rangle_L &= \int_\Omega \sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u}_n \cdot \varphi \mathsf{u}_n \, d\mathbf{x}\,, \\ &= -\frac{1}{2} \int_\Omega \partial_k \varphi \sum_{k=1}^d \mathbf{A}_k \mathsf{u}_n \cdot \mathsf{u}_n \, d\mathbf{x} \end{aligned}$$

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Theorem (Quadratic theorem)

For $\mathbf{A}_k \in \mathrm{M}_{q,p}(\mathbf{R})$ *let* $\Lambda := \Big\{ \boldsymbol{\lambda} \in \mathbf{R}^p \,:\, (\exists\, \boldsymbol{\xi} \neq \mathbf{0}) \quad \sum_{k=1}^d \xi_k \mathbf{A}_k \boldsymbol{\lambda} = \mathbf{0} \Big\}$

$$Q(\boldsymbol{\lambda}) := \mathbf{Q}\boldsymbol{\lambda} \cdot \boldsymbol{\lambda}, \ \textit{such that } Q = 0 \textit{ on } \Lambda\,,$$

$$\mathsf{u}_n \rightharpoonup \mathsf{u} \quad \textit{weakly in} \quad \mathrm{L}^2(\Omega; \mathbf{R}^p)\,,$$

$$\Big(\sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u}_n \Big) \quad \textit{is relatively compact in} \quad \mathrm{H}^{-1}(\Omega; \mathbf{R}^q)\,.$$

Then $Q \circ \mathsf{u}_n \rightharpoonup Q \circ \mathsf{u}$ *in* $\mathcal{D}'(\Omega)\,.$

Compensated compactness

$$\begin{aligned}\langle\, T_0 \mathsf{u}_n \mid \varphi \mathsf{u}_n \,\rangle_L &= \int_\Omega \sum_{k=1}^d \mathbf{A}_k \partial_k \mathsf{u}_n \cdot \varphi \mathsf{u}_n \, d\mathbf{x}\,, \qquad p = q = d+1 \\ &= -\frac{1}{2} \int_\Omega \partial_k \varphi \sum_{k=1}^d \overbrace{\mathbf{A}_k \mathsf{u}_n \cdot \mathsf{u}_n}^{Q \circ \mathsf{u}_n} d\mathbf{x}\end{aligned}$$

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Proof of (K2)

$$\sum_{k=1}^{d} \xi_k \mathbf{A}_k \boldsymbol{\lambda} = \begin{bmatrix} \lambda_{d+1}\xi_1 \\ \vdots \\ \lambda_{d+1}\xi_d \\ \sum_{k=1}^{d} \lambda_k \xi_k \end{bmatrix} \quad \Longrightarrow \quad \Lambda \ldots \lambda_{d+1} = 0$$

$$Q(\lambda) = \mathbf{A}_i \boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = 2\lambda_i \lambda_{d+1} = 0\,, \quad \lambda \in \Lambda$$

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$\checkmark$

Comparison with classical H-convergence

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c_n \end{bmatrix} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega)$$

$$\iff \begin{cases} \mathbf{C}_n(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} & \geq \quad \alpha|\boldsymbol{\xi}|^2 \\ \mathbf{C}_n(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} & \geq \quad \frac{1}{\beta}|\mathbf{C}_n(\mathbf{x})\boldsymbol{\xi}|^2 \end{cases}$$

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At a subsequence $\mathbf{C}_n \xrightarrow{H} \mathbf{C}$, by compactness theorem.

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– Has the limit $\mathbf{C}$ the same structure?

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At a subsequence $\mathbf{C}_n \xrightarrow{H} \mathbf{C}$, by compactness theorem.

- Has the limit $\mathbf{C}$ the same structure?
- Can we make a connection with H-converging (in classical sense) subsequence of $(\mathbf{A}^n)$?

Characterisation of the H-limit

Theorem

For the Friedrichs system corresponding to the stationary diffusion equation, a sequence $(\mathbf{C}_n)$ in $\mathcal{M}_{d+1}(\alpha, \beta; \Omega)$ of the form

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & 0 \\ 0^\top & c_n \end{bmatrix} .$$

H-converges with respect to $\mathcal{L}_0$ and V_D if and only if $(\mathbf{A}^n)$ classically H-converges to some $\mathbf{A}$ and (c_n) L^∞ weakly $*$ converges to some c. In that case, the H-limit is the matrix function

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0^\top & c \end{bmatrix} ,$$

Heat equation as Friedrichs system

$\Omega \subseteq \mathbf{R}^d$ open and bounded set with Lipschitz boundary Γ, $T > 0$ and $\Omega_T := \Omega \times \langle 0, T\rangle$

$$\partial_t u_n - \mathsf{div}_{\mathbf{x}}(\mathbf{A}^n \nabla_{\mathbf{x}} u_n) + c u_n = f \qquad \text{in } \Omega_T \, ,$$

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The matrices $\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R})$, $k = 1, \ldots d$, $\mathbf{A}_{d+1} = \mathbf{e}_{d+1} \otimes \mathbf{e}_{d+1}$ and

$$\mathbf{C}_n = \begin{bmatrix} (\mathbf{A}^n)^{-1} & \mathsf{0} \\ \mathsf{0}^\top & c \end{bmatrix}$$

$$T_0 \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} \\ \partial_t u_{d+1} + \mathsf{div}_{\mathbf{x}} \mathsf{u}_d \end{bmatrix} .$$

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Graph space

$$W = \left\{ \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \right\} .$$

Compactness result

Dirichlet boundary conditions with zero initial value:

$$V = \Big\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. on } \Omega \Big\},$$

$$\widetilde{V} = \Big\{ \mathsf{v} \in W : v^u \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v^u(\cdot,T) = 0 \text{ a.e. on } \Omega \Big\}.$$

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(K1):

$${}_{W'}\langle\, D\mathsf{u}, \mathsf{u} \,\rangle_W = \|u_{d+1}(\cdot,T)\|^2_{\mathrm{L}^2(\Omega)}\,.$$

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$\implies \quad \mathcal{M}_{d+1}(\alpha,\beta;\Omega)$ is compact with H-topology for given $\mathcal{L}_0$ and V

Comparison with classical parabolic H-convergence. . .

Compactness result

Dirichlet boundary conditions with zero initial value:

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$$\widetilde{V} = \Big\{ \mathsf{v} \in W : v^u \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v^u(\cdot,T) = 0 \text{ a.e. on } \Omega \Big\}.$$

(K1):

$$_{W'}\langle D\mathsf{u}, \mathsf{u} \rangle_W = \|u_{d+1}(\cdot,T)\|^2_{\mathrm{L}^2(\Omega)}.$$

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(K2): similarly to stationary diffusion equation: $\Lambda = \{\boldsymbol{\lambda} \in \mathbf{R}^{d+1} : \lambda_{d+1} = 0\}$

✓

$\Longrightarrow \quad \mathcal{M}_{d+1}(\alpha,\beta;\Omega)$ is compact with H-topology for given $\mathcal{L}_0$ and V

Comparison with classical parabolic H-convergence... similarly as for stationary diffusion equation.

G-convergence

Instead of $\mathbf{C}_n \in \mathcal{M}_r(\alpha, \beta; \Omega)$ we take

$$\mathcal{C}_n \in \mathcal{F}(\alpha, \beta; \Omega) := \Big\{ \mathcal{C} \in \mathcal{L}(L) : (\forall\, \mathsf{u} \in L) \quad \langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L \geq \alpha \|\mathsf{u}\|_L^2 \quad \& \quad \langle\, \mathcal{C}\mathsf{u} \mid \mathsf{u} \,\rangle_L \geq \frac{1}{\beta} \|\mathcal{C}\mathsf{u}\|_L^2 \Big\} .$$

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Definition (G-convergence for Friedrichs systems)

For $\mathcal{C}_n \in \mathcal{F}(\alpha, \beta; \Omega)$, we say that a sequence of isomorphisms $T_n := T_0 + \mathcal{C}_n : V \to L$ G-converges to an isomorphism $T := T_0 + \mathcal{C} : V \to L$, for some $\mathcal{C} \in \mathcal{F}(\alpha', \beta'; \Omega)$ if

$$(\forall\, \mathsf{f} \in L) \quad T_n^{-1}\mathsf{f} \rightharpoonup T^{-1}\mathsf{f} \text{ in } W .$$

Theorem

For fixed T_0 and V, if family $\mathcal{F}(\alpha, \beta; \Omega)$ satisfies (K1), then for any sequence $(\mathcal{C}_n)$ in $\mathcal{F}(\alpha, \beta; \Omega)$ there exists a subsequence of $T_n := T_0 + \mathcal{C}_n$ which G-converges to $T := T_0 + \mathcal{C}$ with $\mathcal{C} \in \mathcal{F}(\alpha, \beta; \Omega)$.

Open problems . . .

– Find all representations of a particular equation in the form of a Friedrichs system.

– Application to other equations of practical importance (mixed-type problems).

– Compare the results to those already known in the classical setting.

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