Friedrichs systems

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Joint work with Krešimir Burazin, Marko Vrdoljak and Marko Erceg
Why should one be interested in Friedrichs systems?
  Symmetric hyperbolic systems
  Symmetric positive systems

Classical theory
  Boundary conditions for Friedrichs systems
  Existence, uniqueness, well-posedness

Abstract formulation
  Graph spaces
  Cone formalism of Ern, Guermond and Caplain
  Interdependence of different representations of boundary conditions

Kreĭn space formalism
  Kreĭn spaces
  Equivalence of boundary conditions

What can we say for the Friedrichs operator now?
  Sufficient assumptions
  An example: elliptic equation
  Other second order equations
  Two-field theory
  Non-stationary theory

Homogenisation of Friedrichs systems
  Homogenisation
  Examples: Stationary diffusion and heat equation

Concluding remarks
Friedrichs’ system (KOF1958)

Assumptions:
\[ d, r \in \mathbb{N}, \Omega \subseteq \mathbb{R}^d \text{ open and bounded with Lipschitz boundary } \Gamma; \]
Friedrichs’ system (KOF1958)

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d, r ∈ N, Ω ⊆ R^d open and bounded with Lipschitz boundary Γ;
A_k ∈ W^{1,\infty}(\Omega; M_r(\mathbb{R})), k \in 1..d, and C ∈ L^\infty(\Omega; M_r(\mathbb{R}))
Friedrichs’ system (KOF1958)

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\( A_k \in W^{1,\infty}(\Omega; M_r(\mathbb{R})), k \in 1..d, \) and \( C \in L^\infty(\Omega; M_r(\mathbb{R})) \) satisfying
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\[(F1) \quad \text{matrix functions } A_k \text{ are symmetric: } A_k = A_k^\top;\]

\[(F2) \quad (\exists \mu_0 > 0) \quad C + C^\top + \sum_{k=1}^{d} \partial_k A_k \geq 2\mu_0 \mathbf{I} \quad \text{(ae on } \Omega).\]
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The operator \(\mathcal{L} : L^2(\Omega; \mathbb{R}^r) \rightarrow D'(\Omega; \mathbb{R}^r)\)

\[ \mathcal{L}u := \sum_{k=1}^d \partial_k(A_ku) + Cu \]

is called \textit{the symmetric positive operator} or \textit{the Friedrichs operator},
Friedrichs’ system (KOF1958)

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is called the symmetric positive operator or the Friedrichs operator, and

\(\mathcal{L}u = f\)

the symmetric positive system or the Friedrichs system.
Symmetric hyperbolic systems (KOF1954)

Summing over repeated indices:

\[ A^k \partial_k u + Bu = f. \]

In divergence form:

\[ \partial_k (A^k u) + (B - \partial_k A^k)u = f. \]
Symmetric hyperbolic systems (KOF1954)

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\[ A^k \partial_k u + Bu = f . \]

In divergence form:

\[ \partial_k (A^k u) + Cu = f . \]

It is symmetric if all matrices \( A^k \) are symmetric; and hyperbolic (Friedrichs) if one of the matrices is even positive definite.
The wave equation

In $d$-dimensional space:

$$(\rho u')' - \text{div} (A \nabla u) = g .$$

Time $t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$:

$$(*) \quad \partial_0 (\rho \partial_0 u) - \sum_{i,j=1}^{d} \partial_i (a^{ij} \partial_j u) = g .$$
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New variables: $v_j := \partial_j u$, $j \in 0..d$ give vector unknown $u = [u, v_0, \ldots, v_d]^\top$, and with: $a^{00} := -\rho$, $a^{0i} := a^{i0} := 0$ we have

$$-\partial_i (a^{ij} v_j) = g .$$
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This transformation gives us only one equation. For a system with $d + 2$ unknowns to be formally deterministic, we need $d + 1$ more equations. Clearly, defining equations for $v^i$ would lead to a formally deterministic system, which is not symmetric.
The wave equation (cont.)

We also have \((d + 1)(d + 2)/2\) symmetry relations \(\partial_i v_j = \partial_j v_i\).
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Take the derivatives of the products in \((\ast)\):

\[
\rho \partial_0 v_0 - a^{ij} \partial_i v_j + \partial_0 \rho v_0 - (\partial_i a^{ij}) v_j = g.
\]

This will be the second equation of the system.
The wave equation (cont.)

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The remaining \(d\) equations will be the Schwarz symmetry relations, with one index being 0, but multiplied by \(A^\top\):

\[
\begin{align*}
\partial_0 u - v_0 &= 0 \\
\rho \partial_0 v_0 - a^{ij} \partial_i v_j + b^j v_j &= g \\
a^{ij} \partial_0 v_i - a^{ij} \partial_i v_0 &= 0 ,
\end{align*}
\]

where \(b^0 := \partial_0 \rho, b^j := -\partial_i a^{ij} = [-\text{div} A^\top]^j, \) for \(j \in 1..d\).
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\]

where \(b^0 := \partial_0 \rho, b^j := -\partial_i a^{ij} = [-\text{div} A^\top]^j\), for \(j \in 1..d\).

Actually, we can take \(v_0 = \partial_0 u\) as a definition of \(u\), and solve first for the remaining unknowns.
The wave equation in the required form

\[
\begin{bmatrix}
\rho & 0 & \cdots & 0 \\
0 & \mathbf{A}^\top & & \\
\vdots & & \ddots & \\
0 & 0 & \cdots & 0
\end{bmatrix} \partial_0 u + \sum_{i=1}^{d} \begin{bmatrix}
0 & -a_{11} & \cdots & -a_{in} \\
-a_{11} & \ddots & \cdots & \\
\vdots & \ddots & 0 & \\
-a_{in} & \cdots & -a_{nn} & 0
\end{bmatrix} \partial_i u \\
+ \begin{bmatrix}
b_0 & b_1 & \cdots & b_n \\
0 & \ddots & \cdots & \\
\vdots & \ddots & \ddots & \\
0 & 0 & \cdots & 0
\end{bmatrix} u = \begin{bmatrix}
g \\
0 \\
\vdots \\
0
\end{bmatrix}.
\]
The wave equation in the required form

\[
\begin{bmatrix}
  \rho & 0 & \cdots & 0 \\
  0 & & & \\
  \vdots & & & \mathbf{A}^\top \\
  0 & & & 
\end{bmatrix}
\partial_0 \mathbf{u} + \sum_{i=1}^{d} \begin{bmatrix}
  0 & -a^{i1} & \cdots & -a^{in} \\
  -a^{i1} & & & \\
  \vdots & & & 0 \\
  -a^{in} & & & 
\end{bmatrix}
\partial_i \mathbf{u} + \begin{bmatrix}
  b^0 & b^1 & \cdots & b^n \\
  0 & & & \\
  \vdots & & & 0 \\
  0 & & & 
\end{bmatrix} \mathbf{u} = \begin{bmatrix}
  g \\
  0 \\
  \vdots \\
  0 
\end{bmatrix}
\]

\( A^i \) are symmetric, \( A^0 \) is even positive definite (\( \rho > 0 \) and \( \mathbf{A} \) is p.d.).
In particular, the system to which we reduced the wave equation is hyperbolic in the sense of Petrovski.
The wave equation (cont.)

For initial data \( u(0, .) = u_0 \) and \( u'(0, .) = u_1 \), take:

\[
\begin{align*}
( & \quad u(0, .) = u_0 \\
\partial_0 u(0, .) = u_1 \\
\partial_i u(0, .) = \partial_i u_0 , \text{ for } i \in 1..d
\end{align*}
\]

as the initial data for the system.

\( u_0 \) is defined on \( \mathbb{R}^d \), so we can compute its derivatives in the spatial directions.
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\( u_0 \) is defined on \( \mathbb{R}^d \), so we can compute its derivatives in the spatial directions. 

To check:

the identities defining \( v_i \) (and therefore the symmetry relations).

For \( i \in 1..d \):

\[
\partial_0 v_i = \partial_i v_0 = \partial_i \partial_0 u = \partial_0 \partial_i u.
\]

(The first equality follows from the regularity of \( A^\top \), because \( A^\top (\partial_0 v - \nabla v_0) = 0 \) implies \( \partial_0 v_i = \partial_i v_0 \).)

Now, we have that \( \partial_0(v_i - \partial_i u) = 0 \), and \( v_i - \partial_i u = 0 \) at \( t = 0 \), and we conclude that the last identity holds for any \( t > 0 \).
In a material with electric permeability $\epsilon$, conductivity $\sigma$ and magnetic susceptibility $\mu$

\[ D' = \text{rot} \, H - J + F \]
\[ B' = -\text{rot} \, E + G , \]
Maxwell’s systems

In a material with electric permeability $\epsilon$, conductivity $\sigma$ and magnetic susceptibility $\mu$

$$D' = \text{rot } H - J + F$$
$$B' = -\text{rot } E + G,$$

together with $\text{div } D = \rho$ and $\text{div } B = 0$, and with the constitutive laws:

$$D(., t) = \epsilon E(., t)$$
$$J(., t) = \sigma E(., t)$$
$$B(., t) = \mu H(., t).$$
Maxwell’s systems (cont.)

E and H as variables, \( u := \begin{bmatrix} E \\ H \end{bmatrix} \), the system can be written in the form of a symmetric system:

\[
\sum_{i=0}^{3} A^i \partial_i u + Bu = f ,
\]

where:

\[
A^0 = \epsilon_0 \mu - , \quad A^1 := 0 Q^\top Q^1 0 - , \quad A^2 := 0 Q^\top Q^2 0 - , \quad A^3 := 0 Q^\top Q^3 0 - .
\]
Maxwell’s systems (cont.)

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$$\sum_{i=0}^{3} A^i \partial_i u + Bu = f,$$

where:

$$A^0 = \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}, A^1 := \begin{bmatrix} 0 & Q_1^T \\ Q_1 & 0 \end{bmatrix}, A^2 := \begin{bmatrix} 0 & Q_2^T \\ Q_2 & 0 \end{bmatrix}, A^3 := \begin{bmatrix} 0 & Q_3^T \\ Q_3 & 0 \end{bmatrix}.$$
Maxwell’s systems (cont.)

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A^3 := \begin{bmatrix} 0 & Q_3^T \\ Q_3 & 0 \end{bmatrix}.
\]

The constant antisymmetric matrices \( Q_k \) are given by:

\[
Q_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},
Q_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},
Q_3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Maxwell’s systems (cont.)

\[ \mathbf{B} = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{while the right hand side is } f = \begin{bmatrix} F \\ G \end{bmatrix}. \]
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In the above we have used the fact that the rotator (curl) of a vector field \( E \) can be written as:

\[
\text{rot } E = \begin{bmatrix} \partial_2 E^3 - \partial_3 E^2 \\ \partial_3 E^1 - \partial_1 E^3 \\ \partial_1 E^2 - \partial_2 E^1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \partial_1 E \\
+ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \partial_2 E + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial_3 E.
\]
Maxwell’s systems (cont.)

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\text{rot } \mathbf{E} = \begin{bmatrix}
\partial_2 E^3 - \partial_3 E^2 \\
\partial_3 E^1 - \partial_1 E^3 \\
\partial_1 E^2 - \partial_2 E^1
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix} \partial_1 \mathbf{E}
\]

\[+ \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix} \partial_2 \mathbf{E} + \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \partial_3 \mathbf{E}.\]

If we assume the uniform boundedness and symmetry of the permeability and susceptibility tensors, the above system is even symmetric hyperbolic.
Friedrichs systems

Introduced in:
Friedrichs systems

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Goal:
– treating the equations of mixed type, such as the Tricomi equation:

\[ y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \]
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– unified treatment of equations and systems of different type.
Friedrichs systems

Introduced in:
K. O. Friedrichs: Symmetric positive linear differential equations,
Communications on Pure and Applied Mathematics 11 (1958), 333–418

Goal:
– treating the equations of mixed type, such as the Tricomi equation:

\[ y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \]

– unified treatment of equations and systems of different type.
– still it does not cover all of Gårding’s theory of general elliptic equations, or
Lerray’s of general hyperbolic equations.
Example – heat equation, first form

Heat equation with lower order terms ($\Omega \subseteq \mathbb{R}^d$, $T > 0$ and $\Omega_T := \langle 0, T \rangle \times \Omega$):

$$\partial_t u - \text{div} \left( A \nabla u \right) + b \cdot \nabla u + cu = f \quad \text{in } \Omega_T,$$

where $f \in L^2(\Omega_T)$, $c \in L^\infty(\Omega_T)$, $b \in L^\infty(\Omega_T; \mathbb{R}^d)$ and $A \in L^\infty(\Omega_T; \mathbb{M}_d(\mathbb{R}))$ is symmetric with eigenvalues between $\alpha > 0$ and $\beta \geq \alpha$ a.e. on $\Omega_T$. 
Heat equation with lower order terms ($\Omega \subseteq \mathbb{R}^d$, $T > 0$ and $\Omega_T := (0, T) \times \Omega$):

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Similarly as for the wave equation: $A = [a^1 \cdots a^d]$ and $w = \nabla u$

$$\begin{bmatrix} 1 & 0^\top \\ 0 & 0 \end{bmatrix} \partial_t \begin{bmatrix} u \\ w \end{bmatrix} - \sum_{i=1}^d \begin{bmatrix} \text{div } a^i & (a^i)^\top \\ a^i & 0 \end{bmatrix} \partial_{x^i} \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} c & b^\top \\ 0 & A \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$
Heat equation with lower order terms \((\Omega \subseteq \mathbb{R}^d, T > 0 \text{ and } \Omega_T := \langle 0, T \rangle \times \Omega)\):

\[
\partial_t u - \text{div} (A \nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega_T,
\]

where \(f \in L^2(\Omega_T), c \in L^\infty(\Omega_T), b \in L^\infty(\Omega_T; \mathbb{R}^d)\) and \(A \in L^\infty(\Omega_T; M_d(\mathbb{R}))\) is symmetric with eigenvalues between \(\alpha > 0\) and \(\beta \geq \alpha\) a.e. on \(\Omega_T\).

Similarly as for the wave equation: \(A = [a^1 \cdots a^d]\) and \(w = \nabla u\)

\[
\begin{bmatrix}
1 & 0^\top \\
0 & 0
\end{bmatrix}
\partial_t
\begin{bmatrix}
u \\
w
\end{bmatrix}
- \sum_{i=1}^d
\begin{bmatrix}
\text{div } a^i & (a^i)^\top \\
a^i & 0
\end{bmatrix}
\partial_{x^i}
\begin{bmatrix}
u \\
w
\end{bmatrix}
+ \begin{bmatrix}
c & b^\top \\
0 & A
\end{bmatrix}
\begin{bmatrix}
u \\
w
\end{bmatrix}
= \begin{bmatrix}
f \\
0
\end{bmatrix}.
\]

It is clearly symmetric; positivity should be checked.
Example – heat equation, second form

New unknown vector function taking values in $\mathbb{R}^{d+1}$:

$$u = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ -A\nabla u \end{bmatrix}.$$
Example – heat equation, second form

New unknown vector function taking values in $\mathbb{R}^{d+1}$:

\[
\begin{bmatrix}
 u \\
 v
\end{bmatrix} = \begin{bmatrix}
 u \\
 -A \nabla u
\end{bmatrix}.
\]

Then the heat equation can be written as a first-order system

\[
\begin{cases}
 \partial_t u + \text{div} \, v + cu - A^{-1} b \cdot v = f \\
 \nabla u + A^{-1} v = 0
\end{cases}
\]
Example – heat equation, second form

New unknown vector function taking values in $\mathbb{R}^{d+1}$:

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Then the heat equation can be written as a first-order system

$$\begin{cases} \partial_t u + \text{div} \, v + cu - A^{-1} b \cdot v = f \\ \nabla u + A^{-1} v = 0 \end{cases},$$

which is a Friedrichs system

$$\begin{bmatrix} 1 & 0^\top \\ 0 & 0 \end{bmatrix} \partial_t \begin{bmatrix} u \\ v \end{bmatrix} + \sum_{i=1}^d \begin{bmatrix} 1 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \partial_{x_i} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c & -A^{-1} b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. $$
Example – heat equation, second form

New unknown vector function taking values in $\mathbb{R}^{d+1}$:

$$u = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ -A \nabla u \end{bmatrix}.$$ 

Then the heat equation can be written as a first-order system

$$\begin{cases} \partial_t u + \text{div} \ v + cu - A^{-1} b \cdot v = f \\ \nabla u + A^{-1} v = 0 \end{cases},$$

which is a Friedrichs system

$$\left[ \begin{array}{cc} 1 & 0^T \\ 0 & 0 \end{array} \right] \partial_t \begin{bmatrix} u \\ v \end{bmatrix} + \sum_{i=1}^d \begin{bmatrix} 1 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \partial_{x_i} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c & -A^{-1} b \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$ 

The condition (F1) holds. The positivity condition $C + C^T \succeq 2\mu_0 I$ is fulfilled if and only if $c - \frac{1}{4} A^{-1} b \cdot b$ is uniformly positive.
Tricomi’s equation

\[ y\partial_x^2 u + \partial_y^2 u = 0. \]
Tricomi’s equation

\[ y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]

The Tricomi equation is of mixed type. The standard procedure for classification gives us \( ac - b^2 = y \), so the equation is \textit{elliptic} for \( y > 0 \), parabolic on the line \( y = 0 \) and hyperbolic in the lower half plane \( y < 0 \).
The Tricomi equation is of mixed type. The standard procedure for classification gives us $ac - b^2 = y$, so the equation is *elliptic* for $y > 0$, parabolic on the line $y = 0$ and hyperbolic in the lower half plane $y < 0$.

Two unknown functions:

\[ v := \partial_x u \]
\[ w := \partial_y u , \]

lead to the form:

\[ y \partial_x v - \partial_y w = 0 , \]

which gives a formally deterministic system, but not symmetric.
Tricomi’s equation

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lead to the form:

\[ y \partial_x v - \partial_y w = 0 , \]

which gives a formally deterministic system, but not symmetric. The Schwarz symmetries give us more equations, and the following choice leads to a symmetric system:

\[
\begin{align*}
\partial_x u - v &= 0 \\
- y \partial_x v - \partial_y w &= 0 \\
\partial_x w - \partial_y v &= 0 .
\end{align*}
\]
Tricomi’s equation

Again, eliminate $u$ and solve the system of two remaining equations, with unknowns $v$ and $w$: $u_1 := v, u_2 := w$. 
Tricomi's equation

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Any solution of this equation satisfies the symmetric system:

$$A^1 \partial_x u + A^2 \partial_y u = 0,$$

where the matrices are given by:

$$A^1 := \begin{bmatrix} -y & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A^2 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$
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Clearly, \( A^1 \) and \( A^2 \) are symmetric, and for \( y < 0 \) the matrix \( A^1 \) is positive definite — its (simple) eigenvalues are 1 and \(-y\).
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Thus, a symmetric hyperbolic system corresponds to the Tricomi’s equation in the lower half plane.
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Thus, a symmetric *hyperbolic* system corresponds to the Tricomi’s equation in the *lower* half plane.

It is not positive ([KOF1958] — a transformation providing the right form).
Why should one be interested in Friedrichs systems?
- Symmetric hyperbolic systems
- Symmetric positive systems

Classical theory
- Boundary conditions for Friedrichs systems
- Existence, uniqueness, well-posedness

Abstract formulation
- Graph spaces
- Cone formalism of Ern, Guermond and Caplain
- Interdependence of different representations of boundary conditions

Kreĭn space formalism
- Kreĭn spaces
- Equivalence of boundary conditions

What can we say for the Friedrichs operator now?
- Sufficient assumptions
- An example: elliptic equation
- Other second order equations
- Two-field theory
- Non-stationary theory

Homogenisation of Friedrichs systems
- Homogenisation
- Examples: Stationary diffusion and heat equation

Concluding remarks
Boundary conditions

Boundary conditions are enforced via matrix valued boundary field:
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$$A_\nu := \sum_{k=1}^{d} \nu_k A_k \in L^\infty(\Gamma; M_r(\mathbb{R})),$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_d)$ is the outward unit normal on $\Gamma$,.
Boundary conditions are enforced via matrix valued boundary field:

$$A_{\nu} := \sum_{k=1}^{d} \nu_k A_k \in L^\infty (\Gamma; M_r(\mathbb{R})),$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_d)$ is the outward unit normal on $\Gamma$, and

$$M \in L^\infty (\Gamma; M_r(\mathbb{R})).$$
Boundary conditions are enforced via matrix valued boundary field:

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Boundary condition

\[ (A_\nu - M)u|_\Gamma = 0 \]
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Boundary condition

\[ (A_\nu - M)u|_{\Gamma} = 0 \]

allows the treatment of different types of usual boundary conditions.
Assumptions on the boundary matrix $M$

We assume (for ae $x \in \Gamma$) \cite{KOF1958}

\begin{align*}
\text{(FM1)} \quad (\forall \xi \in \mathbb{R}^r) \quad M(x)\xi \cdot \xi & \geq 0, \\
\end{align*}
Assumptions on the boundary matrix $M$

We assume (for ae $x \in \Gamma$) \[^{[KOF1958]}\]

(FM1) \((\forall \xi \in \mathbb{R}^r) \quad M(x)\xi \cdot \xi \geq 0\),

(FM2) \[R^r = \ker\left(A_\nu(x) - M(x)\right) + \ker\left(A_\nu(x) + M(x)\right).\]
Assumptions on the boundary matrix $M$

We assume (for ae $x \in \Gamma$) \cite{KOF1958}

(FM1) \hspace{1cm} \forall \xi \in \mathbb{R}^r \hspace{1cm} M(x)\xi \cdot \xi \geq 0,

(FM2) \hspace{1cm} \mathbb{R}^r = \ker(A_\nu(x) - M(x)) + \ker(A_\nu(x) + M(x)).

Such $M$ is called \textit{the admissible boundary condition}.
Assumptions on the boundary matrix $M$

We assume (for all $x \in \Gamma$) \[ (FM1) \quad (\forall \xi \in \mathbb{R}^r) \quad M(x)\xi \cdot \xi \geq 0, \]

\[ (FM2) \quad \mathbb{R}^r = \ker \left( A_\nu(x) - M(x) \right) + \ker \left( A_\nu(x) + M(x) \right). \]

Such $M$ is called the admissible boundary condition.

The boundary problem: for given $f \in L^2(\Omega; \mathbb{R}^r)$ find $u$ such that

\[
\begin{cases}
  \mathcal{L}u = f \\
  (A_\nu - M)u|_\Gamma = 0
\end{cases}
\]
Different ways to enforce boundary conditions

Instead of

\[(A_\nu - M)u = 0 \quad \text{on } \Gamma,\]

Lax proposed boundary conditions with

\[u(x) \in N(x), \quad x \in \Gamma,\]

where \(N = \{N(x) : x \in \Gamma\}\) is a family of subspaces of \(\mathbb{R}^r\).
Different ways to enforce boundary conditions

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Boundary problem:

\[
\begin{aligned}
\mathcal{L}u &= f \\
\text{s.t.} \quad u(x) &\in N(x), \quad x \in \Gamma.
\end{aligned}
\]
Assumptions on $N$

$maximal boundary conditions:$ (for ae $x \in \Gamma$) \[PDL\]

(FX1) $N(x)$ is non-negative with respect to $A_\nu(x)$:

$$(\forall \xi \in N(x)) \ A_\nu(x)\xi \cdot \xi \geq 0;$$

(FX2) there is no non-negative subspace with respect to $A_\nu(x)$, which contains $N(x)$;
Assumptions on $\mathcal{N}$

**maximal boundary conditions:** (for ae $x \in \Gamma$) [PDL]

(FX1) $\mathcal{N}(x)$ is non-negative with respect to $A_\nu(x)$:

$$\forall \xi \in \mathcal{N}(x) \quad A_\nu(x)\xi \cdot \xi \geq 0;$$

(FX2) there is no non-negative subspace with respect to $A_\nu(x)$, which contains $\mathcal{N}(x)$;

or [RSP&LS1966]

Let $\mathcal{N}(x)$ and $\tilde{\mathcal{N}}(x) := (A_\nu(x)\mathcal{N}(x))^\perp$ satisfy (for ae $x \in \Gamma$)

(FV1) $\forall \xi \in \mathcal{N}(x) \quad A_\nu(x)\xi \cdot \xi \geq 0$

(FV2) $\forall \xi \in \tilde{\mathcal{N}}(x) \quad A_\nu(x)\xi \cdot \xi \leq 0$

(FV2) $\tilde{\mathcal{N}}(x) = (A_\nu(x)\mathcal{N}(x))^\perp$ and $\mathcal{N}(x) = (A_\nu(x)\tilde{\mathcal{N}}(x))^\perp$. 

Equivalence of different descriptions of boundary conditions

**Theorem.** It holds

$$(FM1)–(FM2) \iff (FX1)–(FX2) \iff (FV1)–(FV2),$$

with

$$N(x) := \ker\left(A_\nu(x) - M(x)\right).$$
Theorem. It holds

\[(FM1)\text{–}(FM2) \iff (FX1)\text{–}(FX2) \iff (FV1)\text{–}(FV2),\]

with

\[N(x) := \ker\left(A_\nu(x) - M(x)\right).\]

In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].
Classical results on well-posedness

Friedrichs:
- uniqueness of the classical solution
- existence of a weak solution (under some additional assumptions)
Classical results on well-posedness

Friedrichs:
– uniqueness of the classical solution
– existence of a weak solution (under some additional assumptions)

Contributions:
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
Friedrichs:
- uniqueness of the classical solution
- existence of a *weak* solution (under some additional assumptions)

Contributions:
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
- the meaning of traces for functions in the graph space
- weak well-posedness results under additional assumptions (on $A_\nu$)
- regularity of solution
- numerical treatment
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  Symmetric positive systems

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Concluding remarks
New approach...


– abstract setting (operators on Hilbert spaces)
New approach...


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– intrinsic criterion for the bijectivity of *Friedrichs’ operator*
New approach...


– abstract setting (operators on Hilbert spaces)

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– avoiding the question of traces for functions in the graph space
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– investigation of different formulations of boundary conditions
New approach...


– abstract setting (operators on Hilbert spaces)
– intrinsic criterion for the bijectivity of *Friedrichs’* operator
– avoiding the question of traces for functions in the graph space
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... and new open questions.
Assumptions

\[ L \] — real Hilbert space \((L' \equiv L)\),
\[ \mathcal{D} \subseteq L \] — dense subspace,
Assumptions

$L$ — real Hilbert space \((L' \equiv L)\),
\(\mathcal{D} \subseteq L\) — dense subspace,
\(T, \tilde{T} : \mathcal{D} \rightarrow L\) — linear unbounded operators satisfying
Assumptions

$L$ — real Hilbert space ($L' \equiv L$),
$\mathcal{D} \subseteq L$ — dense subspace,
$T, \tilde{T} : \mathcal{D} \rightarrow L$ — linear unbounded operators satisfying

(T1) \hspace{1cm} (\forall \varphi, \psi \in \mathcal{D}) \hspace{0.5cm} \langle T\varphi | \psi \rangle_L = \langle \varphi | \tilde{T}\psi \rangle_L ;
Assumptions

$L$ — real Hilbert space ($L' \equiv L$),
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$T, \tilde{T} : \mathcal{D} \to L$ — linear unbounded operators satisfying

(T1) \hspace{1cm} (\forall \varphi, \psi \in \mathcal{D}) \hspace{0.5cm} \langle T\varphi \mid \psi \rangle_L = \langle \varphi \mid \tilde{T}\psi \rangle_L ;

(T2) \hspace{1cm} (\exists c > 0)(\forall \varphi \in \mathcal{D}) \hspace{0.5cm} \|(T + \tilde{T})\varphi\|_L \leq c\|\varphi\|_L ;

(T3) \hspace{1cm} (\exists \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \hspace{0.5cm} \langle (T + \tilde{T})\varphi \mid \varphi \rangle_L \geq 2\mu_0\|\varphi\|^2_L.$
Let $\mathcal{D} := C_c^\infty(\Omega; \mathbb{R}^r)$, $L = L^2(\Omega; \mathbb{R}^r)$ and $T, \tilde{T} : \mathcal{D} \to L$ be defined by

$$Tu := \sum_{k=1}^{d} \partial_k (A_k u) + Cu,$$

$$\tilde{T}u := -\sum_{k=1}^{d} \partial_k (A_k^T u) + (C^T + \sum_{k=1}^{d} \partial_k A_k^T)u,$$

where $A_k$ and $C$ are as above (they satisfy (F1)–(F2)).
The Friedrichs operator

Let $\mathcal{D} := C_c^\infty(\Omega; \mathbb{R}^r)$, $L = L^2(\Omega; \mathbb{R}^r)$ and $T, \tilde{T} : \mathcal{D} \to L$ be defined by

$$Tu := \sum_{k=1}^{d} \partial_k (A_k u) + Cu,$$

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where $A_k$ and $C$ are as above (they satisfy (F1)–(F2)).

Then $T$ and $\tilde{T}$ satisfy (T1)–(T3)
The Friedrichs operator

Let $\mathcal{D} := C^\infty_c(\Omega; \mathbb{R}^r)$, $L = L^2(\Omega; \mathbb{R}^r)$ and $T, \tilde{T} : \mathcal{D} \rightarrow L$ be defined by

$$Tu := \sum_{k=1}^{d} \partial_k(A_k u) + Cu,$$

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where $A_k$ and $C$ are as above (they satisfy (F1)–(F2)).

Then $T$ and $\tilde{T}$ satisfy (T1)–(T3)

... fits in this framework.
(\mathcal{D}, \langle \cdot | \cdot \rangle_T) is an inner product space, where
\langle \cdot | \cdot \rangle_T := \langle \cdot | \cdot \rangle_L + \langle T\cdot | T\cdot \rangle_L.

\| \cdot \|_T is called graph norm.
Prolongations

\((\mathcal{D}, \langle \cdot | \cdot \rangle_T)\) is an inner product space, where

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\(W_0\) — the completion of \(\mathcal{D}\) in the graph norm
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\(W_0\) — the completion of \(\mathcal{D}\) in the graph norm

\(T, \tilde{T} : \mathcal{D} \longrightarrow L\) are continuous with respect to \((\| \cdot \|_T, \| \cdot \|_L)\) . . . extension by density to \(\mathcal{L}(W_0; L)\).
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\(T, \tilde{T} : \mathcal{D} \rightarrow L\) are continuous with respect to \((\| \cdot \|_T, \| \cdot \|_L)\) ... extension by density to \(\mathcal{L}(W_0; L)\).

The following embedding are dense and continuous:

\[ W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'. \]
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Let \(\tilde{T}^* \in \mathcal{L}(L; W_0')\) be the adjoint operator of \(\tilde{T} : W_0 \rightarrow L\)

\[(\forall u \in L)(\forall v \in W_0) \quad W_0' \langle \tilde{T}^* u, v \rangle_{W_0} = \langle u | \tilde{T} v \rangle_L.\]
Prolongations

$(\mathcal{D}, \langle \cdot \mid \cdot \rangle_T)$ is an inner product space, where

$$\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle_L + \langle T \cdot \mid T \cdot \rangle_L.$$ 

$\| \cdot \|_T$ is called **graph norm**.

$W_0$ — the completion of $\mathcal{D}$ in the graph norm

$T, \tilde{T} : \mathcal{D} \longrightarrow L$ are continuous with respect to $(\| \cdot \|_T, \| \cdot \|_L)$ ... extension by density to $\mathcal{L}(W_0; L)$.

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Therefore $T = \tilde{T}^* |_{W_0}$
Prolongations

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\(T, \tilde{T} : \mathcal{D} \rightarrow L\) are continuous with respect to \((\| \cdot \|_T, \| \cdot \|_L)\) ... extension by density to \(\mathcal{L}(W_0; L)\).

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\[
W_0 \hookrightarrow L \equiv L' \hookrightarrow W'_0.
\]

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\[
(\forall u \in L)(\forall v \in W_0) \quad W'_0 \langle \tilde{T}^* u, v \rangle_{W_0} = \langle u | \tilde{T} v \rangle_L.
\]

Therefore \(T = \tilde{T}^*|_{W_0}\), and analogously \(\tilde{T} = T^*|_{W_0}\).
Prolongations

\((\mathcal{D}, \langle \cdot | \cdot \rangle_T)\) is an inner product space, where

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\]

\(\| \cdot \|_T\) is called graph norm.

\(W_0\) — the completion of \(\mathcal{D}\) in the graph norm

\(T, \tilde{T} : \mathcal{D} \to L\) are continuous with respect to \((\| \cdot \|_T, \| \cdot \|_L)\) ... extension by density to \(L(W_0; L)\).

The following embedding are dense and continuous:

\[
W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'.
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Let \(\tilde{T}^* \in L(L; W_0')\) be the adjoint operator of \(\tilde{T} : W_0 \to L\)

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\]

Therefore \(T = \tilde{T}^*|_{W_0}\), and analogously \(\tilde{T} = T^*|_{W_0}\).

Abusing notation: \(T, \tilde{T} \in L(L; W_0')\) ... (T1)–(T3)
Lemma. The graph space

\[ W := \{ u \in L : Tu \in L \} = \{ u \in L : \tilde{T}u \in L \}, \]

is a Hilbert space with respect to \( \langle \cdot | \cdot \rangle_T \).
Lemma. The graph space

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Problem: for given \( f \in L \) find \( u \in W \) such that \( Tu = f \).
Lemma. The graph space

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is a Hilbert space with respect to \( \langle \cdot | \cdot \rangle_T \).

Problem: for given \( f \in L \) find \( u \in W \) such that \( Tu = f \).

Find sufficient conditions on \( V \subseteq W \) such that \( T|_V : V \rightarrow L \) is an isomorphism.
Boundary operator \( D \in \mathcal{L}(W; W') \):

\[
W' \langle Du, v \rangle_W := \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L , \quad u, v \in W .
\]
Boundary operator $D \in \mathcal{L}(W; W')$:

$$W' \langle Du, v \rangle_W := \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L, \quad u, v \in W.$$  

**Lemma.** $D$ is symmetric and satisfies

$$\ker D = W_0$$

$$\text{im } D = W_0^0 := \{ g \in W' : (\forall u \in W_0) \quad W' \langle g, u \rangle_W = 0 \} .$$

In particular, **im $D$ is closed in $W'$**.
Boundary operator $D \in \mathcal{L}(W; W')$:

$$W'\langle Du, v \rangle_W := \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L, \quad u, v \in W.$$ 

Lemma. $D$ is symmetric and satisfies

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$$\text{im } D = W_0^0 := \{ g \in W' : (\forall u \in W_0) \ W'\langle g, u \rangle_W = 0 \}.$$ 

In particular, $\text{im } D$ is closed in $W'$.

If $T$ is the Friedrichs operator $\mathcal{L}$, then for $u, v \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^r)$ we have

$$W'\langle Du, v \rangle_W = \int \mathbf{A}_\nu(x) u_{\mid \Gamma}(x) \cdot v_{\mid \Gamma}(x) dS(x).$$
Well-posedness theorem

Let $V$ and $\tilde{V}$ be subspaces of $W$ that satisfy

(V1) \[ (\forall u \in V) \quad \langle Du, u \rangle_W \geq 0 \]

(V2) \[ (\forall v \in \tilde{V}) \quad \langle Dv, v \rangle_W \leq 0 \]

(V2) \[ V = D(\tilde{V})^0, \quad \tilde{V} = D(V)^0. \]

(cone formalism)
Well-posedness theorem

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\begin{align*}
(V1) & \quad \forall u \in V \quad \langle Du, u \rangle_W \geq 0 \\
& \quad \forall v \in \tilde{V} \quad \langle Dv, v \rangle_W \leq 0
\end{align*}

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(V2) & \quad V = D(\tilde{V})^0, \quad \tilde{V} = D(V)^0.
\end{align*}

(cone formalism)

**Theorem.** Under assumptions $(T1) - (T3)$ and $(V1) - (V2)$, the operators $T|_V : V \to L$ and $\tilde{T}|_{\tilde{V}} : \tilde{V} \to L$ are isomorphisms.

[AE&JLG&GC2007]
Correspondence with *classical* assumptions

(V1) \( (\forall u \in V) \quad w' \langle Du, u \rangle_W \geq 0, \)

(V2) \( (\forall v \in \tilde{V}) \quad w' \langle Dv, v \rangle_W \leq 0, \)

(V2) \( V = D(\tilde{V})^0, \quad \tilde{V} = D(V)^0, \)
Correspondence with classical assumptions

(V1) 
\[ (\forall u \in V) \quad \langle Du, u \rangle_W \geq 0, \]
\[ (\forall v \in \tilde{V}) \quad \langle Dv, v \rangle_W \leq 0, \]

(V2) 
\[ V = D(\tilde{V})^0, \quad \tilde{V} = D(V)^0, \]

(FV1) 
\[ (\forall \xi \in N(x)) \quad A_\nu(x)\xi \cdot \xi \geq 0, \]
\[ (\forall \xi \in \tilde{N}(x)) \quad A_\nu(x)\xi \cdot \xi \leq 0, \]

(FV2) 
\[ \tilde{N}(x) = (A_\nu(x)N(x))^\perp \quad \text{and} \quad N(x) = (A_\nu(x)\tilde{N}(x))^\perp, \]
\[ \text{(for ae } x \in \Gamma') \]
Other sets of conditions in the classical setting (recall)

**maximal boundary conditions:** (for ae $x \in \Gamma$)

(FX1) \hspace{1cm} (\forall \xi \in N(x)) \hspace{0.5cm} A_\nu(x)\xi \cdot \xi \geq 0,

(FX2) \hspace{1cm} \text{there is no non-negative subspace with respect to } A_\nu(x), \text{ which contains } N(x),

**admissible boundary conditions:** there exists a matrix function $M : \Gamma \rightarrow M_r(\mathbb{R})$ such that (for ae $x \in \Gamma$)

(FM1) \hspace{1cm} (\forall \xi \in \mathbb{R}^r) \hspace{0.5cm} M(x)\xi \cdot \xi \geq 0,

(FM2) \hspace{1cm} \mathbb{R}^r = \ker\left(A_\nu(x) - M(x)\right) + \ker\left(A_\nu(x) + M(x)\right).
maximal boundary conditions: (for ae \( x \in \Gamma \))

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Correspondence — maximal b.c.

**maximal boundary conditions:** (for ae \( x \in \Gamma \))

\[(\text{FX1}) \quad (\forall \xi \in N(x)) \quad A_\nu(x) \xi \cdot \xi \geq 0,\]

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**subspace \( V \) is maximal non-negative with respect to \( D \):**

\[(\text{X1}) \quad V \text{ is non-negative with respect to } D: \quad (\forall v \in V) \quad W' \langle Dv, v \rangle_W \geq 0,\]

\[(\text{X2}) \quad \text{there is no non-negative subspace with respect to } D \text{ that contains } V.\]
Correspondence — admissible b.c.

**admissible boundary condition:** there exist a matrix function $M : \Gamma \longrightarrow \mathbb{M}_r(\mathbb{R})$ such that (for ae $x \in \Gamma$)

(FM1) $\quad (\forall \xi \in \mathbb{R}^r) \quad M(x)\xi \cdot \xi \geq 0$,

(FM2) $\quad \mathbb{R}^r = \ker(A_{\nu}(x) - M(x)) + \ker(A_{\nu}(x) + M(x))$. 
Correspondence — admissible b.c.

**admissible boundary condition:** there exist a matrix function $M : \Gamma \rightarrow M_r(\mathbb{R})$ such that (for all $x \in \Gamma$)

(FM1) \hspace{1cm} (\forall \xi \in \mathbb{R}^r) \hspace{0.5cm} M(x)\xi \cdot \xi \geq 0,

(FM2) \hspace{1cm} \mathbb{R}^r = \ker\left(A_{\nu}(x) - M(x)\right) + \ker\left(A_{\nu}(x) + M(x)\right).

**admissible boundary condition:** there exist $M \in \mathcal{L}(W; W')$ that satisfy

(M1) \hspace{1cm} (\forall u \in W) \hspace{0.5cm} \langle Mu, u \rangle_W \geq 0,

(M2) \hspace{1cm} W = \ker(D - M) + \ker(D + M).
Theorem. (classical) It holds

\[(FM1)–(FM2) \iff (FV1)–(FV2) \iff (FX1)–(FX2),\]

with

\[N(x) := \ker \left( A_\nu(x) - M(x) \right).\]
Equivalence of different descriptions of b.c.

Theorem. (classical) It holds

\[(FM1)–(FM2) \iff (FV1)–(FV2) \iff (FX1)–(FX2),\]

with

\[N(x) := \ker\left(A_\nu(x) - M(x)\right).\]

Theorem. (A. Ern, J.-L. Guermond, G. Caplain) It holds

\[(M1)–(M2) \implies (V1)–(V2) \implies (X1)–(X2),\]

with

\[V := \ker(D - M).\]
Theorem. Let $V$ and $\tilde{V}$ satisfy $(V1)$–$(V2)$, and suppose that there exist operators $P \in \mathcal{L}(W; V)$ and $Q \in \mathcal{L}(W; \tilde{V})$ such that

$$(\forall v \in V) \quad D(v - Pv) = 0,$$

$$(\forall v \in \tilde{V}) \quad D(v - Qv) = 0,$$

$$DPQ = DQP.$$ 

Let us define $M \in \mathcal{L}(W; W')$ (for $u, v \in W$) with

$$w'\langle Mu, v \rangle_w = w'\langle DPu, Pv \rangle_w - w'\langle DQu, Qv \rangle_w$$

$$+ w'\langle D(P + Q - PQ)u, v \rangle_w - w'\langle Du, (P + Q - PQ)v \rangle_w.$$

Then $V := \ker(D - M), \tilde{V} := \ker(D + M^*), \text{ and } M \text{ satisfies (M1)–(M2)}.$
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Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and $M$ satisfies (M1)–(M2).

Lemma. Suppose additionally that $V + \tilde{V}$ is closed. Then the operators $P$ and $Q$ from previous theorem do exist.
Lemma. (K. Burazin, N.A.)

If $\text{codim } W_0 (= \dim W/W_0)$ is finite, then the set $V + \tilde{V}$ is closed whenever $V$ and $\tilde{V}$ satisfy (V1)–(V2).
When this is satisfied?

Lemma. (K. Burazin, N.A.)
If \( \text{codim} \, W_0(= \dim \, W/W_0) \) is finite, then the set \( V + \tilde{V} \) is closed whenever \( V \) and \( \tilde{V} \) satisfy (V1)–(V2).

In one dimension (ode-s) this is the case.
The classification of admissible conditions can be given.
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However, in general this is not true, and for many interesting situations \( V + \tilde{V} \) is NOT closed.
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Sufficient conditions for a counter example:
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Sufficient conditions for a counter example:

Theorem. (K. Burazin, N.A.)
Let subspaces \( V \) and \( \tilde{V} \) of space \( W \) satisfy (V1)–(V2), \( V \cap \tilde{V} = W_0 \), and \( W \neq V + \tilde{V} \).
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Sufficient conditions for a counter example:

Theorem. (K. Burazin, N.A.)

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Then \( V + \tilde{V} \) is not closed in \( W \).

Moreover, there do not exist operators \( P \) and \( Q \) with desired properties.
Counter example

Let $\Omega \subseteq \mathbb{R}^2$, $\mu > 0$ and $f \in L^2(\Omega)$ be given. Scalar elliptic equation

$$-\Delta u + \mu u = f$$
Countereample

Let \( \Omega \subseteq \mathbb{R}^2 \), \( \mu > 0 \) and \( f \in L^2(\Omega) \) be given. Scalar elliptic equation

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\]

can be written as Friedrichs’ system:

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\begin{cases}
p + \nabla u = 0 \\
\mu u + \text{div}p = f
\end{cases}
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Counter example

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$$\begin{cases} p + \nabla u = 0 \\ \mu u + \text{div} p = f \end{cases}.$$

Then $W = L^2_{\text{div}}(\Omega) \times H^1(\Omega)$. 
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Then $W = L^2_{\text{div}}(\Omega) \times H^1(\Omega)$. For $\alpha > 0$ we define (Robin b. c.)

$$V := \{(p, u)^\top \in W : \mathcal{T}_{\text{div}} p = \alpha \mathcal{T}_{H^1} u\},$$

$$\tilde{V} := \{(r, v)^\top \in W : \mathcal{T}_{\text{div}} r = -\alpha \mathcal{T}_{H^1} v\}.$$
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Lemma.

The above $V$ and $\tilde{V}$ satisfy (V1)-(V2), $V \cap \tilde{V} = W_0$ and $V + \tilde{V} \neq W$.  

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Let \( \Omega \subseteq \mathbb{R}^2 \), \( \mu > 0 \) and \( f \in L^2(\Omega) \) be given. Scalar elliptic equation

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**Lemma.**

The above \( V \) and \( \tilde{V} \) satisfy (V1)-(V2), \( V \cap \tilde{V} = W_0 \) and \( V + \tilde{V} \neq W \).

There exists an operator \( M \in \mathcal{L}(W; W') \), that satisfies (M1)–(M2) and \( V = \ker(D - M) \).

\[\square\]
Why should one be interested in Friedrichs systems?
- Symmetric hyperbolic systems
- Symmetric positive systems

Classical theory
- Boundary conditions for Friedrichs systems
- Existence, uniqueness, well-posedness

Abstract formulation
- Graph spaces
- Cone formalism of Ern, Guermond and Caplain
- Interdependence of different representations of boundary conditions

Kreõn space formalism
- Kreõn spaces
- Equivalence of boundary conditions

What can we say for the Friedrichs operator now?
- Sufficient assumptions
- An example: elliptic equation
- Other second order equations
- Two-field theory
- Non-stationary theory

Homogenisation of Friedrichs systems
- Homogenisation
- Examples: Stationary diffusion and heat equation

Concluding remarks
New notation

\[
[u \mid v] := \langle Du, v \rangle_W = \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L, \quad u, v \in W
\]

is an indefinite inner product on \(W\).
New notation

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(V1) \hspace{1cm} (\forall v \in V) \quad [v \mid v] \geq 0,

(V2) \hspace{1cm} V = \tilde{V}^{[\perp]}, \quad \tilde{V} = V^{[\perp]}.

([\perp] \text{ stands for } [\cdot \mid \cdot]-\text{orthogonal complement})
New notation

\[
[u \mid v] := W' \langle Du, v \rangle_W = \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L, \quad u, v \in W
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is an indefinite inner product on \( W \).

(V1) \[(\forall v \in V) \ [v \mid v] \geq 0, \]
\[(\forall v \in \tilde{V}) \ [v \mid v] \leq 0; \]

(V2) \[V = \tilde{V}^{[\perp]}, \quad \tilde{V} = V^{[\perp]} \cdot \]

([\perp] stands for \([\cdot \mid \cdot]\)-orthogonal complement)

**subspace \( V \) is maximal non-negative in \((W, [\cdot \mid \cdot])\):**

(X1) \[V \text{ is non-negative in } (W, [\cdot \mid \cdot]): \ (\forall v \in V) \ [v \mid v] \geq 0, \]

(X2) \[\text{there is no non-negative subspace in } (W, [\cdot \mid \cdot]) \text{ containing } V. \]
(\(W, [\cdot | \cdot]\)) is not a Krešin space – it is a degenerate space, because its Gramm operator \(G := j \circ D\) (\(j : W' \rightarrow W\) is the canonical isomorphism) has large kernel:

\[
\ker G = W_0.
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**Theorem.** If \(G\) is the Gramm operator of the space \(W\), then the quotient space \(\hat{W} := W/\ker G\) is a Krešn space if and only if \(\text{im} G\) is closed.
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\( \hat{W} := W/W_0 \) is the Kreĭn space, with

\[ [ \hat{u} | \hat{v} ] := [ u | v ], \quad u, v \in W. \]
(\(W, [\cdot | \cdot]\)) is not a Krešin space – it is a degenerate space, because its Gramm operator \(G := j \circ D\) \((j : W' \longrightarrow W\) is the canonical isomorphism) has large kernel:

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\(\hat{W} := W/W_0\) *is the Krešin space, with*

\[
[\hat{u} | \hat{v}] := [u | v], \quad u, v \in W.
\]

Important: \(\text{im} D\) is closed and \(\ker D = W_0\).
**Lemma.** Let $U \supseteq W_0$ and $Y$ be subspaces of $W$. Then

a) $U$ is closed if and only if $\widehat{U} := \{\hat{v} : v \in U\}$ is closed in $\hat{W}$;

b) $(\widehat{U + Y}) = \{u + v + W_0 : u \in U, v \in Y\} = \widehat{U} + \widehat{Y}$;

c) $U + Y$ is closed if and only if $\widehat{U} + \widehat{Y}$ is closed;

d) $(\widehat{Y})^{\perp} = \widehat{Y}^{\perp}$.

e) if $Y$ is maximal non-negative (non-positive) in $W$, than $\widehat{Y}$ is maximal non-negative (non-positive) in $\hat{W}$;

f) if $\widehat{U}$ is maximal non-negative (non-positive) in $\hat{W}$, then $U$ is maximal non-negative (non-positive) in $W$. 

\[\blacksquare\]
\[(V1)-(V2) \iff (X1)-(X2)\]

**Theorem.**  
a) If subspaces \( V \) and \( \tilde{V} \) satisfy \((V1)-(V2)\), then \( V \) is maximal non-negative in \( W \) (satisfies \((X1)-(X2)\)) and \( \tilde{V} \) is maximal non-positive in \( W \).

b) If \( V \) is maximal non-negative in \( W \), then \( V \) and \( \tilde{V} := V^{\perp} \) satisfy \((V1)-(V2)\).

\[\blacksquare\]
\[(M1)-(M2) \implies (V1)-(V2) \quad \text{(recall)}\]

**Theorem.** \([EGC]\) \((T1)-(T3)\) and \(M \in \mathcal{L}(W; W')\) satisfy \((M)\) imply

\[V := \ker(D - M) \quad \text{and} \quad \tilde{V} := \ker(D + M^*) \quad \text{satisfy} \ (V).\]

**Corollary.** Under above assumptions

\[T|_{\ker(D-M)} : \ker(D - M) \longrightarrow L \quad \text{and} \quad \tilde{T}|_{\ker(D+M^*)} : \ker(D + M^*) \longrightarrow L\]

are isomorphisms.
(M1)–(M2) ← (V1)–(V2) (recall)

**Theorem.** Let $V$ and $\tilde{V}$ satisfy (V1)–(V2), and suppose that there exist operators $P \in \mathcal{L}(W; V)$ and $Q \in \mathcal{L}(W; \tilde{V})$ such that

\[
\begin{align*}
(\forall v \in V) \quad D(v - Pv) &= 0, \\
(\forall v \in \tilde{V}) \quad D(v - Qv) &= 0,
\end{align*}
\]

\[DPQ = DQP.\]

Let us define $M \in \mathcal{L}(W; W')$ (for $u, v \in W$) with

\[
w'(Mu, v)_W = w'(DPu, Pv)_W - w'(DQu, Qv)_W \\
+ w'(D(P + Q - PQ)u, v)_W - w'(Du, (P + Q - PQ)v)_W.
\]

Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and $M$ satisfies (M1)–(M2).

**Lemma.** Suppose additionally that $V + \tilde{V}$ is closed. Then the operators $P$ and $Q$ from previous theorem do exist.
\((\text{M1})–(\text{M2}) \leftarrow (\text{V1})–(\text{V2}) \) (recall)

**Theorem.** Let \( V \) and \( \tilde{V} \) satisfy \((\text{V1})–(\text{V2})\), and suppose that there exist operators \( P \in \mathcal{L}(W;V) \) and \( Q \in \mathcal{L}(W;\tilde{V}) \) such that

\[
(\forall v \in V) \quad D(v - Pv) = 0,
\]

\[
(\forall v \in \tilde{V}) \quad D(v - Qv) = 0,
\]

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DPQ = DQP.
\]

Let us define \( M \in \mathcal{L}(W;W') \) (for \( u,v \in W \)) with

\[
W' \langle Mu,v \rangle_W = W' \langle DPu,Pv \rangle_W - W' \langle Du,(P + Q - PQ)v \rangle_W.
\]

Then \( V := \ker(D - M) \), \( \tilde{V} := \ker(D + M^*) \), and \( M \) satisfies \((\text{M1})–(\text{M2})\). \( \blacksquare \)

**Lemma.** Suppose additionally that \( V + \tilde{V} \) is closed. Then the operators \( P \) and \( Q \) from previous theorem do exist. \( \blacksquare \)

Closedness of \( V + \tilde{V} \) is actually equivalent to the existence of operators \( P \) and \( Q \).
On existence of $P$ and $Q$

Our original approach was indirect:
Firstly, the existence of $P$ and $Q$ implies the existence of certain projectors in the quotient Kreĭn space; more precisely:

$$\hat{P} \hat{w} := \overline{P} w, \quad \hat{Q} \hat{w} := \overline{Q} w, \quad w \in W$$

the projectors $\hat{P}, \hat{Q} : \hat{W} \rightarrow \hat{W}$ are defined, satisfying

$$\hat{P}^2 = \hat{P} \quad \text{and} \quad \hat{Q}^2 = \hat{Q},$$

$$\text{im} \hat{P} = \hat{V} \quad \text{and} \quad \text{im} \hat{Q} = \hat{V},$$

$$\hat{P} \hat{Q} = \hat{Q} \hat{P}.$$ 

Secondly, this allowed us to prove the existence of corresponding projectors on $W$. 
Theorem. \( V, \tilde{V} \) are two closed subspaces of \( W \) that satisfy \( W_0 \subseteq V \cap \tilde{V} \), then the following statements are equivalent:

\begin{itemize}
  \item[a)] There exist operators \( P \in \mathcal{L}(W; V) \) and \( Q \in \mathcal{L}(W; \tilde{V}) \), such that
    \[
    (\forall v \in V) \quad D(v - Pv) = 0 , \\
    (\forall v \in \tilde{V}) \quad D(v - Qv) = 0 , \\
    DPQ = DQP .
    \]
\end{itemize}
(M1)–(M2) ⇐ (V1)–(V2) (direct proof)

**Theorem.** If \( V, \tilde{V} \) are two closed subspaces of \( W \) that satisfy \( W_0 \subseteq V \cap \tilde{V} \), then the following statements are equivalent:

a) There exist operators \( P \in \mathcal{L}(W; V) \) and \( Q \in \mathcal{L}(W; \tilde{V}) \), such that

\[
\forall v \in V \quad D(v - P v) = 0, \\
\forall v \in \tilde{V} \quad D(v - Q v) = 0, \\
DPQ = DQP.
\]

b) There exist projectors \( P', Q' \in \mathcal{L}(W; W) \), such that

\[
P'^2 = P' \quad \text{and} \quad Q'^2 = Q', \\
\text{im} P' = V \quad \text{and} \quad \text{im} Q' = \tilde{V}, \\
P'Q' = Q'P'.
\]
Theorem. If $V, \tilde{V}$ are two closed subspaces of $W$ that satisfy $W_0 \subseteq V \cap \tilde{V}$, then the following statements are equivalent:

a) There exist operators $P \in \mathcal{L}(W; V)$ and $Q \in \mathcal{L}(W; \tilde{V})$, such that

\begin{align*}
(\forall v \in V) & \quad D(v - Pv) = 0, \\
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DPQ &= DQP.
\end{align*}

b) There exist projectors $P', Q' \in \mathcal{L}(W; W)$, such that

\begin{align*}
P'^2 &= P' \quad \text{and} \quad Q'^2 = Q', \\
im P' &= V \quad \text{and} \quad im Q' = \tilde{V}, \\
P'Q' &= Q'P'.
\end{align*}

(b) is equivalent to closedness of $V + \tilde{V}$. 

$(M1)-(M2) \iff (V1)-(V2)$  (direct proof)
Theorem.

a) $V, \tilde{V} \leq W$ satisfy (V), and exists a closed subspace $W_2 \subseteq C^\tau$ of $W$, $V \dot{\bot} W_2 = W$, then there exist an operator $M \in L(W; W')$ satisfying (M) and $V = \ker(D - M)$.

If we define $W_1$ as orthogonal complement of $W_0$ in $V$, so that $W = W_1 \dot{\bot} W_0 \dot{\bot} W_2$, and denote by $R_1, R_0, R_2$ projectors that correspond to above direct sum, then one such operator is given with $M = D(R_1 - R_2)$. 
Theorem.

a) $V, \tilde{V} \leq W$ satisfy (V), and exists a closed subspace $W_2 \subseteq C^-$ of $W$, $V \dot{+} W_2 = W$, then there exist an operator $M \in \mathcal{L}(W; W')$ satisfying (M) and $V = \ker(D - M)$.

If we define $W_1$ as orthogonal complement of $W_0$ in $V$, so that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and denote by $R_1, R_0, R_2$ projectors that correspond to above direct sum, then one such operator is given with $M = D(R_1 - R_2)$.

b) $M \in \mathcal{L}(W; W')$ an operator satisfying (M1)–(M2), $V := \ker(D - M)$.

For $W_2$, the orthogonal complement of $W_0$ in $\ker(D + M)$, $W_2 \subseteq C^-$ is closed, $V \dot{+} W_2 = W$, and $M$ coincide with the operator in (a).
Theorem. 

a) $V, \tilde{V} \leq W$ satisfy (V), and exists a closed subspace $W_2 \subseteq C^-$ of $W$, $V \dot{+} W_2 = W$, then there exist an operator $M \in \mathcal{L}(W; W')$ satisfying (M) and $V = \ker(D - M)$. 

If we define $W_1$ as orthogonal complement of $W_0$ in $V$, so that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and denote by $R_1, R_0, R_2$ projectors that correspond to above direct sum, then one such operator is given with $M = D(R_1 - R_2)$. 

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For $W_2$, the orthogonal complement of $W_0$ in $\ker(D + M)$, $W_2 \subseteq C^-$ is closed, $V \dot{+} W_2 = W$, and $M$ coincide with the operator in (a). 

Lemma. Let $W''_2 \leq W$ satisfies $W''_2 \subseteq C^-$ and $W''_2 \dot{+} V = W$. Then there is a closed subspace $W_2$ of $W$, such that $W_2 \subseteq C^-$ and $W_2 \dot{+} V = W$. 

(M1)–(M2) \iff (V1)–(V2) (cont.)
(M1)–(M2) ⇐ (V1)–(V2)  (cont.)

**Lemma.** If \( U_1 + U_2 = W \) for some subspaces \( U_1 \subseteq C^+ \) and \( U_2 \subseteq C^- \) of \( W \), then \( U_1 \cap U_2 \subseteq W_0 \).

*If additionally \( U_1 \) is maximal nonnegative and \( U_2 \) maximal nonpositive, then \( U_1 \cap U_2 = W_0 \).*

\[ \square \]
Lemma. If $U_1 + U_2 = W$ for some subspaces $U_1 \subseteq C^+$ and $U_2 \subseteq C^-$ of $W$, then $U_1 \cap U_2 \subseteq W_0$.

If additionally $U_1$ is maximal nonnegative and $U_2$ maximal nonpositive, then $U_1 \cap U_2 = W_0$.

Theorem. For a maximal nonnegative subspace $V$ of $W$, it is equivalent:

a) There is a maximal nonpositive subspace $W_2$ of $W$, such that $W_2 + V = W$;
b) There is a nonpositive subspace $W_2$ of $\hat{W}$, such that $W_2 + \hat{V} = \hat{W}$.
Lemma. If $U_1 + U_2 = W$ for some subspaces $U_1 \subseteq C^+$ and $U_2 \subseteq C^-$ of $W$, then $U_1 \cap U_2 \subseteq W_0$.

If additionally $U_1$ is maximal nonnegative and $U_2$ maximal nonpositive, then $U_1 \cap U_2 = W_0$.

Theorem. For a maximal nonnegative subspace $V$ of $W$, it is equivalent:

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b) There is a nonpositive subspace $W_2$ of $\hat{W}$, such that $W_2 + \hat{V} = \hat{W}$.

Corollary. The conditions (V) and (M) are equivalent.
Some used properties

**Theorem.**

a) $[\cdot | \cdot]$-orthogonal complement of a maximal non-negative (non-positive) subspace is non-positive (non-negative).

b) Each maximal semi-definite subspace contains all isotropic vectors in $W$.

c) If $L$ is a non-negative (non-positive) subspace of a Krein space, such that $L^{[\perp]}$ is non-positive (non-negative), then $\text{Cl} \setminus L$ is maximal non-negative (non-positive).

d) Each maximal semi-definite subspace of a Krein space is closed.

e) A subspace $L$ of a Krein space is closed if and only if $L = L^{[\perp][\perp]}$.

f) For a subspace $L$ of a Krein space $W$ it holds

$$L \cap L^{[\perp]} = \{0\} \iff \text{Cl} (L + L^{[\perp]}) = W.$$
Why should one be interested in Friedrichs systems?
  Symmetric hyperbolic systems
  Symmetric positive systems

Classical theory
  Boundary conditions for Friedrichs systems
  Existence, uniqueness, well-posedness

Abstract formulation
  Graph spaces
  Cone formalism of Ern, Guermond and Caplain
  Interdependence of different representations of boundary conditions

Kreĭn space formalism
  Kreĭn spaces
  Equivalence of boundary conditions

What can we say for the Friedrichs operator now?
  Sufficient assumptions
  An example: elliptic equation
  Other second order equations
  Two-field theory
  Non-stationary theory

Homogenisation of Friedrichs systems
  Homogenisation
  Examples: Stationary diffusion and heat equation

Concluding remarks
Posing and solving the problem

Problem: for given $f \in L$ find $u \in W$ such that $Tu = f$. 
Posing and solving the problem

*Problem*: for given $f \in L$ find $u \in W$ such that $Tu = f$.

*Boundary operator* $D \in \mathcal{L}(W; W')$:

$$W' \langle Du, v \rangle_W := \langle Tu | v \rangle_L - \langle u | \tilde{T}v \rangle_L, \quad u, v \in W.$$
Posing and solving the problem

**Problem:** for given $f \in L$ find $u \in W$ such that $Tu = f$.

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$$W'\langle Du, v \rangle_W := \langle Tu | v \rangle_L - \langle u | \tilde{T}v \rangle_L, \quad u, v \in W.$$

**Theorem.** Assume $(T1) - (T3)$ and the existence of $M \in \mathcal{L}(W; W')$ satisfying

(M1) \hspace{1cm} (\forall u \in W) \quad W'\langle Mu, u \rangle_W \geq 0,

(M2) \hspace{1cm} W = \ker(D - M) + \ker(D + M).

Then the operator $T|_{\ker(D - M)} : \ker(D - M) \longrightarrow L$ is an isomorphism.
Application to the classical theory

Let \( \mathcal{D} := C_c^\infty(\Omega; \mathbb{R}^r) \), \( L = L^2(\Omega; \mathbb{R}^r) \) and \( T, \tilde{T} : \mathcal{D} \to L \) be defined by

\[
Tu := \sum_{k=1}^d \partial_k (A_k u) + Cu,
\]

\[
\tilde{T}u := - \sum_{k=1}^d \partial_k (A_k^\top u) + (C^\top + \sum_{k=1}^d \partial_k A_k^\top)u,
\]

where \( A_k \) and \( C \) are as before (they satisfy (F1)–(F2)).

Then \( T \) and \( \tilde{T} \) satisfy (T1)–(T3)
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\]

where \( A_k \) and \( C \) are as before (they satisfy (F1)–(F2)).

Then \( T \) and \( \tilde{T} \) satisfy (T1)–(T3) and

\[
W = \left\{ u \in L^2(\Omega; \mathbb{R}^r) : \sum_{k=1}^{d} \partial_k(A_k u) + Cu \in L^2(\Omega; \mathbb{R}^r) \right\}.
\]
Correlation of boundary conditions

Classical theory: \[(A_\nu - M)u_\Gamma = 0,\]
Correlation of boundary conditions

**Classical theory:** \[(A_\nu - M)u|_\Gamma = 0,\]

with \(M \in L^\infty(\partial\Omega; M_r(R))\) satisfying (for ae \(x \in \Gamma\))

\[
(FM1) \quad (\forall \xi \in R^r) \quad M(x)\xi \cdot \xi \geq 0,
\]

\[
(FM2) \quad R^r = \ker(A_\nu(x) - M(x)) + \ker(A_\nu(x) + M(x)).
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Correlation of boundary conditions

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*Abstract theory:* \(u \in \ker(D - M),\)
Correlation of boundary conditions

**Classical theory:** \((A_\nu - M)u|_\Gamma = 0,\)
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**Abstract theory:** \(u \in \ker(D - M),\)
with \(M \in \mathcal{L}(W; W')\) satisfying

\[(M1) \quad (\forall u \in W) \quad W'\langle Mu, u\rangle_W \geq 0,\]

\[(M2) \quad W = \ker(D - M) + \ker(D + M).\]
Correlation of boundary conditions

**Classical theory:** \((A_{\nu} - M)u|_{\Gamma} = 0,\)

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(FM1) \((\forall \xi \in R^r)\quad M(x)\xi \cdot \xi \geq 0,\)

(FM2) \(R^r = \ker(A_{\nu}(x) - M(x)) + \ker(A_{\nu}(x) + M(x)).\)

**Abstract theory:** \(u \in \ker(D - M),\)

with \(M \in \mathcal{L}(W; W')\) satisfying

(M1) \((\forall u \in W)\quad W'(Mu, u)_{W} \geq 0,\)

(M2) \(W = \ker(D - M) + \ker(D + M).\)

For given matrix field \(M\) is there an operator \(\tilde{M}\) determined by \(M\) in some natural way?
What is a natural way?

Abstract well-posedness result:

\[ T|_{\ker(D - M)} : \ker(D - M) \rightarrow L \text{ is an isomorphism.} \]
Abstract well-posedness result:

\[ T \big|_{\ker(D-M)} : \ker(D - M) \longrightarrow L \text{ is an isomorphism.} \]

should correspond to the

Weak well-posedness result for the original problem:

\[
\begin{cases}
T u = f \\
(A_\nu - M)u|_\Gamma = 0
\end{cases}
\]

meaning that any smooth weak solution is also a classical solution.
Abstract well-posedness result:

\[ T\big|_{\ker(D-M)} : \ker(D-M) \longrightarrow L \text{ is an isomorphism.} \]

should correspond to the

Weak well-posedness result for the original problem:

\[
\begin{cases}
Tu = f \\
(A_\nu - M)u|_\Gamma = 0
\end{cases}
\]

meaning that any smooth weak solution is also a classical solution

i.e. smooth \( u \in \ker(A-M) \) should satisfy \((A_\nu - M)u|_{\partial\Omega} = 0\)
For $u, v \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^r)$ we have

$$W' \langle Du, v \rangle_W = \int_\Gamma A_\nu(x) u|_\Gamma(x) \cdot v|_\Gamma(x) dS(x).$$
Representation of $D$ and $M$ via matrix fields

For $u, v \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^r)$ we have

$$W' \langle Du, v \rangle_W = \int_{\Gamma} A_{\nu}(x)u|_{\Gamma}(x) \cdot v|_{\Gamma}(x) dS(x).$$

For a given field $M$, it is reasonable to seek an operator $M$ of the form

$$(m) \quad W' \langle Mu, v \rangle_W = \int_{\Gamma} M(x)u|_{\Gamma}(x) \cdot v|_{\Gamma}(x) dS(x).$$
Representation of $D$ and $M$ via matrix fields

For $u, v \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^r)$ we have

$$W' \langle Du, v \rangle_W = \int_\Gamma A_\nu(x) u|_\Gamma(x) \cdot v|_\Gamma(x) dS(x).$$

For a given field $M$, it is reasonable to seek an operator $M$ of the form

(m) \quad W' \langle M u, v \rangle_W = \int_\Gamma M(x) u|_\Gamma(x) \cdot v|_\Gamma(x) dS(x).

\ldots then smooth $u \in \ker(D - M)$ would satisfy $(A_\nu - M)u|_\Gamma = 0$
Representation of $D$ and $M$ via matrix fields

For $u, v \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^r)$ we have

$$W' \langle Du, v \rangle_W = \int_{\Gamma} A_\nu(x)u_{\Gamma}(x) \cdot v_{\Gamma}(x)dS(x).$$

For a given field $M$, it is reasonable to seek an operator $M$ of the form

(\text{m})  

$$W' \langle Mu, v \rangle_W = \int_{\Gamma} M(x)u_{\Gamma}(x) \cdot v_{\Gamma}(x)dS(x).$$

... then smooth $u \in \ker(D - M)$ would satisfy $(A_\nu - M)u_{\Gamma} = 0$

**Question:** Do (FM) and (m) define $M \in \mathcal{L}(W; W')$ satisfying (M)?
Representation of $D$ and $M$ via matrix fields

For $u, v \in C_\infty^\infty(\mathbb{R}^d; \mathbb{R}^r)$ we have

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Question: Do $(FM)$ and $(m)$ define $M \in \mathcal{L}(W; W')$ satisfying $(M)$?

Answer: not in general (by a counterexample)
Representation of $D$ and $M$ via matrix fields

For $u, v \in C_c^\infty (\mathbb{R}^d; \mathbb{R}^r)$ we have

$$W' \langle Du, v \rangle_W = \int_\Gamma A_\nu (x) u|_\Gamma (x) \cdot v|_\Gamma (x) dS(x).$$

For a given field $M$, it is reasonable to seek an operator $M$ of the form

$$(m) \quad W' \langle Mu, v \rangle_W = \int_\Gamma M(x) u|_\Gamma (x) \cdot v|_\Gamma (x) dS(x).$$

... then smooth $u \in \ker (D - M)$ would satisfy $(A_\nu - M)u|_\Gamma = 0$

Question: Do (FM) and (m) define $M \in \mathcal{L}(W; W')$ satisfying (M)?

Answer: not in general (by a counterexample)

Question: ... perhaps under some additional assumptions...?
Idea: represent $M$ by $A_\nu$.

**Lemma.** If $M$ satisfies $(FM)$, then (for $a \epsilon x \in \Gamma$) there is a pair of projectors $S_+(x), S_-(x)$
(i.e. $S_+(x) + S_-(x) = I$ and $S_+(x)S_-(x) = S_-(x)S_+(x) = 0$), s.t.

$$(A_\nu + M)(x) = 2S_+^T(x)A_\nu(x) \quad \& \quad (A_\nu - M)(x) = 2S_-^T(x)A_\nu(x).$$

$\blacksquare$
Idea: represent $M$ by $A_\nu$

Lemma. If $M$ satisfies $(FM)$, then (for $ae x \in \Gamma$) there is a pair of projectors $S_+(x), S_-(x)$
(i.e. $S_+(x) + S_-(x) = I$ and $S_+(x)S_-(x) = S_-(x)S_+(x) = 0$), s.t.

$$(A_\nu + M)(x) = 2S_+^T(x)A_\nu(x) \quad \& \quad (A_\nu - M)(x) = 2S_-^T(x)A_\nu(x).$$

Therefore

$$M(x) = \left(I - 2S_-^T(x)\right)A_\nu(x).$$
Idea: represent $M$ by $A_\nu$ 

**Lemma.** If $M$ satisfies $(FM)$, then (for all $x \in \Gamma$) there is a pair of projectors $S_+(x), S_-(x)$ (i.e. $S_+(x) + S_-(x) = I$ and $S_+(x)S_-(x) = S_-(x)S_+(x) = 0$), s.t. 

$$(A_\nu + M)(x) = 2S_+^T(x)A_\nu(x) \quad \& \quad (A_\nu - M)(x) = 2S_-^T(x)A_\nu(x).$$

Therefore 

$$M(x) = \left(I - 2S_-^T(x)\right)A_\nu(x).$$

\ldots under additional regularity on $S_-$ expect continuity of $M$ \ldots
Idea: represent $\mathbf{M}$ by $\mathbf{A}_\nu$

**Lemma.** If $\mathbf{M}$ satisfies \((FM)\), then (for all $\mathbf{x} \in \Gamma$) there is a pair of projectors $\mathbf{S}_+(\mathbf{x}), \mathbf{S}_-(\mathbf{x})$
(i.e. $\mathbf{S}_+(\mathbf{x}) + \mathbf{S}_-(\mathbf{x}) = \mathbf{I}$ and $\mathbf{S}_+(\mathbf{x})\mathbf{S}_-(\mathbf{x}) = \mathbf{S}_-(\mathbf{x})\mathbf{S}_+(\mathbf{x}) = \mathbf{0}$), s.t.

$$(\mathbf{A}_\nu + \mathbf{M})(\mathbf{x}) = 2\mathbf{S}_+^T(\mathbf{x})\mathbf{A}_\nu(\mathbf{x}) \quad \& \quad (\mathbf{A}_\nu - \mathbf{M})(\mathbf{x}) = 2\mathbf{S}_-^T(\mathbf{x})\mathbf{A}_\nu(\mathbf{x}) \; .$$

Therefore

$$\mathbf{M}(\mathbf{x}) = \left( \mathbf{I} - 2\mathbf{S}_-^T(\mathbf{x}) \right)\mathbf{A}_\nu(\mathbf{x}) \; .$$

...under additional regularity on $\mathbf{S}_-$ expect continuity of $\mathbf{M}$ ...

...(M1) then trivially follows from \((FM1)\)
Idea: represent $\mathbf{M}$ by $\mathbf{A}_\nu$

**Lemma.** If $\mathbf{M}$ satisfies (FM), then (for all $x \in \Gamma$) there is a pair of projectors $\mathbf{S}_+(x), \mathbf{S}_-(x)$

(i.e. $\mathbf{S}_+(x) + \mathbf{S}_-(x) = \mathbf{I}$ and $\mathbf{S}_+(x)\mathbf{S}_-(x) = \mathbf{S}_-(x)\mathbf{S}_+(x) = \mathbf{0}$), s.t.

$$(\mathbf{A}_\nu + \mathbf{M})(x) = 2\mathbf{S}_+(x)\mathbf{A}_\nu(x) \quad \& \quad (\mathbf{A}_\nu - \mathbf{M})(x) = 2\mathbf{S}_-(x)\mathbf{A}_\nu(x).$$

Therefore

$$\mathbf{M}(x) = \left(\mathbf{I} - 2\mathbf{S}_-(x)\right)\mathbf{A}_\nu(x).$$

... under additional regularity on $\mathbf{S}_-$ expect continuity of $\mathbf{M}$ ...

...(M1) then trivially follows from (FM1)...

... perhaps this regularity is strong enough to derive (M2) from (FM2)?
Idea: represent $M$ by $A_\nu$.

**Lemma.** If $M$ satisfies $(FM)$, then (for all $x \in \Gamma$) there is a pair of projectors $S_+(x), S_-(x)$
(i.e. $S_+(x) + S_-(x) = I$ and $S_+(x)S_-(x) = S_-(x)S_+(x) = 0$), s.t.

$$(A_\nu + M)(x) = 2S_+^T(x)A_\nu(x) \quad \& \quad (A_\nu - M)(x) = 2S_-^T(x)A_\nu(x).$$

Therefore

$$M(x) = \left(I - 2S_-^T(x)\right)A_\nu(x).$$

... under additional regularity on $S_-$ expect continuity of $M$ ...

... $(M1)$ then trivially follows from $(FM1)$ ...

... perhaps this regularity is strong enough to derive $(M2)$ from $(FM2)$?

Idea: represent $M$ by $A_\nu$

Lemma. If $M$ satisfies $(FM)$, then (for all $x \in \Gamma$) there is a pair of projectors $S_+ (x), S_- (x)$

(i.e. $S_+ (x) + S_- (x) = I$ and $S_+ (x)S_- (x) = S_- (x)S_+ (x) = 0$), s.t.

$$(A_\nu + M)(x) = 2S^\top_+(x)A_\nu(x) \quad \& \quad (A_\nu - M)(x) = 2S^\top_-(x)A_\nu(x).$$

Therefore

$$M(x) = \left(I - 2S^\top_-(x)\right)A_\nu(x).$$

... under additional regularity on $S_-$ expect continuity of $M$ ...

...(M1) then trivially follows from (FM1)...

... perhaps this regularity is strong enough to derive (M2) from (FM2)?


... not good enough for applications to hyperbolic equations
\textbf{Lemma}

For a matrix field $M$ the following statements are equivalent.

- $M$ satisfies \((FM2)\).
- There is a matrix field $P$ such that $M = A_\nu(I - 2P)$ and
  
  $\ker(A_\nu P) + \ker(A_\nu (I - P)) = \mathbb{R}^r \text{ ae in } \partial \Omega$. 

$P$ is not necessarily a projector
Main result for Friedrichs systems

**Theorem.** Let matrix field $M \in L^\infty(\Gamma; M_r(\mathbb{R}))$ satisfy $(FM)$, and let $S_-$ be extendable to a measurable function on $\partial \Omega$, and satisfy:

$(S1)$ The multiplication operator $S_{-,p}$ is in $\mathcal{L}(W)$. 

$(S2)$ $(\forall v \in H^1(\Omega; \mathbb{R}^r))$ $S_{-,p}v \in H^1(\Omega; \mathbb{R}^r)$ & $\mathcal{T}_{H^1}(S_{-,p}v) = S_- \mathcal{T}_{H^1}v$. 

Test on examples . . . assumptions are reasonable . . .
Main result for Friedrichs systems

**Theorem.** Let matrix field $M \in L^\infty(\Gamma; M_r(\mathbb{R}))$ satisfy (FM), and let $S_-$ be extendable to a measurable function on $\text{Cl}\ \Omega$, and satisfy:

(S1) The multiplication operator $S_{-,p}$ is in $\mathcal{L}(W)$.

$$ (S_{-,p}(v) := S_{-,p}v \text{ for } v \in W) $$

(S2) $(\forall v \in H^1(\Omega; \mathbb{R}^r))$ $S_{-,p}v \in H^1(\Omega; \mathbb{R}^r)$ & $T_{H^1}(S_{-,p}v) = S_- T_{H^1}v$.

Then (m) defines operator $M \in \mathcal{L}(W; W')$ satisfying (M1).
Main result for Friedrichs systems

**Theorem.** Let matrix field $M \in L^\infty(\Gamma; M_r(\mathbb{R}))$ satisfy $(FM)$, and let $S_-$ be extendable to a measurable function on $\text{Cl} \setminus \Omega$, and satisfy:

$(S1)$ The multiplication operator $S_{-,p}$ is in $\mathcal{L}(W)$.

$$(S_{-,p}(v) := S_{-,p}v \text{ for } v \in W)$$

$(S2)$ $(\forall v \in H^1(\Omega; \mathbb{R}^r))$ $S_{-,p}v \in H^1(\Omega; \mathbb{R}^r)$ & $T_{H^1}(S_{-,p}v) = S_-T_{H^1}v$.

Then (m) defines operator $M \in \mathcal{L}(W; W')$ satisfying $(M1)$.

Furthermore, such $M$ satisfies $(M2)$. 

\[ \square \]
Main result for Friedrichs systems

**Theorem.** Let matrix field $M \in L^\infty(\Gamma; M_r(\mathbb{R}))$ satisfy \((FM)\), and let $S_-$ be extendable to a measurable function on $\text{Cl}\Omega$, and satisfy:

\((S1)\) The multiplication operator $S_-,p$ is in $L(W)$.
\[(S_-,p(v) := S_-,pv \text{ for } v \in W)\]

\((S2)\) $(\forall v \in H^1(\Omega; \mathbb{R}^r))$ $S_-,pv \in H^1(\Omega; \mathbb{R}^r)$ & $T_{H^1}(S_-,pv) = S_- T_{H^1}v$.

Then \((m)\) defines operator $M \in L(W; W')$ satisfying \((M1)\).

Furthermore, such $M$ satisfies \((M2)\).

Test on examples . . .
Main result for Friedrichs systems

**Theorem.** Let matrix field $\mathbf{M} \in L^{\infty}(\Gamma; \mathbf{M}_r(\mathbb{R}))$ satisfy (FM), and let $S_-$ be extendable to a measurable function on $\partial \Omega$, and satisfy:
(S1) The multiplication operator $S_{-,p}$ is in $\mathcal{L}(W)$.

\[ (S_{-,p}(v) := S_{-,p} v \text{ for } v \in W) \]

(S2) $(\forall v \in H^1(\Omega; \mathbb{R}^r))$ $S_{-,p} v \in H^1(\Omega; \mathbb{R}^r)$ & $T_{H^1}(S_{-,p} v) = S_+ T_{H^1} v$.

Then (m) defines operator $\mathbf{M} \in \mathcal{L}(W; W')$ satisfying (M1).

Furthermore, such $\mathbf{M}$ satisfies (M2).

Test on examples . . . assumptions are reasonable . . .
An example – scalar elliptic equation

\[ \Omega \subseteq \mathbb{R}^2, \mu > 0 \text{ and } f \in L^2(\Omega) \text{ given.} \]

\[ -\Delta u + \mu u = f \]
An example – scalar elliptic equation

\( \Omega \subseteq \mathbb{R}^2, \mu > 0 \) and \( f \in L^2(\Omega) \) given.

\[-\Delta u + \mu u = f\]

can be written as a first-order system

\[
\begin{cases}
p + \nabla u = 0 \\
\mu u + \text{div}p = f
\end{cases}
\]
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\[
\begin{aligned}
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\]

which is a Friedrichs system with the choice of

\[
A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.
\]
An example – scalar elliptic equation

\( \Omega \subseteq \mathbb{R}^2, \mu > 0 \) and \( f \in L^2(\Omega) \) given.

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\end{bmatrix}, \quad A_2 = \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & \mu
\end{bmatrix}.
\]

Note

\[
A_\nu = \nu_1 A_1 + \nu_2 A_2 = \begin{bmatrix}
    0 & 0 & \nu_1 \\
    0 & 0 & \nu_2 \\
    \nu_1 & \nu_2 & 0
\end{bmatrix}.
\]
Elliptic equation – different boundary conditions

\[
\begin{bmatrix}
0 & 0 & -\nu_1 \\
0 & 0 & -\nu_2 \\
\nu_1 & \nu_2 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 2\nu_1 \\
0 & 0 & 2\nu_2 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p \\ u
\end{bmatrix}_{|\Gamma} = 0
\]

\[u_{|\Gamma} = 0\]
Elliptic equation – different boundary conditions

\[
\begin{align*}
\begin{bmatrix}
0 & 0 & -\nu_1 \\
0 & 0 & -\nu_2 \\
\nu_1 & \nu_2 & 0
\end{bmatrix} & \quad \begin{bmatrix}
0 & 0 & 2\nu_1 \\
0 & 0 & 2\nu_2 \\
0 & 0 & 0
\end{bmatrix} & \quad \begin{bmatrix}
p \\ u
\end{bmatrix}
\bigg|_\Gamma = 0 \\
\begin{bmatrix}
0 & 0 & \nu_1 \\
0 & 0 & \nu_2 \\
-\nu_1 & -\nu_2 & 0
\end{bmatrix} & \quad \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
2\nu_1 & 2\nu_2 & 0
\end{bmatrix} & \quad \nu \cdot (\nabla u) \bigg|_\Gamma = 0
\end{align*}
\]
Elliptic equation – different boundary conditions

\[
\begin{align*}
M & = 
\begin{bmatrix}
0 & 0 & -\nu_1 \\
0 & 0 & -\nu_2 \\
\nu_1 & \nu_2 & 0
\end{bmatrix}
\quad & A_\nu - M & =
\begin{bmatrix}
0 & 0 & 2\nu_1 \\
0 & 0 & 2\nu_2 \\
0 & 0 & 0
\end{bmatrix}
\quad & (A_\nu - M) \begin{bmatrix}
p \\
u
\end{bmatrix}_\Gamma = 0
\end{align*}
\]

\[
\begin{align*}
M & = 
\begin{bmatrix}
0 & 0 & \nu_1 \\
0 & 0 & \nu_2 \\
-\nu_1 & -\nu_2 & 0
\end{bmatrix}
\quad & A_\nu - M & =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
2\nu_1 & 2\nu_2 & 0
\end{bmatrix}
\quad & \mathbf{\nu} \cdot (\nabla u)_\Gamma = 0
\end{align*}
\]

\[
\begin{align*}
M & = 
\begin{bmatrix}
0 & 0 & \nu_1 \\
0 & 0 & \nu_2 \\
-\nu_1 & -\nu_2 & 2\alpha
\end{bmatrix}
\quad & A_\nu - M & =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
2\nu_1 & 2\nu_2 & 2\alpha
\end{bmatrix}
\quad & \mathbf{\nu} \cdot (\nabla u)_\Gamma + \alpha u|_\Gamma = 0
\end{align*}
\]
Elliptic equation – different boundary conditions

\[
\begin{align*}
M &= \begin{bmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix} \\
A_\nu - M &= \begin{bmatrix} 0 & 0 & 2\nu_1 \\ 0 & 0 & 2\nu_2 \\ 0 & 0 & 0 \end{bmatrix} \\
(A_\nu - M) \begin{bmatrix} p \\ u \end{bmatrix}\big|_\Gamma &= 0 \\
\nu \cdot (\nabla u)\big|_\Gamma &= 0 \\
\nu \cdot (\nabla u)\big|_\Gamma + \alpha u\big|_\Gamma &= 0
\end{align*}
\]

All above matrices \( M \) satisfy \((FM)\).
Elliptic equation – projector $S_-$

Dirichlet:

$$S_- = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}$$
Elliptic equation – projector $S_-$

Dirichlet:

$$S_- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Neumann:

$$S_- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Constants can easily be extended, but we need $\nu \colon \Gamma \rightarrow \mathbb{R}$ to be Lipschitz in order to have bounded multiplication for the Robin b.c.
Elliptic equation – projector \( \mathbf{S}_- \)

Dirichlet:

\[
\mathbf{S}_- = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Neumann:

\[
\mathbf{S}_- = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Robin:

\[
\mathbf{S}_- = \begin{bmatrix}
0 & 0 & -\alpha \nu_1 \\
0 & 0 & -\alpha \nu_2 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Constants can easily be extended, but we need \( \nu : \Gamma \rightarrow \mathbb{R} \) to be Lipschitz in order to have bounded multiplication for the Robin b.c.
Elliptic equation – projector $S_-$

Dirichlet:

$$S_- = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$S_- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Robin:

$$S_- = \begin{bmatrix} 0 & 0 & -\alpha \nu_1 \\ 0 & 0 & -\alpha \nu_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Constants can easily be extended, but we need $\nu : \Gamma \rightarrow \mathbb{R}^r$ to be Lipschitz in order to have bounded multiplication for the Robin b.c.
Practical sufficient conditions

Lemma
For constant $A_k \in M_r(\mathbb{R})$ and $P \in M_r(\mathbb{R})$ the multiplication operator $u \mapsto Pu$ belongs to $\mathcal{L}(W)$ if and only if there exists $S \in M_r(\mathbb{R})$ such that $A_k P = S A_k$ for $k \in 1..d$. 
Practical sufficient conditions

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Theorem (sufficient conditions)
Let $P : \text{Cl} \Omega \longrightarrow M_r(\mathbb{R})$ be a Lipschitz matrix function satisfying:
- $(\exists S \in W^{1,\infty}(\Omega; M_r(\mathbb{R}))) (\forall k \in 1..d) \quad A_k P = S A_k$
Practical sufficient conditions

Lemma

For constant $A_k \in M_r(\mathbb{R})$ and $P \in M_r(\mathbb{R})$ the multiplication operator $u \mapsto Pu$ belongs to $L(W)$ if and only if there exists $S \in M_r(\mathbb{R})$ such that $A_k P = SA_k$ for $k \in 1..d$.

Theorem (sufficient conditions)

Let $P : \text{Cl} \Omega \rightarrow M_r(\mathbb{R})$ be a Lipschitz matrix function satisfying:

- $(\exists S \in W^{1,\infty}(\Omega; M_r(\mathbb{R}))) (\forall k \in 1..d) \quad A_k P = SA_k$,
- for almost every $x \in \partial \Omega$ the matrix $A_{\nu}(x)(I - 2P(x))$ is positive semidefinite
Practical sufficient conditions

Lemma

For constant $\mathbf{A}_k \in \mathbb{M}_r(\mathbb{R})$ and $\mathbf{P} \in \mathbb{M}_r(\mathbb{R})$ the multiplication operator
$u \mapsto \mathbf{P} u$ belongs to $\mathcal{L}(\mathcal{W})$ if and only if there exists $\mathbf{S} \in \mathbb{M}_r(\mathbb{R})$ such that
$\mathbf{A}_k \mathbf{P} = \mathbf{S}\mathbf{A}_k$ for $k \in 1..d$.

Theorem (sufficient conditions)

Let $\mathbf{P} : \text{Cl}\, \Omega \longrightarrow \mathbb{M}_r(\mathbb{R})$ be a Lipschitz matrix function satisfying:

1. $(\exists \mathbf{S} \in W^{1,\infty}(\Omega; M_r(\mathbb{R}))) (\forall k \in 1..d) \quad \mathbf{A}_k \mathbf{P} = \mathbf{S}\mathbf{A}_k$,
2. for almost every $x \in \partial\Omega$ the matrix $\mathbf{A}_\nu(x)(\mathbf{I} - 2\mathbf{P}(x))$ is positive semidefinite, and
3. for almost every $x \in \partial\Omega$ it holds
   \[
   \ker\left(\mathbf{A}_\nu(x)\mathbf{P}(x)\right) + \ker\left((\mathbf{A}_\nu(x)(\mathbf{I} - \mathbf{P}(x)))\right) = \mathbb{R}^{r}.
   \]
Practical sufficient conditions

Lemma
For constant $A_k \in M_r(R)$ and $P \in M_r(R)$ the multiplication operator $u \mapsto Pu$ belongs to $L(W)$ if and only if there exists $S \in M_r(R)$ such that $A_kP = SA_k$ for $k \in 1..d$.

Theorem (sufficient conditions)
Let $P : \text{Cl} \Omega \longrightarrow M_r(R)$ be a Lipschitz matrix function satisfying:
- $(\exists S \in W^{1,\infty}(\Omega; M_r(R))) (\forall k \in 1..d)$ $A_kP = SA_k$,
- for almost every $x \in \partial \Omega$ the matrix $A_\nu(x)(I - 2P(x))$ is positive semidefinite, and
- for almost every $x \in \partial \Omega$ it holds
  \[
  \ker(A_\nu(x)P(x)) + \ker((A_\nu(x)(I - P(x))) = R^r.
  \]

Then formula $(m)$, for $M(x) := A_\nu(x)(I - 2P(x))$ on $\partial \Omega$, defines a bounded operator $M \in L(W; W')$ satisfying $(M)$. 
Tests on examples

Applications on hyperbolic equations (transport and wave equation)
Tests on examples

Applications on hyperbolic equations (transport and wave equation)

Tests on examples

Applications on hyperbolic equations (transport and wave equation)


...still unable do get good results for mixed type problems
Heat equation

... with zero initial and Dirichlet boundary condition:

\[
\begin{align*}
\frac{\partial}{\partial t} u - \text{div}_x (A \nabla_x u) + b \cdot \nabla_x u + cu &= f \quad \text{in } \Omega_T \\
0 &= \text{in } \partial \Omega \\
0 &= \text{in } \Omega_T \\
0 &= \text{on } \partial \Omega \times \langle 0, T \rangle \\
0 &= \text{on } \Omega
\end{align*}
\]
... with zero initial and Dirichlet boundary condition:

\[
\begin{aligned}
\partial_t u - \text{div}_x (A \nabla_x u) + b \cdot \nabla_x u + cu &= f \text{ in } \Omega_T \\
 u &= 0 \text{ on } \partial \Omega \times ]0, T[ \\
 u(\cdot, 0) &= 0 \text{ on } \Omega
\end{aligned}
\]

...as a Friedrichs system:

\[
\begin{aligned}
\nabla_x u_{d+1} + A^{-1} u_d &= 0 \\
\partial_t u_{d+1} + \text{div}_x u_d + c u_{d+1} - A^{-1} b \cdot u_d &= f
\end{aligned}
\]

(note that we use \( u = (u_d, u_{d+1})^T \)).
The operator $T$ is given by

$$T \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_x u_{d+1} + A^{-1} u_d \\ \partial_t u_{d+1} + \text{div}_x u_d + c u_{d+1} - A^{-1} b \cdot u_d \end{bmatrix},$$
Friedrichs operator and the graph space

The operator $T$ is given by

$$T \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_x u_{d+1} + A^{-1} u_d \\ \partial_t u_{d+1} + \text{div} x u_d + c u_{d+1} - A^{-1} b \cdot u_d \end{bmatrix},$$

while the corresponding graph space is

$$W = \left\{ u \in L^2(\Omega_T; \mathbb{R}^{d+1}) : \nabla_x u_{d+1} \in L^2(\Omega_T; \mathbb{R}^d) \right\} \cap \left\{ u \in L^2(\Omega_T; \text{div}) : \partial_t u_{d+1} + \text{div} x u_d \in L^2(\Omega_T) \right\} \cap \left\{ u \in L^2(\Omega_T) : u_{d+1} \in L^2(0, T; H^1(\Omega)) \right\}.$$
Properties of the last component

Lemma. The projection \( u = (u_d, u_{d+1})^\top \mapsto u_{d+1} \) is a continuous linear operator from \( W \) to \( W(0, T) \), which is continuously embedded to \( C([0, T]; L^2(\Omega)) \).
Lemma. The projection $u = (u_d, u_{d+1})^\top \mapsto u_{d+1}$ is a continuous linear operator from $W$ to $W(0, T)$, which is continuously embedded to $C([0, T]; L^2(\Omega))$.

The space

$$W(0, T) = \left\{ u \in L^2(0, T; H^1(\Omega)) : \partial_t u \in L^2(0, T; H^{-1}(\Omega)) \right\},$$

is a Banach space when equipped by norm

$$\|u\|_{W(0, T)} = \sqrt{\|u\|_{L^2(0, T; H^1(\Omega))}^2 + \|\partial_t u\|_{L^2(0, T; H^{-1}(\Omega))}^2}.$$
Main result

Let

\[ V = \left\{ u \in W : u_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \right\}, \]
\[ \widetilde{V} = \left\{ v \in W : v_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad v_{d+1}(\cdot, T) = 0 \text{ a.e. on } \Omega \right\}. \]
Main result

Let
\[ V = \left\{ u \in W : u_{d+1} \in L^2(0, T; H_0^1(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \right\}, \]
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Do they satisfy (V1)–(V2)?
Main result

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Do they satisfy (V1)–(V2)? Technical...
Main result

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\[ \tilde{V} = \left\{ v \in W : v_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad v_{d+1}(\cdot, T) = 0 \ \text{a.e. on } \Omega \right\}. \]

Do they satisfy (V1)–(V2)? Technical...

**Theorem**

*The above \( V \) and \( \tilde{V} \) satisfy (V1)–(V2), and therefore the operator \( T_{V} : V \rightarrow L \) is an isomorphism.*
Two-field theory...

Heat equation with \( b = 0 \) and \( c = 0 \):

\[
\begin{aligned}
\partial_t u - \text{div}_x (A \nabla_x u) &= f \quad \text{in } \Omega_T \\
u &= 0 \quad \text{on } \Gamma \times \langle 0, T \rangle \\
u(\cdot, 0) &= 0 \quad \text{on } \Omega
\end{aligned}
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Two-field theory...

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Two field theory:
Two-field theory... 

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 u(\cdot, 0) &= 0 \text{ on } \Omega
\end{aligned}
$$

Two field theory:

developed by Ern and Guermond for elliptic problems
Heat equation with $b = 0$ and $c = 0$:

\[
\begin{align*}
\frac{\partial t}{\partial t} u - \text{div}_x (A \nabla_x u) &= f \text{ in } \Omega_T \\
u &= 0 \text{ on } \Gamma \times \langle 0, T \rangle \\
u(\cdot, 0) &= 0 \text{ on } \Omega
\end{align*}
\]

Two field theory:

developed by Ern and Guermond for elliptic problems

matrices need to be of the form

\[
A^k = \begin{bmatrix} 0 \\ (B^k)^\top \\ a^k \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C^d \\ 0 \end{bmatrix}^\top \frac{\partial t}{\partial t} c^{d+1},
\]

where $B^k \in \mathbb{R}^d$ are constant vectors, $a^k \in W^{1, \infty}(\Omega_T)$, $C^d \in L^\infty(\Omega_T; M_d(\mathbb{R}))$ and $c^{d+1} \in L^\infty(\Omega_T)$, $k \in 1..(d + 1)$.
Two-field theory...

Heat equation with $b = 0$ and $c = 0$:

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\partial_t u - \text{div}_x (A \nabla_x u) &= f \quad \text{in } \Omega_T \\
u &= 0 \quad \text{on } \Gamma \times (0, T) \\
u(\cdot, 0) &= 0 \quad \text{on } \Omega
\end{aligned}
\]

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A^k = \begin{bmatrix} 0 & B^k \\ (B^k)^\top & a^k \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix},
\]

where $B^k \in \mathbb{R}^d$ are constant vectors, $a^k \in W^{1,\infty}(\Omega_T)$, $C^d \in L^\infty(\Omega_T; M_d(\mathbb{R}))$ and $c^{d+1} \in L^\infty(\Omega_T)$, $k \in 1..(d + 1)$.

For the heat equation matrices have this form!
Instead of coercivity (positivity) condition (F2), the following is required:

$$\left( \exists \mu_1 > 0 \right) \forall \xi = (\xi_d, \xi_{d+1}) \in \mathbb{R}^{d+1}$$

$$\left( C + C^T + \sum_{k=1}^{d+1} \partial_k A_k \right) \xi \cdot \xi \geq 2\mu_1 |\xi_d|^2 \quad \text{(a.e. on } \Omega) ,$$
Instead of coercivity (positivity) condition (F2), the following is required:

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(\exists \mu_1 > 0) (\forall \xi = (\xi_d, \xi_{d+1}) \in \mathbb{R}^{d+1}) \quad \left( C + C^T + \sum_{k=1}^{d+1} \partial_k A_k \right) \xi \cdot \xi \geq 2\mu_1 |\xi_d|^2 \quad \text{(a.e. on } \Omega),
\]

\[
(\exists \mu_2 > 0) (\forall u \in V \cup \tilde{V}) \quad \sqrt{\langle \mathcal{L}u | u \rangle_{L^2(\Omega_T;\mathbb{R}^{d+1})} + \|Bu_{d+1}\|_{L^2(\Omega_T;\mathbb{R}^d)} } \geq \mu_2 \|u_{d+1}\|_{L^2(\Omega_T)},
\]

where \(Bu_{d+1} := \sum_{k=1}^{d+1} B^k \partial_k u_{d+1} = \nabla_x u_{d+1} \).
Instead of coercivity (positivity) condition (F2), the following is required:

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(\exists \mu_1 > 0)(\forall \xi = (\xi_d, \xi_{d+1}) \in \mathbb{R}^{d+1}) \quad \left( C + C^T + \sum_{k=1}^{d+1} \partial_k A_k \right) \xi \cdot \xi \geq 2\mu_1 |\xi_d|^2 \quad \text{(a.e. on } \Omega),
\]

\[
(\exists \mu_2 > 0)(\forall u \in V \cup \tilde{V}) \quad \sqrt{\langle \mathcal{L} u \mid u \rangle_{L^2(\Omega_T;\mathbb{R}^{d+1})} + \|Bu_{d+1}\|_{L^2(\Omega_T;\mathbb{R}^d)}} \geq \mu_2 \|u_{d+1}\|_{L^2(\Omega_T)},
\]

where \(Bu_{d+1} := \sum_{k=1}^{d+1} B^k \partial_k u_{d+1} = \nabla_x u_{d+1}\).

For our system both conditions are trivially fulfilled.
Instead of coercivity (positivity) condition (F2), the following is required:

\[
(\exists \mu_1 > 0)(\forall \xi = (\xi_d, \xi_{d+1}) \in \mathbb{R}^{d+1}) \quad \left( C + C^\top + \sum_{k=1}^{d+1} \partial_k A_k \right) \xi \cdot \xi \geq 2\mu_1 |\xi_d|^2 \quad \text{(a.e. on } \Omega),
\]

\[
(\exists \mu_2 > 0)(\forall u \in V \cup \tilde{V}) \quad \sqrt{\langle Lu | u \rangle_{L^2(\Omega_T;\mathbb{R}^{d+1})} + \|Bu_{d+1}\|_{L^2(\Omega_T;\mathbb{R}^d)}} \geq \mu_2 \|u_{d+1}\|_{L^2(\Omega_T)},
\]

where \( Bu_{d+1} := \sum_{k=1}^{d+1} B_k \partial_k u_{d+1} = \nabla_x u_{d+1} \).

For our system both conditions are trivially fulfilled.

Therefore, we have the well-posedness result.
An example – stationary diffusion equation

We consider the equation

$$- \text{div}(A \nabla u) + cu = f$$

in $\Omega \subseteq \mathbb{R}^d$, where $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$ with $1/\beta' \leq c \leq 1/\alpha'$, for some $\beta' \geq \alpha' > 0$, and $A \in \mathbb{M}_d(\alpha', \beta'; \Omega) := \{A \in L^\infty(\Omega; \mathbb{M}_d) : (\forall \xi \in \mathbb{R}^d) A \xi \cdot \xi \geq \alpha' |\xi|^2 \& A \xi \cdot \xi \geq 1/\beta' |A \xi|^2\}$.

New unknown vector function taking values in $\mathbb{R}^{d+1}$:

$$\mathbf{u} = \mathbf{u}_d \mathbf{u}_d + 1 - \nabla x \mathbf{u} - A \nabla u = 0 \quad \text{div } u_d + cu_d + 1 = f.$$
An example – stationary diffusion equation

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\[-\text{div}(A \nabla u) + cu = f\]

in \(\Omega \subseteq \mathbb{R}^d\), where \(f \in L^2(\Omega)\), \(c \in L^\infty(\Omega)\) with \(\frac{1}{\beta'} \leq c \leq \frac{1}{\alpha'}\), for some \(\beta' \geq \alpha' > 0\),
An example – stationary diffusion equation

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\[
A \in \mathcal{M}_d(\alpha', \beta'; \Omega) := \left\{ A \in L^\infty(\Omega; \mathbb{M}_d(\mathbb{R})) : \right. \\
\left. (\forall \xi \in \mathbb{R}^d) \ A\xi \cdot \xi \geq \alpha' |\xi|^2 \ & A\xi \cdot \xi \geq \frac{1}{\beta'} |A\xi|^2 \right\}
\]
An example – stationary diffusion equation

We consider the equation

$$-\text{div} (A \nabla u) + cu = f$$

in $\Omega \subseteq \mathbb{R}^d$, where $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$ with $\frac{1}{\beta'} \leq c \leq \frac{1}{\alpha'}$, for some $\beta' \geq \alpha' > 0$, and

$$A \in M_d(\alpha', \beta'; \Omega) := \left\{ A \in L^\infty(\Omega; M_d(\mathbb{R})) : \right. \\
(\forall \xi \in \mathbb{R}^d) \ A\xi \cdot \xi \geq \alpha' |\xi|^2 \& \ A\xi \cdot \xi \geq \frac{1}{\beta'} |A\xi|^2 \left. \right\}$$

New unknown vector function taking values in $\mathbb{R}^{d+1}$:

$$u = \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} -A \nabla_x u \\ u \end{bmatrix}.$$
An example – stationary diffusion equation

We consider the equation

\[- \text{div} (A \nabla u) + cu = f\]

in \( \Omega \subseteq \mathbb{R}^d \), where \( f \in L^2(\Omega) \), \( c \in L^\infty(\Omega) \) with \( \frac{1}{\beta'} \leq c \leq \frac{1}{\alpha'} \), for some \( \beta' \geq \alpha' > 0 \), and

\[ A \in \mathcal{M}_d(\alpha', \beta'; \Omega) := \left\{ A \in L^\infty(\Omega; M_d(\mathbb{R})) : \right\}

\[ \left( \forall \xi \in \mathbb{R}^d \right) A\xi \cdot \xi \geq \alpha'|\xi|^2 \& A\xi \cdot \xi \geq \frac{1}{\beta'}|A\xi|^2 \right\} \]

New unknown vector function taking values in \( \mathbb{R}^{d+1} \):

\[ u = \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} -A \nabla x u \\ u \end{bmatrix}. \]

Then the starting equation can be written as a first-order system

\[ \begin{cases} 
\nabla_x u_{d+1} + A^{-1} u_d = 0 \\
\text{div} u_d + cu_{d+1} = f
\end{cases}, \]
An example – stationary diffusion equation (cont.)

which is a Friedrichs system with the choice of

\[ A_k = e_k \otimes e_{d+1} + e_{d+1} \otimes e_k \in M_{d+1}(\mathbb{R}), \quad C = \begin{bmatrix} A^{-1} & 0 \\ 0 & c \end{bmatrix}. \]
An example – stationary diffusion equation (cont.)

which is a Friedrichs system with the choice of

\[ A_k = e_k \otimes e_{d+1} + e_{d+1} \otimes e_k \in M_{d+1}(\mathbb{R}), \]

\[ C = \begin{bmatrix} A^{-1} & 0 \\ 0 & c \end{bmatrix}. \]

The graph space: \( W = L^2_{\text{div}}(\Omega) \times H^1(\Omega). \)
An example – stationary diffusion equation (cont.)

which is a Friedrichs system with the choice of

\[
A_k = e_k \otimes e_{d+1} + e_{d+1} \otimes e_k \in M_{d+1}(\mathbb{R}), \quad C = \begin{bmatrix} A^{-1} & 0 \\ 0 & c \end{bmatrix}.
\]

The graph space: \( W = L^2_{\text{div}}(\Omega) \times H^1(\Omega) \).

Dirichlet, Neumann and Robin boundary conditions are imposed by the following choice of \( V \) and \( \tilde{V} \):

\[
V_D = \tilde{V}_D := L^2_{\text{div}}(\Omega) \times H^1_0(\Omega),
\]

\[
V_N = \tilde{V}_N := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = 0\},
\]

\[
V_R := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = a u_{d+1}|_{\Gamma}\},
\]

\[
\tilde{V}_R := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = -a u_{d+1}|_{\Gamma}\}.
\]
Non-stationary problem

Marko Erceg, Krešimir Burazin: *Non-stationary abstract Friedrichs systems via semigroup theory*, submitted
Non-stationary problem

Marko Erceg, Krešimir Burazin: *Non-stationary abstract Friedrichs systems via semigroup theory*, submitted

$L$ real Hilbert space, as before ($L' \equiv L$), $T > 0$

We consider an abstract Cauchy problem in $L$:

\[
\begin{aligned}
(P) \quad \left\{ 
& u'(t) + T u(t) = f(t) \\
& u(0) = u_0
\right. \\
\end{aligned}
\]
Non-stationary problem

Marko Erceg, Krešimir Burazin: *Non-stationary abstract Friedrichs systems via semigroup theory*, submitted

$L$ real Hilbert space, as before ($L' \equiv L$), $T > 0$

We consider an abstract Cauchy problem in $L$:

\[
(P) \quad \begin{cases} 
    u'(t) + Tu(t) = f(t) \\ 
    u(0) = u_0
\end{cases},
\]

where

- $f : \langle 0, T \rangle \rightarrow L$, $u_0 \in L$ are given,
- $T$ (not depending on $t$) satisfies (T1), (T2) and

\[
(T3') \quad (\forall \varphi \in \mathcal{D}) \quad \langle (T + \tilde{T})\varphi \mid \varphi \rangle_L \geq 0,
\]

- $u : [0, T] \rightarrow L$ is unknown.
Non-stationary problem

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$L$ real Hilbert space, as before ($L' \equiv L$), $T > 0$

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$$\begin{cases} u'(t) + Tu(t) = f(t) \\ u(0) = u_0 \end{cases},$$

where

- $f : \langle 0, T \rangle \rightarrow L$, $u_0 \in L$ are given,
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$$(T3') \quad (\forall \varphi \in \mathcal{D}) \quad \langle (T + \tilde{T})\varphi | \varphi \rangle_L \geq 0,$$

- $u : [0, T] \rightarrow L$ is unknown.

Numerics:

Semigroup setting

A priori estimate:

$$(\forall t \in [0, T]) \quad \|u(t)\|_L^2 \leq e^t \left( \|u_0\|_L^2 + \int_0^t \|f(s)\|_L^2 \right).$$
Semigroup setting

A priori estimate:

\[
(\forall t \in [0, T]) \quad \|u(t)\|_L^2 \leq e^t \left( \|u_0\|_L^2 + \int_0^t \|f(s)\|_L^2 \right).
\]

Let \( A : V \subseteq L \to L \), \( A := -T|_V \)

Then (P) becomes:

(P') \[
\begin{cases}
    u'(t) - Au(t) = f(t) \\
    u(0) = u_0
\end{cases}
\]
Semigroup setting

A priori estimate:

\[(\forall t \in [0, T]) \quad \| u(t) \|_L^2 \leq e^t \left( \| u_0 \|_L^2 + \int_0^t \| f(s) \|_L^2 \right) .\]

Let \( A : V \subseteq L \rightarrow L, A := -T|_V \)

Then (P) becomes:

\((P')\) \quad \begin{cases} u'(t) - Au(t) = f(t) \\ u(0) = u_0 \end{cases} .

**Theorem.** *The operator \( A \) is an infinitesimal generator of a \( C_0 \)-semigroup on \( L \).*
Existence and uniqueness result

**Corollary.** Let $T$ be an operator that satisfies $(T1)$–$(T2)$ and $(T3)'$, let $V$ be a subspace of its graph space satisfying $(V1)$–$(V2)$, and $f \in L^1(\langle 0, T \rangle; L)$. Then for every $u_0 \in L^1$ the problem (P) has the unique mild solution $u \in C([0, T]; L)$ given with $u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds$, $t \in [0, T]$, where $(T(t))_{t \geq 0}$ is the semigroup generated by $A$. If additionally $f \in C([0, T]; L) \cap W^{1,1}(\langle 0, T \rangle; V)$ with $V$ equipped with the graph norm and $u_0 \in V$, then the above mild solution is the classical solution of (P) on $[0, T)$. 
**Existence and uniqueness result**

**Corollary.** Let $T$ be an operator that satisfies $(T1)$–$(T2)$ and $(T3)'$, let $V$ be a subspace of its graph space satisfying $(V1)$–$(V2)$, and $f \in L^1(\langle 0, T \rangle; L)$. Then for every $u_0 \in L$ the problem $(P)$ has the unique mild solution $u \in C([0, T]; L)$ given with

$$u(t) = T(t)u_0 + \int_0^t T(t - s)f(s)ds, \quad t \in [0, T],$$

where $(T(t))_{t \geq 0}$ is the semigroup generated by $A$. 
Corollary. Let $T$ be an operator that satisfies $(T1)$–$(T2)$ and $(T3)'$, let $V$ be a subspace of its graph space satisfying $(V1)$–$(V2)$, and $f \in L^1(\langle 0, T \rangle; L)$. Then for every $u_0 \in L$ the problem $(P)$ has the unique mild solution $u \in C([0, T]; L)$ given with

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \in [0, T],$$

where $(T(t))_{t \geq 0}$ is the semigroup generated by $A$. If additionally $f \in C([0, T]; L) \cap \left(W^{1,1}(\langle 0, T \rangle; L) \cup L^1(\langle 0, T \rangle; V)\right)$ with $V$ equipped with the graph norm and $u_0 \in V$, then the above mild solution is the classical solution of $(P)$ on $[0, T]$. 

\end{proof}
Mild solution

**Theorem.** Let \( u_0 \in L, f \in L^1(\langle 0, T \rangle; L) \) and let

\[
u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \in [0, T],
\]

be the mild solution of \((P)\).
**Theorem.** Let \( u_0 \in L, \ f \in L^1(\langle 0, T \rangle; L) \) and let

\[
u(t) = T(t)u_0 + \int_0^t T(t - s)f(s)ds, \quad t \in [0, T],
\]

be the mild solution of (P).

Then \( u', Tu, f \in L^1(\langle 0, T \rangle; W_0') \) and

\[
u' + Tu = f,
\]

in \( L^1(\langle 0, T \rangle; W_0'). \)
Bound on solution

From

\[ u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)f(s)ds, \quad t \in [0, T], \]

we get:

\[ (\forall t \in [0, T]) \quad \|u(t)\|_L \leq \|u_0\|_L + \int_0^t \|f(s)\|_L ds. \]
Bound on solution

From
\[ u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t - s)f(s)ds, \quad t \in [0, T], \]
we get:
\[ (\forall t \in [0, T]) \quad \|u(t)\|_L \leq \|u_0\|_L + \int_0^t \|f(s)\|_L ds. \]

A priori estimate was:
\[ (\forall t \in [0, T]) \quad \|u(t)\|_L^2 \leq e^t \left( \|u_0\|_L^2 + \int_0^t \|f(s)\|_L^2 ds \right). \]
Let \( \Omega \subseteq \mathbb{R}^3 \) be open and bounded with a Lipschitz boundary \( \Gamma \), \( \mu, \varepsilon \in W^{1,\infty}(\Omega) \) positive and away from zero, \( \Sigma_{ij} \in L^\infty(\Omega; M_3(\mathbb{R})) \), \( i, j \in \{1, 2\} \), and \( f_1, f_2 \in L^1([0, T]; L^2(\Omega; \mathbb{R}^3)) \).
Non-stationary Maxwell system 1/5

Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with a Lipschitz boundary $\Gamma$, $\mu, \varepsilon \in W^{1,\infty}(\Omega)$ positive and away from zero, $\Sigma_{ij} \in L^\infty(\Omega; M_3(\mathbb{R}))$, $i, j \in \{1, 2\}$, and $f_1, f_2 \in L^1(\langle 0, T\rangle; L^2(\Omega; \mathbb{R}^3))$.

We consider a generalized non-stationary Maxwell system

\[
(\text{MS}) \quad \begin{cases} 
\mu \partial_t H + \text{rot}\ E + \Sigma_{11} H + \Sigma_{12} E = f_1 \\
\varepsilon \partial_t E - \text{rot}\ H + \Sigma_{21} H + \Sigma_{22} E = f_2 
\end{cases} \quad \text{in } \langle 0, T\rangle \times \Omega,
\]
Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with a Lipschitz boundary $\Gamma$, $\mu, \varepsilon \in W^{1,\infty}(\Omega)$ positive and away from zero, $\Sigma_{ij} \in L^\infty(\Omega; M_3(\mathbb{R}))$, $i, j \in \{1, 2\}$, and $f_1, f_2 \in L^1(\langle 0, T \rangle; L^2(\Omega; \mathbb{R}^3))$.

We consider a generalized non-stationary Maxwell system

\[
\begin{aligned}
\mu \partial_t H + \text{rot} E + \Sigma_{11} H + \Sigma_{12} E &= f_1 \\
\varepsilon \partial_t E - \text{rot} H + \Sigma_{21} H + \Sigma_{22} E &= f_2
\end{aligned}
\]

in $\langle 0, T \rangle \times \Omega$,

where $E, H : [0, T] \times \Omega \to \mathbb{R}^3$ are unknown functions.
Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with a Lipschitz boundary $\Gamma$, $\mu, \varepsilon \in W^{1,\infty}(\Omega)$ positive and away from zero, $\Sigma_{ij} \in L^\infty(\Omega; M_3(\mathbb{R}))$, $i, j \in \{1, 2\}$, and $f_1, f_2 \in L^1(\langle 0, T \rangle; L^2(\Omega; \mathbb{R}^3))$.

We consider a generalized non-stationary Maxwell system

\begin{equation}
\begin{cases}
\mu \partial_t H + \text{rot } E + \Sigma_{11} H + \Sigma_{12} E = f_1 \\
\varepsilon \partial_t E - \text{rot } H + \Sigma_{21} H + \Sigma_{22} E = f_2
\end{cases}
\end{equation}

in $\langle 0, T \rangle \times \Omega$,

where $E, H : [0, T] \times \Omega \rightarrow \mathbb{R}^3$ are unknown functions.

Change of variable

\[
u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\mu} H \\ \sqrt{\varepsilon} E \end{bmatrix}, \quad c := \frac{1}{\sqrt{\mu \varepsilon}} \in W^{1,\infty}(\Omega),
\]
Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded with a Lipschitz boundary $\Gamma$, $
abla, \varepsilon \in W^{1,\infty}(\Omega)$ positive and away from zero, $\Sigma_{ij} \in L^\infty(\Omega; M_3(\mathbb{R}))$, $i,j \in \{1,2\}$, and $f_1, f_2 \in L^1([0,T]; L^2(\Omega; \mathbb{R}^3))$.

We consider a generalized non-stationary Maxwell system

\[ \begin{align*}
\mu \partial_t H + \text{rot} E + \Sigma_{11} H + \Sigma_{12} E &= f_1 \\
\varepsilon \partial_t E - \text{rot} H + \Sigma_{21} H + \Sigma_{22} E &= f_2
\end{align*} \text{ in } [0,T) \times \Omega,
\]

where $E, H : [0,T) \times \Omega \rightarrow \mathbb{R}^3$ are unknown functions.

Change of variable

\[ u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\mu} H \\ \sqrt{\varepsilon} E \end{bmatrix}, \quad c := \frac{1}{\sqrt{\mu \varepsilon}} \in W^{1,\infty}(\Omega), \]

turns (MS) to the Friedrichs system

\[ \partial_t u + T u = F, \]
with

\[
A_1 := c \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}, \quad A_2 := c \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad A_3 := c \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad F = \begin{bmatrix}
\frac{1}{\sqrt{\mu}} f_1 \\
\frac{1}{\sqrt{\varepsilon}} f_2
\end{bmatrix}, \quad C := \ldots.
\]
Non-stationary Maxwell system 2/5

with

\[ A_1 := c \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad A_2 := c \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \]

\[ A_3 := c \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \frac{1}{\sqrt{\mu}} f_1 \\ \frac{1}{\sqrt{\varepsilon}} f_2 \end{bmatrix}, \quad C := \ldots. \]

(F1) and (F2) are satisfied (with change \( v := e^{-\lambda t} u \) for large \( \lambda > 0 \), if needed)
The spaces involved:

\[ L = L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3), \]
\[ W = L^2_{\text{rot}}(\Omega; \mathbb{R}^3) \times L^2_{\text{rot}}(\Omega; \mathbb{R}^3), \]
\[ W_0 = L^2_{\text{rot},0}(\Omega; \mathbb{R}^3) \times L^2_{\text{rot},0}(\Omega; \mathbb{R}^3) = \text{Cl}_{W^C_c}(\Omega; \mathbb{R}^6), \]

where \( L^2_{\text{rot}}(\Omega; \mathbb{R}^3) \) is the graph space of the rot operator.
The spaces involved:

\[
L = L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3),
\]
\[
W = L^2_{\text{rot}}(\Omega; \mathbb{R}^3) \times L^2_{\text{rot}}(\Omega; \mathbb{R}^3),
\]
\[
W_0 = L^2_{\text{rot},0}(\Omega; \mathbb{R}^3) \times L^2_{\text{rot},0}(\Omega; \mathbb{R}^3) = \text{Cl}_{W} C^\infty_c (\Omega; \mathbb{R}^6),
\]

where \(L^2_{\text{rot}}(\Omega; \mathbb{R}^3)\) is the graph space of the rot operator.

The boundary condition

\[
\nu \times E|_{\Gamma} = 0
\]

corresponds to the following choice of spaces \(V, \tilde{V} \subseteq W:\)
The spaces involved:

\[ L = L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3), \]
\[ W = L^2_{\text{rot}}(\Omega; \mathbb{R}^3) \times L^2_{\text{rot}}(\Omega; \mathbb{R}^3), \]
\[ W_0 = L^2_{\text{rot},0}(\Omega; \mathbb{R}^3) \times L^2_{\text{rot},0}(\Omega; \mathbb{R}^3) = \text{Cl}_{W} C^\infty_c(\Omega; \mathbb{R}^6), \]

where \( L^2_{\text{rot}}(\Omega; \mathbb{R}^3) \) is the graph space of the \text{rot} operator.

The boundary condition

\[ \nu \times E|_\Gamma = 0 \]

corresponds to the following choice of spaces \( V, \tilde{V} \subseteq W: \)

\[ V = \tilde{V} = \{ u \in W : \nu \times u_2 = 0 \} \]
\[ = \{ u \in W : \nu \times E = 0 \} \]
\[ = L^2_{\text{rot}}(\Omega; \mathbb{R}^3) \times L^2_{\text{rot},0}(\Omega; \mathbb{R}^3). \]
Theorem. Let $E_0 \in L^2_{\text{rot},0}(\Omega; \mathbb{R}^3)$, $H_0 \in L^2_{\text{rot}}(\Omega; \mathbb{R}^3)$ and let $f_1, f_2 \in C([0, T]; L^2(\Omega; \mathbb{R}^3))$ satisfy at least one of the following conditions:

- $f_1, f_2 \in W^{1,1}(\langle 0, T \rangle; L^2(\Omega; \mathbb{R}^3))$;
- $f_1 \in L^1(\langle 0, T \rangle; L^2_{\text{rot}}(\Omega; \mathbb{R}^3))$, $f_2 \in L^1(\langle 0, T \rangle; L^2_{\text{rot},0}(\Omega; \mathbb{R}^3))$. 

Then the abstract initial-boundary value problem

\[
\mu H' + \text{rot } E + \Sigma_{11} H + \Sigma_{12} E = f_1 \\
\text{ε } E' - \text{rot } H + \Sigma_{21} H + \Sigma_{22} E = f_2 \\
E(0) = E_0 \\
H(0) = H_0 \\
\nu \times E|_{\Gamma} = 0
\]


Theorem. Let \( E_0 \in L^2_{\text{rot},0}(\Omega; \mathbb{R}^3) \), \( H_0 \in L^2_{\text{rot}}(\Omega; \mathbb{R}^3) \) and let \( f_1, f_2 \in C([0, T]; L^2(\Omega; \mathbb{R}^3)) \) satisfy at least one of the following conditions:
- \( f_1, f_2 \in W^{1,1}(\langle 0, T \rangle; L^2(\Omega; \mathbb{R}^3)) \);
- \( f_1 \in L^1(\langle 0, T \rangle; L^2_{\text{rot}}(\Omega; \mathbb{R}^3)), f_2 \in L^1(\langle 0, T \rangle; L^2_{\text{rot},0}(\Omega; \mathbb{R}^3)) \).

Then the abstract initial-boundary value problem

\[
\begin{align*}
\mu H' + \text{rot} E + \Sigma_{11} H + \Sigma_{12} E &= f_1 \\
\varepsilon E' - \text{rot} H + \Sigma_{21} H + \Sigma_{22} E &= f_2 \\
E(0) &= E_0 \\
H(0) &= H_0 \\
\nu \times E|_{\Gamma} &= 0
\end{align*}
\]
Theorem. ...has unique classical solution given by

\[
\begin{bmatrix} H \\ E \end{bmatrix}(t) = \begin{bmatrix} \frac{1}{\sqrt{\mu}} I & 0 \\ 0 & \frac{1}{\sqrt{\varepsilon}} I \end{bmatrix} \mathcal{T}(t) \begin{bmatrix} \sqrt{\mu} H_0 \\ \sqrt{\varepsilon} E_0 \end{bmatrix} \\
\quad + \begin{bmatrix} \frac{1}{\sqrt{\mu}} I & 0 \\ 0 & \frac{1}{\sqrt{\varepsilon}} I \end{bmatrix} \int_0^t \mathcal{T}(t-s) \begin{bmatrix} \frac{1}{\sqrt{\mu}} f_1(s) \\ \frac{1}{\sqrt{\varepsilon}} f_2(s) \end{bmatrix} ds, \quad t \in [0, T],
\]

where \((\mathcal{T}(t))_{t \geq 0}\) is the contraction \(C_0\)-semigroup generated by \(-\mathcal{T}\).
Other examples

- Symmetric hyperbolic system

\[
\begin{aligned}
&\partial_t u + \sum_{k=1}^{d} \partial_k (A_k u) + Cu = f \quad \text{in } (0, T) \times \mathbb{R}^d, \\
u(0, \cdot) = u_0
\end{aligned}
\]
Other examples

- Symmetric hyperbolic system

\[
\begin{aligned}
\frac{\partial}{\partial t} u + \sum_{k=1}^{d} \frac{\partial}{\partial k} (A_k u) + Cu &= f \quad \text{in } \langle 0, T \rangle \times \mathbb{R}^d, \\
u(0, \cdot) &= u_0
\end{aligned}
\]

- Non-stationary div-grad problem

\[
\begin{aligned}
\frac{\partial}{\partial t} q + \nabla p &= f_1 \quad \text{in } \langle 0, T \rangle \times \Omega, \quad \Omega \subseteq \mathbb{R}^d, \\
\frac{1}{c_0^2} \frac{\partial}{\partial t} p + \text{div } q &= f_2 \quad \text{in } \langle 0, T \rangle \times \Omega, \\
|\partial p| &= 0, \quad p(0) = p_0, \quad q(0) = q_0
\end{aligned}
\]
Other examples

- Symmetric hyperbolic system

\[
\begin{aligned}
\partial_t u + \sum_{k=1}^{d} \partial_k (A_k u) + C u &= f \quad \text{in } \langle 0, T \rangle \times \mathbb{R}^d,
\end{aligned}
\]

\[
\begin{aligned}
u(0, \cdot) &= u_0
\end{aligned}
\]

- Non-stationary div-grad problem

\[
\begin{aligned}
&\partial_t q + \nabla p = f_1 \quad \text{in } \langle 0, T \rangle \times \Omega, \quad \Omega \subseteq \mathbb{R}^d, \\
&\frac{1}{c_0^2} \partial_t p + \text{div } q = f_2 \quad \text{in } \langle 0, T \rangle \times \Omega, \\
&p|_{\partial \Omega} = 0, \quad p(0) = p_0, \quad q(0) = q_0
\end{aligned}
\]

- Wave equation

\[
\begin{aligned}
&\partial_{tt} u - c^2 \Delta u = f \quad \text{in } \langle 0, T \rangle \times \mathbb{R}^d \\
u(0, \cdot) &= u_0, \quad \partial_t u(0, \cdot) = u_0^1
\end{aligned}
\]
Friedrichs systems in a complex Hilbert space

Let $L$ be a complex Hilbert space, $L' \equiv L$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : L \rightarrow L$ linear operators that satisfy (T1)–(T3) (or T3' instead of T3).
Friedrichs systems in a complex Hilbert space

Let $L$ be a complex Hilbert space, $L' \equiv L$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : L \rightarrow L$ linear operators that satisfy (T1)–(T3) (or T3' instead of T3).

Technical differences with respect to the real case, but results remain the same. . .
Friedrichs systems in a complex Hilbert space

Let $L$ be a complex Hilbert space, $L' \equiv L$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : L \rightarrow L$ linear operators that satisfy (T1)–(T3) (or T3' instead of T3).

Technical differences with respect to the real case, but results remain the same.

For the classical Friedrichs operator we require

(F1) matrix functions $A_k$ are selfadjoint: $A_k = A_k^*$,
Friedrichs systems in a complex Hilbert space

Let $L$ be a complex Hilbert space, $L' \equiv L$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : L \to L$ linear operators that satisfy (T1)–(T3) (or T3' instead of T3).

Technical differences with respect to the real case, but results remain the same . . .

For the classical Friedrichs operator we require

\((F1)\)

\[
\text{matrix functions } A_k \text{ are selfadjoint: } A_k = A_k^* ,
\]

\((F2)\)

\[
(\exists \mu_0 > 0) \quad C + C^* + \sum_{k=1}^{d} \partial_k A_k \geq 2\mu_0 I \quad (ae \text{ on } \Omega) ,
\]
Let $L$ be a complex Hilbert space, $L' \equiv L$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : L \to L$ linear operators that satisfy (T1)–(T3) (or T3' instead of T3).

Technical differences with respect to the real case, but results remain the same.

For the classical Friedrichs operator we require

(F1) matrix functions $A_k$ are selfadjoint: $A_k = A_k^*$,

(F2) $\forall \mu_0 > 0$ $C + C^* + \sum_{k=1}^{d} \partial_k A_k \geq 2\mu_0 I$ (ae on $\Omega$),

and again (F1)–(F2) imply (T1)–(T3).
Application to Dirac system 1/2

We consider the Cauchy problem
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\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_{t}u + \sum_{k=1}^{3} A_k \partial_{k}u + Cu = f \quad \text{in } \langle 0, T \rangle \times \mathbb{R}^3, \\
u(0) = u_0, 
\end{array} \right.
\end{aligned}
\]

(DS)
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u(0) &= u_0,
\end{aligned}
\]

where \( u : [0, T) \times \mathbb{R}^3 \to C^4 \) is an unknown function, \( u_0 : \mathbb{R}^3 \to C^4 \), \( f : [0, T) \times \mathbb{R}^3 \to C^4 \) are given, and \( A_k := \sigma_k \sigma_k \), \( C := \begin{pmatrix} c_1 I & 0 \\ 0 & c_2 I \end{pmatrix} \), where \( \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), and \( c_1, c_2 \in L^\infty(\mathbb{R}^3; C) \).
We consider the Cauchy problem

\[
\begin{aligned}
(Tu) \\
\begin{cases}
\partial_t u + \sum_{k=1}^3 A_k \partial_k u + Cu = f & \text{in } (0, T) \times \mathbb{R}^3, \\
u(0) = u_0,
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where \( u : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{C}^4 \) is an unknown function, \( u_0 : \mathbb{R}^3 \rightarrow \mathbb{C}^4 \), \( f : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{C}^4 \) are given, and
Application to Dirac system 1/2

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\[
A_k := \begin{bmatrix}
    0 & \sigma_k \\
    \sigma_k & 0
\end{bmatrix}, \quad k \in 1..3, \quad C := \begin{bmatrix}
    c_1 \mathbf{I} & 0 \\
    0 & c_2 \mathbf{I}
\end{bmatrix},
\]

are Pauli matrices, and \( c_1, c_2 \in L_\infty(\mathbb{R}^3; \mathbb{C}) \).
We consider the Cauchy problem

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where \( u : [0, T] \times \mathbb{R}^3 \rightarrow C^4 \) is an unknown function, \( u_0 : \mathbb{R}^3 \rightarrow C^4 \), \( f : [0, T] \times \mathbb{R}^3 \rightarrow C^4 \) are given, and

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A_k := \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad k \in \{1, 2, 3\}, \quad C := \begin{bmatrix} c_1 \mathbf{I} & 0 \\ 0 & c_2 \mathbf{I} \end{bmatrix},
\]

where

\[
\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

are Pauli matrices, and \( c_1, c_2 \in L^\infty(\mathbb{R}^3; C) \).
Theorem. Let \( u_0 \in W \) and let \( f \in C([0,T]; L^2(\mathbb{R}^3; \mathbb{C}^4)) \) satisfies at least one of the following conditions:

- \( f \in W^{1,1}(\langle 0, T \rangle; L^2(\mathbb{R}^3; \mathbb{C}^4)) \);
- \( f \in L^1(\langle 0, T \rangle; W) \).

Then the abstract Cauchy problem

\[
\begin{cases}
\frac{\partial}{\partial t} u + \sum_{k=1}^{3} A_k \partial_k u + Cu = f \\
u(0) = u_0
\end{cases}
\]

has unique classical solution given with

\[
u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds , \quad t \in [0, T] ,
\]

where \((T(t))_{t \geq 0}\) is the contraction \( C_0 \)-semigroup generated by \(-T\).
The operator $T$ depends on $t$ (i.e. the matrix coefficients $A_k$ and $C$ depend on $t$ if $T$ is a classical Friedrichs operator):

\[
\begin{cases}
    u'(t) + T(t)u(t) = f(t) \\
    u(0) = u_0
\end{cases}
\]
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$$\begin{align*}
  u'(t) + T(t)u(t) &= f(t) \\
  u(0) &= u_0
\end{align*}$$

– Semigroup theory can treat time-dependent case, but conditions that ensure existence/uniqueness result are rather complicated to verify…
Consider

\[ \begin{aligned}
    u'(t) + Tu(t) &= f(t, u(t)) \\
    u(0) &= u_0 
\end{aligned} \]

where \( f : [0, T] \times L \rightarrow L \).
TODO: Semilinear problem

Consider

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\begin{cases}
    u'(t) + Tu(t) = f(t, u(t)) \\
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– semigroup theory gives existence and uniqueness of solution
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- semigroup theory gives existence and uniqueness of solution
- it requires (locally) Lipschitz continuity of \( f \) in variable \( u \)
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where \( f : [0, T] \times L \to L \).

- semigroup theory gives existence and uniqueness of solution
- it requires (locally) Lipschitz continuity of \( f \) in variable \( u \)
- if \( L = L^2 \) it is not appropriate assumption, as power functions do not satisfy it; \( L = L^\infty \) is better...
TODO: Banach space setting

\[ \text{Let } L \text{ be a reflexive complex Banach space, } L' \text{ its antidual, } D \subseteq L, T, \tilde{T}: D \rightarrow L' \text{ linear operators that satisfy a modified versions of (T1)–(T3), e.g. (T1):} \]

\[ \forall \phi, \psi \in D, \langle T\phi, \psi \rangle_L = \langle \tilde{T}\psi, \phi \rangle_L. \]

Technical differences with Hilbert space case, but results remain essentially the same for the stationary case.

Problems:
- in the classical case (F1)–(F2) need not to imply (T2) and (T3): instead of (T3) we get
  \[ L^p \langle (T + \tilde{T})\phi, \phi \rangle_{L^p} \geq \|\phi\|_{L^2}. \]
- for semigroup treatment of non-stationary case we need to have
  \[ T: D \subseteq L \rightarrow L. \]
Let $L$ be a reflexive complex Banach space, $L'$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : \mathcal{D} \rightarrow L'$ linear operators that satisfy a modified version of (T1)–(T3).
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$$(T1) \quad (\forall \varphi, \psi \in \mathcal{D}) \quad L'\langle T\varphi, \psi \rangle_L = L'\langle \tilde{T}\psi, \phi \rangle_L.$$
Let $L$ be a reflexive complex Banach space, $L'$ its antidual, $\mathcal{D} \subseteq L$, $T, \tilde{T} : \mathcal{D} \rightarrow L'$ linear operators that satisfy a modified versions of (T1)–(T3), e.g.

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Let $L$ be a reflexive complex Banach space, $L'$ its antidual, $D \subseteq L$, $T, \tilde{T} : D \rightarrow L'$ linear operators that satisfy a modified versions of (T1)–(T3), e.g.

\[(T1) \quad (\forall \varphi, \psi \in D) \quad L'(T\varphi, \psi)_L = L'(\tilde{T}\psi, \phi)_L.\]

Technical differences with Hilbert space case, but results remain essentially the same for the stationary case.

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- in the classical case (F1)–(F2) need not to imply (T2) and (T3): instead of (T3) we get

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- for semigroup treatment of non-stationary case we need to have $T : D \subseteq L \rightarrow L$
Why should one be interested in Friedrichs systems?
  - Symmetric hyperbolic systems
  - Symmetric positive systems

Classical theory
  - Boundary conditions for Friedrichs systems
  - Existence, uniqueness, well-posedness

Abstract formulation
  - Graph spaces
  - Cone formalism of Ern, Guermond and Caplain
  - Interdependence of different representations of boundary conditions

Kreăăin space formalism
  - Kreăăin spaces
  - Equivalence of boundary conditions

What can we say for the Friedrichs operator now?
  - Sufficient assumptions
  - An example: elliptic equation
  - Other second order equations
  - Two-field theory
  - Non-stationary theory

Homogenisation of Friedrichs systems
  - Homogenisation
  - Examples: Stationary diffusion and heat equation

Concluding remarks
Homogenisation setting

Homogenisation setting


A sequence of Friedrichs systems \( T_n u_n = f, \ f \in L. \)
Homogenisation setting


A sequence of Friedrichs systems $T_n u_n = f$, $f \in L$.

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- \( C \in M_r(\alpha, \beta; \Omega) = \left\{ C \in L^\infty(\Omega; M_r(R)) : (\forall \xi \in \mathbb{R}^d) \right\} \)
  \[ C\xi \cdot \xi \geq \alpha|\xi|^2 \land C\xi \cdot \xi \geq \frac{1}{\beta}|C\xi|^2 \].
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$$T_0u = \sum_{k=1}^{d} \partial_k(A_ku) = \sum_{k=1}^{d} A_k \partial_k u,$$

so that $T := T_0 + C$ is the Friedrichs operator.
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so that $T := T_0 + \mathbf{C}$ is the Friedrichs operator. Its graph space

$$W := \{ u \in L : Tu \in L \}$$
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  \begin{equation*}
  \mathbf{C}\xi \cdot \xi \geq \alpha |\xi|^2 \text{ and } \mathbf{C}\xi \cdot \xi \geq \frac{1}{\beta} |\mathbf{C}\xi|^2
  \end{equation*}

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\[
W := \{ u \in L : Tu \in L \} = \{ u \in L : T_0 u \in L \}.
\]

Moreover, we have equivalence of norms (\( \gamma = \sqrt{\max\{2, 1 + 2\beta^2\}} \)):

\[
\|u\|_T \leq \gamma \|u\|_{T_0} \leq \gamma^2 \|u\|_T, \quad \text{for any } \mathbf{C}.
\]
Boundary operator and a priori bound

The boundary operator $D$ corresponding to the operator $T$ does not depend on particular $C$ from $\mathcal{M}_r(\alpha, \beta; \Omega)$. 

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If $V$ is a subspace of $W$ that satisfies (V), well-posedness result implies that $T|_V : V \rightarrow L$ is an isomorphism, with

$$\|u\|_{T_0} \leq \gamma \|u\|_T \leq \gamma \sqrt{\frac{1}{\alpha^2} + 1} \|Tu\|_L, \quad u \in V.$$
The boundary operator $D$ corresponding to the operator $T$ does not depend on particular $C$ from $\mathcal{M}_r(\alpha, \beta; \Omega)$.
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Therefore, for fixed $T_0$ and $V$ satisfying (V), we have a priori bound

$$(\exists c > 0)(\forall C \in \mathcal{M}_r(\alpha, \beta; \Omega))(\forall u \in V) \quad \|u\|_{T_0} \leq c \|(\mathcal{L}_0 + C)u\|_L.$$ 

Note that constant $c$ depends only on $T_0$, $\alpha$ and $\beta$. 

In the sequel $\mathcal{L}_0 = \sum_{k=1}^{d} A_k \partial_k$ and $V$ are fixed.

**Definition (H-convergence for Friedrichs systems)**

We say that a sequence $(C_n)$ in $\mathcal{M}_r(\alpha, \beta; \Omega)$ $H$-converges to $C \in \mathcal{M}_r(\alpha', \beta'; \Omega)$ with respect to $T_0$ and $V$ if, for any $f \in L$, the sequence $(u_n)$ in $V$ defined by $u_n := T_0^{-1}f \in V$, with $T_n = \mathcal{L}_0 + C_n$, satisfies

$$
\begin{align*}
    u_n &\rightharpoonup u \quad \text{in } L, \\
    C_n u_n &\rightharpoonup Cu \quad \text{in } L,
\end{align*}
$$

where $u = T^{-1}f \in V$, with $T = \mathcal{L}_0 + C$. 

**H-convergence**
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\[
\begin{align*}
\lim_{n \to \infty} u_n &\to u \quad \text{in} \ L, \\
\lim_{n \to \infty} C_n u_n &\to C u \quad \text{in} \ L,
\end{align*}
\]

where $u = T^{-1}f \in V$, with $T = \mathcal{L}_0 + C$.

As $T_0 u_n + C_n u_n = f = T_0 u + C u$, the second convergence implies $T_0 u_n \to T_0 u$ in $L$.
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Definition ($H$-convergence for Friedrichs systems)

We say that a sequence $(C_n)$ in $\mathcal{M}_r(\alpha, \beta; \Omega)$ $H$-converges to $C \in \mathcal{M}_r(\alpha', \beta'; \Omega)$ with respect to $T_0$ and $V$ if, for any $f \in L$, the sequence $(u_n)$ in $V$ defined by $u_n := T_n^{-1} f \in V$, with $T_n = \mathcal{L}_0 + C_n$, satisfies

$$u_n \rightharpoonup u \text{ in } L,$$

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where $u = T^{-1} f \in V$, with $T = \mathcal{L}_0 + C$.

As $T_0 u_n + C_n u_n = f = T_0 u + C u$, the second convergence implies $T_0 u_n \rightharpoonup T_0 u$ in $L$, which gives the weak convergence $u_n \rightharpoonup u$ in $W$. 
**Theorem**

Let $F = \{f_n : n \in \mathbb{N}\}$ be a dense countable family in $L^2(\Omega; \mathbb{R}^r)$, $C, D \in \mathcal{M}_r(\alpha, \beta; \Omega)$, and $u_n, v_n \in V$ solutions of $(T_0 + C)u_n = f_n$ and $(T_0 + D)v_n = f_n$, respectively. Furthermore, let $u_n - v_n \in H^{-1}(\Omega; \mathbb{R})$. 

\[
d(C, D) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{H^{-1}(\Omega; \mathbb{R}^r)} + \|Cu_n - Dv_n\|_{H^{-1}(\Omega; \mathbb{R}^r)}}{\|f_n\|_{L^2(\Omega; \mathbb{R}^r)}}.
\]

Then the function $d : \mathcal{M}_r(\alpha, \beta; \Omega) \times \mathcal{M}_r(\alpha, \beta; \Omega) \rightarrow \mathbb{R}$ forms a metric on the set $\mathcal{M}_r(\alpha, \beta; \Omega)$, and the $H$-convergence is equivalent to the sequential convergence in this metric space.
Compactness assumptions

Additional assumptions: for every sequence $C_n \in \mathcal{M}_r(\alpha, \beta; \Omega)$ and every $f \in L$, the sequence $u_n \in V$ defined by $u_n := (T_0 + C_n)^{-1}f$ satisfies the following: if $(u_n)$ weakly converges to $u$ in $W$, then also

$$(K1) \quad W \langle Du_n, u_n \rangle_W \longrightarrow W \langle Du, u \rangle_W,$$

or
Compactness assumptions

Additional assumptions: for every sequence \( C_n \in \mathcal{M}_r(\alpha, \beta; \Omega) \) and every \( f \in L \), the sequence \( u_n \in V \) defined by \( u_n := (T_0 + C_n)^{-1}f \) satisfies the following: if \((u_n)\) weakly converges to \( u \) in \( W \), then also

\[
\text{(K1)} \quad W \langle Du_n, u_n \rangle_W \longrightarrow W \langle Du, u \rangle_W,
\]

or

\[
\text{(K2)} \quad (\forall \varphi \in C_c^\infty(\Omega)) \quad \langle T_0 u_n | \varphi u_n \rangle_L \longrightarrow \langle T_0 u | \varphi u \rangle_L.
\]
Compactness theorems

Theorem

For fixed $T_0$ and $V$, if family $M_r(\alpha, \beta; \Omega)$ satisfies (K1) and (K2), then it is compact with respect to $H$-convergence, i.e. from any sequence $(C_n)$ in $M_r(\alpha, \beta; \Omega)$ one can extract a $H$-converging subsequence whose limit belongs to $M_r(\alpha, \beta; \Omega)$.

The proof follows the original proof of Spagnolo in the case of parabolic $G$-convergence.
Compactness theorems

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Stationary diffusion equation as Friedrichs system

\[-\text{div} (A \nabla u) + cu = f\]

in \( \Omega \subseteq \mathbb{R}^d \).
Stationary diffusion equation as Friedrichs system

\[-\text{div} (A \nabla u) + cu = f\]
in \(\Omega \subseteq \mathbb{R}^d\), with \(f \in L^2(\Omega)\), \(A \in \mathcal{M}_d(\alpha', \beta'; \Omega)\) and \(c \in L^\infty(\Omega)\) with \(\frac{1}{\beta'} \leq c \leq \frac{1}{\alpha'}\), for some \(\beta' \geq \alpha' > 0\).
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\[C = \begin{bmatrix} A^{-1} & 0 \\ 0 & c \end{bmatrix} \in L^\infty(\Omega; \mathcal{M}_{d+1}(\mathbb{R})) ,\]

Graph space \(W = L^2(\Omega) \times H^1(\Omega)\)
Stationary diffusion equation as Friedrichs system

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\[
C = \begin{bmatrix}
A^{-1} & 0 \\
0 & c
\end{bmatrix} \in L^\infty(\Omega; M_{d+1}(\mathbb{R})) ,
\]

\[
Tu = \sum_{k=1}^d A_k \partial_k u + Cu = f
\]

\[
T_0 \begin{bmatrix}
u_d \\
u_{d+1}
\end{bmatrix} = \begin{bmatrix}
\nabla u_{d+1} \\
\text{div} \ u_d
\end{bmatrix} , \quad Cu = \begin{bmatrix}
A^{-1} u_d \\
cu_{d+1}
\end{bmatrix} .
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Stationary diffusion equation as Friedrichs system

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in \( \Omega \subseteq \mathbb{R}^d \), with \( f \in L^2(\Omega) \), \( A \in M_d(\alpha', \beta'; \Omega) \) and \( c \in L^\infty(\Omega) \) with \( \frac{1}{\beta'} \leq c \leq \frac{1}{\alpha'} \), for some \( \beta' \geq \alpha' > 0 \). \( A_k = e_k \otimes e_{d+1} + e_{d+1} \otimes e_k \in M_{d+1}(\mathbb{R}) \), for \( k = 1, \ldots, d \)

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\end{bmatrix} .
\]

Graph space . . . \( W = L^2_{\text{div}}(\Omega) \times H^1(\Omega) \)
Boundary conditions

Dirichlet

\[ V_D = \tilde{V}_D := L^2_{\text{div}}(\Omega) \times H^1_0(\Omega), \]
Boundary conditions

\[ V_D = \tilde{V}_D := L^2_{\text{div}}(\Omega) \times H^1_0(\Omega), \]

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\[ V_N = \tilde{V}_N := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = 0\}, \]
Boundary conditions

Dirichlet

\[ V_D = \tilde{V}_D := L^2_{\text{div}}(\Omega) \times H^1_0(\Omega), \]

Neumann

\[ V_N = \tilde{V}_N := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = 0\}, \]

Robin

\[ V_R := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = au_{d+1}|_\Gamma\}, \]
\[ \tilde{V}_R := \{(u_d, u_{d+1})^\top \in W : \nu \cdot u_d = -au_{d+1}|_\Gamma\}. \]
Properties (K1) and (K2)

(K1) For any sequence \((u_n)\) in \(V\)

\[
    u_n \rightharpoonup u \implies \langle D u_n, u_n \rangle_W \to \langle D u, u \rangle_W
\]
Properties (K1) and (K2)

(K1) For any sequence \((u_n)\) in \(V\)

\[
\begin{align*}
&u_n \rightarrow u \quad \Rightarrow \quad W'\langle Du_n, u_n \rangle_W \rightarrow W'\langle Du, u \rangle_W \\
\end{align*}
\]

\[
W'\langle Du, u \rangle_W = 2H^{-\frac{1}{2}}\langle \nu \cdot u_d, u_{d+1} \rangle_{H^\frac{1}{2}} = \left\{
\begin{array}{ll}
0 & \ldots \text{Dirichlet or Neumann} \\
2a\|u_{d+1}\|^2_{L^2(\Gamma)} & \ldots \text{Robin} \\
\end{array}
\right.
\]

\(W = L^2_{\text{div}}(\Omega) \times H^1(\Omega)\)
Properties (K1) and (K2)

(K1) For any sequence \((u_n)\) in \(V\)
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W'\langle Du, u \rangle_W = 2H^{-\frac{1}{2}}\langle \nu \cdot u_d, u_{d+1} \rangle_{H^\frac{1}{2}}
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= \begin{cases} 
0 & \ldots \text{Dirichlet or Neumann} \\
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(K1) For any sequence \((u_n)\) in \(V\)

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\]

\[
W'(Du, u)_W = 2H^{-\frac{1}{2}}\langle \nu \cdot u_d, u_{d+1} \rangle_{H^{\frac{1}{2}}}
\]

\[
= \begin{cases} 
0 & \text{Dirichlet or Neumann} \\
2a\|u_{d+1}\|_{L^2(\Gamma)}^2 & \text{Robin} \quad W = L^2_{\text{div}}(\Omega) \times H^1(\Omega)
\end{cases}
\]

(K2) For any sequence \((u_n)\) in \(V\) and any \(\varphi \in C^\infty_c(\Omega)\)

\[
u_n \rightharpoonup u \implies \langle T_0 u_n | \varphi u_n \rangle_L \rightharpoonup \langle T_0 u | \varphi u \rangle_L
\]
Compensated compactness

\[ \langle T_0 u_n | \varphi u_n \rangle_L = \int_{\Omega} \sum_{k=1}^{d} A_k \partial_k u_n \cdot \varphi u_n \, dx , \]

\[ = -\frac{1}{2} \int_{\Omega} \partial_k \varphi \sum_{k=1}^{d} A_k u_n \cdot u_n \, dx \]
Compensated compactness

\[
\langle T_0 u_n \mid \varphi u_n \rangle_{L} = \int_{\Omega} \sum_{k=1}^{d} A_k \partial_k u_n \cdot \varphi u_n \, dx,
\]

\[
= -\frac{1}{2} \int_{\Omega} \varphi \sum_{k=1}^{d} A_k u_n \cdot u_n \, dx.
\]

Theorem (Quadratic theorem)

For \( A_k \in M_{q,p}(\mathbb{R}) \) let \( \Lambda := \left\{ \lambda \in \mathbb{R}^p : (\exists \xi \neq 0) \sum_{k=1}^{d} \xi_k A_k \lambda = 0 \right\} \)

\( Q(\lambda) := Q \lambda \cdot \lambda, \) such that \( Q = 0 \) on \( \Lambda, \)

\( u_n \rightharpoonup u \) weakly in \( L^2(\Omega; \mathbb{R}^p), \)

\[
\left( \sum_{k=1}^{d} A_k \partial_k u_n \right) \text{ is relatively compact in } H^{-1}(\Omega; \mathbb{R}^q).
\]

Then \( Q \circ u_n \rightharpoonup Q \circ u \) in \( \mathcal{D}'(\Omega). \)
Compensated compactness

\[ \langle T_0 u_n \mid \varphi u_n \rangle_L = \int_{\Omega} \sum_{k=1}^{d} A_k \partial_k u_n \cdot \varphi u_n \, dx, \quad p = q = d + 1 \]

\[ = -\frac{1}{2} \int_{\Omega} \partial_k \varphi \sum_{k=1}^{d} A_k u_n \cdot u_n \, dx \]

Theorem (Quadratic theorem)

For \( A_k \in M_{q,p}(\mathbb{R}) \) let

\[ \Lambda := \left\{ \lambda \in \mathbb{R}^p : (\exists \xi \neq 0) \sum_{k=1}^{d} \xi_k A_k \lambda = 0 \right\} \]

\[ Q(\lambda) := Q \lambda \cdot \lambda, \text{ such that } Q = 0 \text{ on } \Lambda, \]

\[ u_n \rightharpoonup u \text{ weakly in } L^2(\Omega;\mathbb{R}^p), \]

\[ \mathcal{L}_0 u_n = \left( \sum_{k=1}^{d} A_k \partial_k u_n \right) \text{ is relatively compact in } H^{-1}(\Omega;\mathbb{R}^q). \]

Then \( Q \circ u_n \rightharpoonup Q \circ u \text{ in } D'(\Omega). \)
Proof of (K2)

\[
\sum_{k=1}^{d} \xi_k A_k \lambda = \begin{bmatrix}
\lambda_{d+1} \xi_1 \\
\vdots \\
\lambda_{d+1} \xi_d \\
\sum_{k=1}^{d} \lambda_k \xi_k
\end{bmatrix} \implies \Lambda \ldots \lambda_{d+1} = 0
\]

\[
Q(\lambda) = A_i \lambda \cdot \lambda = 2\lambda_i \lambda_{d+1} = 0, \quad \lambda \in \Lambda
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Proof of (K2)

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\sum_{k=1}^{d} \xi_k A_k \lambda = \begin{bmatrix}
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Q(\lambda) = A_i \lambda \cdot \lambda = 2 \lambda_i \lambda_{d+1} = 0, \quad \lambda \in \Lambda
\]
Comparison with classical $H$-convergence

\[ C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0 & c_n \end{bmatrix} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega) \]

\[ \iff \begin{cases} C_n(x) \xi \cdot \xi \geq \alpha |\xi|^2 \\ C_n(x) \xi \cdot \xi \geq \frac{1}{\beta} |C_n(x)\xi|^2 \end{cases} \]
Comparison with classical $H$-convergence

\[ C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0^T & c_n \end{bmatrix} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega) \]

\[ \iff \begin{cases} C_n(x)\xi \cdot \xi \geq \alpha|\xi|^2 \\ C_n(x)\xi \cdot \xi \geq \frac{1}{\beta}|C_n(x)\xi|^2 \end{cases} \]

\[ \iff \begin{cases} \alpha \leq c_n(x) \leq \beta \\ A^n(x)\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \\ A^n(x)\xi \cdot \xi \geq \alpha|A^n(x)\xi|^2 \end{cases} \]
Comparison with classical $H$-convergence

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C_n = \begin{bmatrix}
(A^n)^{-1} & 0 \\
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\iff \begin{cases}
C_n(x)\xi \cdot \xi \geq \alpha|\xi|^2 \\
C_n(x)\xi \cdot \xi \geq \frac{1}{\beta}|C_n(x)\xi|^2
\end{cases}
\]

\[
\iff \begin{cases}
\alpha \leq c_n(x) \leq \beta \\
A^n(x)\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \\
A^n(x)\xi \cdot \xi \geq \alpha|A^n(x)\xi|^2
\end{cases}
\]

At a subsequence $C_n \overset{H}{\longrightarrow} C$, by compactness theorem.
Comparison with classical $H$-convergence

$$C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0^\top & c_n \end{bmatrix} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega)$$

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At a subsequence $C_n \xrightarrow{H} C$, by compactness theorem.

- Has the limit $C$ the same structure?
Comparison with classical $H$-convergence

\[
C_n = \begin{bmatrix}
(A_n^{n})^{-1} & 0 \\
0^\top & c_n
\end{bmatrix} \in \mathcal{M}_{d+1}(\alpha, \beta; \Omega)
\]

\[
\iff \begin{cases}
C_n(x)\xi \cdot \xi & \geq \alpha|\xi|^2 \\
C_n(x)\xi \cdot \xi & \geq \frac{1}{\beta}|C_n(x)\xi|^2
\end{cases}
\]

\[
\iff \begin{cases}
\alpha \leq c_n(x) \leq \beta \\
A^n(x)\xi \cdot \xi & \geq \frac{1}{\beta}|\xi|^2 \\
A^n(x)\xi \cdot \xi & \geq \alpha|A^n(x)\xi|^2
\end{cases}
\]

At a subsequence $C_n \xrightarrow{H} C$, by compactness theorem.
- Has the limit $C$ the same structure?
- Can we make a connection with $H$-converging (in classical sense) subsequence of $(A^n)$?
Characterisation of the $H$-limit

**Theorem**

For the Friedrichs system corresponding to the stationary diffusion equation, a sequence $(C_n)$ in $\mathcal{M}_{d+1}(\alpha, \beta; \Omega)$ of the form

$$C_n = \begin{bmatrix} (A_n^{-1}) & 0 \\ 0^\top & c_n \end{bmatrix}.$$  

$H$-converges with respect to $L_0$ and $V_D$ if and only if $(A_n)$ classically $H$-converges to some $A$ and $(c_n)$ $L^\infty$ weakly $\ast$ converges to some $c$. In that case, the $H$-limit is the matrix function

$$C = \begin{bmatrix} A^{-1} & 0 \\ 0^\top & c \end{bmatrix},$$
Heat equation as Friedrichs system

$\Omega \subseteq \mathbb{R}^d$ open and bounded set with Lipschitz boundary $\Gamma$, $T > 0$ and $\Omega_T := \Omega \times [0, T)$

$$\partial_t u_n - \text{div}_x (A^n \nabla_x u_n) + c u_n = f \quad \text{in } \Omega_T,$$

The matrices $A_k = e_k \otimes e_k + e_{d+1} \otimes e_k \in M_{d+1}(\mathbb{R})$, $k = 1, \ldots, d$, $A_{d+1} = e_{d+1} \otimes e_{d+1}$ and $C_n = \nabla_x^\top (A_n)^{-1} 0 0^\top c - T 0 \nabla_x^\top (u_{d+1} - u_0) - \text{div}_x u_{d+1}$. 

Graph space $W = \left\{ u \in L^2(\text{div} (\Omega_T)) : u_{d+1} \in L^2(0, T; H^1(\Omega)) \right\}$. 

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Heat equation as Friedrichs system

\( \Omega \subseteq \mathbb{R}^d \) open and bounded set with Lipschitz boundary \( \Gamma \), \( T > 0 \) and \( \Omega_T := \Omega \times (0, T) \)

\[
\partial_t u_n - \text{div}_x (A^n \nabla_x u_n) + cu_n = f \quad \text{in } \Omega_T,
\]

\[
u_n = \begin{bmatrix} u_{d_n} \\ u_{d+1_n} \end{bmatrix} = \begin{bmatrix} -A^n \nabla_x u_n \\ u_n \end{bmatrix}.
\]
Heat equation as Friedrichs system

\[ \Omega \subseteq \mathbb{R}^d \] open and bounded set with Lipschitz boundary \( \Gamma \), \( T > 0 \) and \( \Omega_T := \Omega \times \langle 0, T \rangle \)

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\( A_{d+1} = e_{d+1} \otimes e_{d+1} \) and

\[ C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0 & c \end{bmatrix} \]

\[ T_0 \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_x u_{d+1} \\ \partial_t u_{d+1} + \text{div}_x u_d \end{bmatrix}. \]
Heat equation as Friedrichs system

\( \Omega \subseteq \mathbb{R}^d \) open and bounded set with Lipschitz boundary \( \Gamma \), \( T > 0 \) and \( \Omega_T := \Omega \times (0, T) \)

\[ \partial_t u_n - \text{div}_x (A^n \nabla_x u_n) + cu_n = f \quad \text{in } \Omega_T, \]

\[ u_n = \begin{bmatrix} u_{dn} \\ u_{d+1n} \end{bmatrix} = \begin{bmatrix} -A^n \nabla_x u_n \\ u_n \end{bmatrix}. \]

The matrices \( A_k = e_k \otimes e_{d+1} + e_{d+1} \otimes e_k \in M_{d+1} (\mathbb{R}) \), \( k = 1, \ldots, d \), \( A_{d+1} = e_{d+1} \otimes e_{d+1} \) and

\[ C_n = \begin{bmatrix} (A^n)^{-1} & 0 \\ 0^\top & c \end{bmatrix} \]

\[ T_0 \begin{bmatrix} u_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_x u_{d+1} \\ \partial_t u_{d+1} + \text{div}_x u_d \end{bmatrix}. \]

Graph space

\( W = \left\{ u \in L^2_\text{div}(\Omega_T): u_{d+1} \in L^2(0, T; H^1(\Omega)) \right\} \).
Compactness result

Dirichlet boundary conditions with zero initial value:

\[ V = \left\{ u \in W : u_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \right\}, \]

\[ \widetilde{V} = \left\{ v \in W : v^u \in L^2(0, T; H^1_0(\Omega)), \quad v^u(\cdot, T) = 0 \text{ a.e. on } \Omega \right\}. \]
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(K1):

$$W'\langle Du, u \rangle_W = \|u_{d+1}(\cdot, T)\|_{L^2(\Omega)}^2.$$
Compactness result

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\[ \tilde{V} = \left\{ v \in W : v^u \in L^2(0, T; H^1_0(\Omega)), \quad v^u(\cdot, T) = 0 \text{ a.e. on } \Omega \right\}. \]

(K1): \[ W'\langle Du, u \rangle_W = \|u_{d+1}(\cdot, T)\|_{L^2(\Omega)}^2. \]

(K2): similarly to stationary diffusion equation: \( \Lambda = \{ \lambda \in \mathbb{R}^{d+1} : \lambda_{d+1} = 0 \} \)
Compactness result

Dirichlet boundary conditions with zero initial value:

\[ V = \left\{ u \in W : u_{d+1} \in L^2(0, T; H^1_0(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \right\}, \]

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(K1):

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(K2): similarly to stationary diffusion equation: \( \Lambda = \{ \lambda \in \mathbb{R}^{d+1} : \lambda_{d+1} = 0 \} \)

\[ \Rightarrow M_{d+1}(\alpha, \beta; \Omega) \text{ is compact with } H\text{-topology for given } \mathcal{L}_0 \text{ and } V \]

Comparison with classical parabolic H-convergence...
Compactness result

Dirichlet boundary conditions with zero initial value:

\[ V = \left\{ u \in W : u_{d+1} \in L^2(0, T; H_0^1(\Omega)), \quad u_{d+1}(\cdot, 0) = 0 \text{ a.e. on } \Omega \right\}, \]

\[ \tilde{V} = \left\{ v \in W : v^u \in L^2(0, T; H_0^1(\Omega)), \quad v^u(\cdot, T) = 0 \text{ a.e. on } \Omega \right\}. \]

(K1):

\[ W'\langle Du, u \rangle_W = \| u_{d+1}(\cdot, T) \|_{L^2(\Omega)}^2. \]

\(\checkmark\)

(K2): similarly to stationary diffusion equation: \( \Lambda = \{ \lambda \in \mathbb{R}^{d+1} : \lambda_{d+1} = 0 \} \)

\(\checkmark\)

\[ \implies \mathcal{M}_{d+1}(\alpha, \beta; \Omega) \text{ is compact with } H\text{-topology for given } L_0 \text{ and } V \]

Comparison with classical parabolic H-convergence. . . similarly as for stationary diffusion equation.
Instead of $C_n \in \mathcal{M}_r(\alpha, \beta; \Omega)$ we take

$$C_n \in \mathcal{F}(\alpha, \beta; \Omega) := \left\{ C \in \mathcal{L}(L) : (\forall u \in L) \langle Cu | u \rangle_L \geq \alpha \|u\|^2_L \quad \& \quad \langle Cu | u \rangle_L \geq \frac{1}{\beta} \|Cu\|^2_L \right\}.$$
\section*{\textit{G}-convergence}

Instead of $C_n \in \mathcal{M}_r(\alpha, \beta; \Omega)$ we take

$$C_n \in \mathcal{F}(\alpha, \beta; \Omega) := \left\{ C \in \mathcal{L}(L) : (\forall u \in L) \begin{array}{l}
\langle Cu | u \rangle_L \geq \alpha \|u\|_L^2 \\
\langle Cu | u \rangle_L \geq \frac{1}{\beta} \|Cu\|_L^2
\end{array} \right\}.$$

\textbf{Definition (\textit{G}-convergence for Friedrichs systems)}

For $C_n \in \mathcal{F}(\alpha, \beta; \Omega)$, we say that a sequence of isomorphisms $T_n := T_0 + C_n : V \rightarrow L$ \textit{G}-converges to an isomorphism $T := T_0 + C : V \rightarrow L$, for some $C \in \mathcal{F}(\alpha', \beta'; \Omega)$ if

$$(\forall f \in L) \quad T_n^{-1} f \rightharpoonup T^{-1} f \text{ in } W.$$ 

\textbf{Theorem}

For fixed $T_0$ and $V$, if family $\mathcal{F}(\alpha, \beta; \Omega)$ satisfies (K1), then for any sequence $(C_n)$ in $\mathcal{F}(\alpha, \beta; \Omega)$ there exists a subsequence of $T_n := T_0 + C_n$ which \textit{G}-converges to $T := T_0 + C$ with $C \in \mathcal{F}(\alpha, \beta; \Omega)$. 
Why should one be interested in Friedrichs systems?
   - Symmetric hyperbolic systems
   - Symmetric positive systems

Classical theory
   - Boundary conditions for Friedrichs systems
   - Existence, uniqueness, well-posedness

Abstract formulation
   - Graph spaces
   - Cone formalism of Ern, Guermond and Caplain
   - Interdependence of different representations of boundary conditions

Kreĭn space formalism
   - Kreĭn spaces
   - Equivalence of boundary conditions

What can we say for the Friedrichs operator now?
   - Sufficient assumptions
   - An example: elliptic equation
   - Other second order equations
   - Two-field theory
   - Non-stationary theory

Homogenisation of Friedrichs systems
   - Homogenisation
   - Examples: Stationary diffusion and heat equation

Concluding remarks
Open problems . . .

– Find all representations of a particular equation in the form of a Friedrichs system.
– Application to other equations of practical importance (mixed-type problems).
– Compare the results to those already known in the classical setting.


Publications


Marko Erceg, Krešimir Burazin: *Non-stationary abstract Friedrichs systems via semigroup theory*, submitted