H-distributions and mixed-norm Lebesgue spaces

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WeConMApp

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- Existence of H-measures
- Propagation property

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- Mixed-norm Lebesgue spaces
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- A compactness result

H-distributions

- Existence
- Localisation principle

Existence of H-measures

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^r)$, *then there exists its subsequence and a complex matrix Radon measure distribution of order zero* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi})\,.$$

∎

There are some other variants: (ultra)parabolic, fractional, one-scale, ...

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^d)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^d)$:

$(M_b u)(\mathbf{x}) := b(\mathbf{x}) u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier $\mathcal{A}_a$, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{\mathcal{A}_a u} = a\hat{u}$.

The norm is again equal to $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Delicate part: a is given only on S^1.

We extend it by the projection p: if α is a function defined on a compact surface, we take $a := \alpha \circ p$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big)$$

The precise scaling is contained in the projections, not the surface.

First commutation lemma

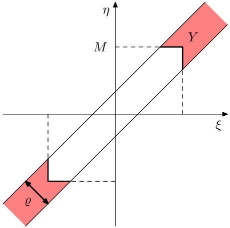
Lemma. (general form of the first commutation lemma — Luc Tartar)
If $b \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfy the condition*

$$(\forall\, \rho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\text{a.e. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \rho)) \;,$$

then $C := [\mathcal{A}_a, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$*.* ∎

For given $M, \rho \in \mathbf{R}^+$ denote the set

$$Y = Y(M, \rho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M \;\&\; |\boldsymbol{\xi} - \boldsymbol{\eta}| \leqslant \rho\} \;.$$



[Some improvements in N.A., M. Mišur, D. Mitrović (2018)]

The importance of First commutation lemma

If we take $\mathsf{u}_n = (u_n, v_n)$, and consider $\mu = \mu_{12}$, we have

$$\begin{aligned}
\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} \psi \, d\boldsymbol{\xi} &= \lim_{n'} \langle \mathcal{A}_\psi(\varphi_1 u_{n'}) | \varphi_2 v_{n'} \rangle \\
&= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} \, d\mathbf{x} \\
&= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} \, d\mathbf{x} = \langle \mu, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle \, .
\end{aligned}$$

Thus the limit is a bilinear functional in $\varphi_1 \bar{\varphi}_2$ and ψ, and we have the bound:

$$\Big| \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} d\mathbf{x} \Big| \leq C \|\psi\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|\varphi_1 \overline{\varphi_2}\|_{\mathrm{C}_0(\mathbf{R}^d)} \, .$$

This form makes sense even for $p < 2$ (for $p > 2$ we use the fact that $u_n \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$).

A class of symbols (L. Tartar)

Actually, we can consider more general operators than $\mathcal{A}_a$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|b_m\|_{\mathrm{C}_0(\mathbf{R}^d)} = k < \infty$.

To such a symbol s, a *standard operator* $S_s \in \mathrm{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$ is assigned by

$$S_s = \sum_m \mathcal{A}_{\alpha_m} M_b \ ,$$

with $\|S_s\|_{\mathrm{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$.

Clearly, S_s does not depend on the above decomposition, as

$$\widehat{S_s u}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} s(\mathbf{x}, \boldsymbol{\xi}/|\boldsymbol{\xi}|) u(\mathbf{x})\, d\mathbf{x} \ ,$$

for u in a dense set of $\mathrm{L}^2(\mathbf{R}^d)$ (e.g. $\mathcal{S}$).

A class of symbols (cont.)

Any operator $A \in \mathrm{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, which differs from S_s only by a compact operator, is an *operator of symbol* s, like

$$L_s = \sum_m M_{b_m} \mathcal{A}_{\alpha_m} \ ,$$

where $\|L_s\|_{\mathrm{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$. Neither L_s depends on the decomposition.

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^r)$*, then there exists its subsequence and an H-measure* $\boldsymbol{\mu}$*, which is a Hermitian non-negative* $r \times r$ *matrix of distributions of order zero on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and any operators* $L_{s_1}, L_{s_2} \in \mathrm{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$*, with symbols* s_1, s_2 *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} L_{s_1}(\varphi_1 u^j_{n'}) \overline{L_{s_2}(\varphi_2 u^k_{n'})} \, d\boldsymbol{\xi} = \langle \mu^{jk}, \varphi_1 s_1 \overline{\varphi_2 s_2} \rangle \ .$$

∎

P. Gérard used a different approach, by using classical symbols. However, it is important to have symbols of lower regularity, as they come in applications from coefficients in PDEs.

We can consider $\Omega \subseteq \mathbf{R}^d$ as a domain, or even a manifold (with a volume form).

Symmetric systems

$$\sum_k \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \quad \mathbf{A}^k \text{ Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0 \qquad \text{(weakly)},$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \qquad \text{(strongly)}.$$

If supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained inside Ω, we can extend them by zero to $\mathbf{R}^d$.

Theorem. (localisation property) *If* $\mathsf{u}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$*, and if* u^n *satisfies:*

$$\partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \to 0 \ \text{ in the space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:*

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}^\top = \mathbf{0} \ .$$

∎

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{P}$ is a singular matrix.)

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \le m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi}\ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).
On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.
X^m is an algebra with respect to the multiplication of functions; it holds:

$$\|f * g\|_{\mathrm{L}^1} \le \|f\|_{\mathrm{L}^1} \|g\|_{\mathrm{L}^1}$$
$$\|\hat{f} \cdot \hat{g}\|_{X^0} \le \|\hat{f}\|_{X^0} \|\hat{g}\|_{X^0}$$

$X^m_{\mathrm{loc}}(\Omega)$: the space of all functions u such that $\varphi u \in X^m$, for $\varphi \in \mathrm{C}^\infty_c(\Omega)$.

Lemma. *Let $\mathcal{A}_\alpha$, M_b be standard operators, with symbols α, b, such that $\alpha \in \mathrm{C}^1(S^{d-1})$ and $b \in X^1$.*
Then $C := [\mathcal{A}_\alpha, M_b] \in \mathcal{L}\big(\mathrm{L}^2(\mathbf{R}^d), \mathrm{H}^1(\mathbf{R}^d)\big)$, and ∇C has a symbol $(\nabla_{\boldsymbol{\xi}} \alpha \cdot \nabla_{\mathbf{x}} b)\boldsymbol{\xi}$.
(we extend α to a homogeneous function on $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{\mathbf{0}\}$)

∎

A smaller class of symbols (L. Tartar)

Corollary. *Under the above assumptions,*

$$\mathcal{A}_\alpha M_b \partial_j u = M_b \partial_j (\mathcal{A}_\alpha u) + Lu, \quad u \in \mathrm{L}^2(\mathbf{R}^d) \ ,$$

where L has a symbol $\xi_j\{\alpha, b\}$. ∎

Actually, we can consider more general operators than $\mathcal{A}_\alpha$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}^1(\mathrm{S}^{d-1})} \|b_m\|_{X^1} < \infty$,
and *standard operators* $S_s = \sum_m \mathcal{A}_{\alpha_m} M_b$.

Lemma. *If S_1, S_2 are standard operators with symbols s_1, s_2 as above, then*

$$\frac{\partial}{\partial x^j}[S_1, S_2] \quad \textit{has symbol} \quad \xi_j\{s_1, s_2\} \ .$$

∎

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \qquad \mathbf{A}^k \text{ Hermitian}$$

Theorem. *Let* $\mathbf{A}^k \in \mathrm{C}_0^1(\Omega; \mathrm{M}_{r\times r})$*.*
If $(\mathsf{u}^n, \mathsf{f}^n)$ *satisfy the above for* $n \in \mathbf{N}$*, and* $\mathsf{u}^n, \mathsf{f}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega)$*, then for any* $\psi \in \mathrm{C}_0^1(\Omega \times S^{d-1})$*, the H-measure associated to sequence* $(\mathsf{u}^n, \mathsf{f}^n)$*:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

satisfies:

$$\left\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - 2\psi \mathbf{S} \right\rangle + \langle 2\mathsf{Re}\,\mathsf{tr}\boldsymbol{\mu}_{12}, \psi \rangle = 0 \;,$$

where $\mathbf{S} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^*)$*, while the Poisson bracket is:*
$\{\phi, Q\} = \nabla_{\boldsymbol{\xi}} \phi \cdot \nabla_{\mathbf{x}} Q - \nabla_{\mathbf{x}} \phi \cdot \nabla_{\boldsymbol{\xi}} Q.$ *[Recall:* $\mathbf{P} = \xi_k \mathbf{A}^k$ *]* ∎

$\boldsymbol{\mu}$ is associated to the pair of sequences $(\mathsf{u}^n, \mathsf{f}^n)$, the block $\boldsymbol{\mu}_{11}$ is determined by u^n, $\boldsymbol{\mu}_{22}$ with f^n, while the non-diagonal blocks correspond to the product of u^n and f^n.

The equation for H-measure

Corollary. *In the sense of distributions on $\Omega \times S^{d-1}$ the H-measure $\boldsymbol{\mu}$ satisfies:*

$$
\begin{aligned}
\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu}_{11} - \partial_t^l(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11})\xi^l \\
+ (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}\ ,
\end{aligned}
$$

where $\partial_t^l := \partial^l - \xi^l \xi_k \partial^k$ is the tangential gradient on the unit sphere. ∎

This allows us to investigate the behaviour of H-measures as solutions of initial-value problems, with appropriate initial conditions.
Besides the wave equations, there are applications to Maxwell's and Dirac's systems, even to the equations that change their type (like the Tricomi equation).

The wave equation

$$(\rho u')' - \mathsf{div}\,(\mathbf{A}\nabla u) = g \ .$$

It can be written as an equivalent symmetric system ($t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$):

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) = g \ .$$

By introducing: $v_j := \partial_j u$, for $j \in 0..d$, we obtain (Schwarz' symmetries!):

$$\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A} & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{v} + \sum_{i=1}^{d} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{id} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{id} & & & \end{bmatrix} \partial_i \mathsf{v}$$

$$+ \begin{bmatrix} b^0 & b^1 & \cdots & b^d \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{v} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ .$$

The symbol of differential operator is:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \xi_k \mathbf{A}^k(\mathbf{x}) = \begin{bmatrix} \xi_0\rho & -(\mathbf{A}\boldsymbol{\xi}')^\top \\ -\mathbf{A}\boldsymbol{\xi}' & \xi_0\mathbf{A} \end{bmatrix} \ .$$

Transport of H-measures associated to the wave equation

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$. For the right hand side of the equation we have:

$$\langle \gamma, \varphi_1 \bar{\varphi}_2 \psi \rangle := \lim_n \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 v_{0,n}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 g_n}(\boldsymbol{\xi})} \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} \ .$$

Theorem. *On $\mathbf{R}^{d+1} \times S^d$ measure ν satisfies ($Q := \rho\xi_0^2 - \mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}'$):*

$$\nabla_{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}}(\xi_0 \nu) - Q\partial_0 \nu + \big(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \mathbf{I}\big)\nabla_{\mathbf{x}} Q \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)\big(\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}\big)(\xi_0 \nu) = 2\mathrm{Re}\,\gamma$$

∎

The equation can be written in a nicer form:

$$\{Q, \xi_0 \nu\} + (\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi})\big[\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)(\xi_0 \nu)\big] - Q\partial_0 \nu = 2\mathrm{Re}\,\gamma \ .$$

Recent Tartar's result (2017)

Theorem. *Let* $u^n \in \mathrm{C}^0(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{C}^1(0,T;\mathrm{L}^2(\Omega))$ *be a sequence of solutions of a "wave equation"*

$$(\rho(u^n)')' - \mathsf{div}\,(\mathbf{A}\nabla u^n) + S^k\partial_k u^n \longrightarrow 0 \qquad \textit{in } \mathrm{L}^2_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)\ ,$$

with $\rho, \mathbf{A}$ *in* $\mathrm{X}^1_{loc} \cap \mathrm{C}^2$, $\rho > 0$ *and* $\mathbf{A}$ *real positive definite (or replace it with its symmetric part, and subsume the lower order terms in the last term), and* S^k *be standard operators with symbols* s^k.
If $u^n \rightharpoonup 0$ *in* $\mathrm{H}^1_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)$, *then* ∇u^n *corresponds to an H-measure* $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\pi$, *then*

$$Q\pi = 0\ ,$$

and

$$\Big\langle \pi, \{\Psi, Q\} + (\xi_k s^k + \xi_k \bar{s}^k)\Psi \Big\rangle = 0\ ,$$

for $\Psi \in \mathrm{C}^1_c(\langle 0,T\rangle \times \Omega \times \mathrm{S}^d)$. ∎

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \ . \end{cases}$$

We have used D'Alembert's formula for solution, our approach and the approach of P. Gérad, obtaining the same result in this special case (which is treatable by both methods, and explicit calculations).

Physically important quantity is *energy density:*

$$d(t,x) := \frac{1}{2}(u_t^2 + u_x^2) \ ,$$

as well as the *energy* at time t: $$e(t) := \int_{\mathbf{R}} d(t,x)\, dx.$$

After simple calculations we get

$$4d(t,x) = \Big(v'(x+t) + w(x+t)\Big)^2 + \Big(v'(x-t) - w(x-t)\Big)^2 \ .$$

Assume that the physical system is modelled by the above wave equation on the microscale. In order to pass to the macroscale, in the spirit of Tatar's programme, we have to pass to the weak limit.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightarrow{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightarrow{\mathrm{L}^2(\mathbf{R})} 0 \; .$$

It follows that

$$u_n \longrightarrow 0 \; ,$$

but $d_n \longrightarrow d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.
$u_t^2 - u_x^2 \longrightarrow 0$;
the kinetic and potential energy are balanced at the macroscopic level.

In order to determine the solution completely, let us take periodically modulated initial conditions (we work in spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ and $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$):

$$v_n(x) := \frac{1}{n}\sin(nx) \qquad \text{and} \qquad w_n(x) := \sin(nx) \; .$$

Simple calculations lead us to: $d_n(t,x) = 1 + \cos 2nx \sin 2nt \longrightarrow 1$,
weak $*$ in the space of Radon measures, therefore in the space of distributions as well.

Even though the sequence of solutions (u_n) weakly converges to zero, the energy density is 1, equally distributed to kinetic and potential energy.

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v\right] ,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v\right] .$$

Convergence of initial data and uniformly compact support imply:

$$u_n \xrightarrow{*} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

The energy density is $d_n = q(\nabla u_n)$.
Goal: compute the distributional limit d_n, i.e. the limit

$$D_n = \int_{\langle 0,T\rangle\times\mathbf{R}^d} d_n\phi\, dtdx \ .$$

Results:

— Gilles Francfort & François Murat (1992): in linear case, C^∞ coefficients

— Patrick Gérard (1996): for constant coefficients, nonlinearity with term $u^p, p \leqslant 5$

— N. A. & Martin Lazar (2002): for symmetric hyperbolic systems

We have attempted to do the same for semilinear wave equation ($d = 3, p = 3$), with variable coefficients. The difficulties led to the study of mixed-norm Lebesgue spaces, and also prompted the introduction of H-distributions.
For nonlinear equations L^2 theory usually does not work; one should try the L^p spaces.

A general view

We can unify the results: consider equations of the form

$$P_0(\varrho P_0 u_n) + \mathsf{P}_1 \cdot \mathbf{A}\mathsf{P}_1 u_n = 0\,,$$

where P_0 and P_1 stand for (pseudo)differential operators in time and space variables, with (principal) symbols p_0 and p_1, and $Q = \varrho p_0^2 + \mathbf{A}\mathsf{p}_1 \cdot \mathsf{p}_1$ being the symbol of the differential operator defining the left-hand side of the equation. For the H-measure $\tilde{\boldsymbol{\mu}}$ associated to $(P_0 u_n, \mathsf{P}_1 u_n)$, converging weakly in L^2 to 0, $\tilde{\boldsymbol{\mu}}$ is of the form

$$\tilde{\boldsymbol{\mu}} = \frac{\overline{\mathsf{p} \otimes \mathsf{p}}}{|\mathsf{p}|^2}\tilde{\nu}\,,$$

where $\tilde{\nu} := \mathrm{tr}\tilde{\boldsymbol{\mu}}$ is a scalar measure, and the localisation principle reads

$$Q\tilde{\nu} = 0\,.$$

Finally, the propagation principle states

$$\Big\langle \frac{\xi_m \tilde{\nu}}{|\mathsf{p}|^2}, \{\phi, Q\} \Big\rangle + \Big\langle \frac{\nu}{|\mathsf{p}|^2}, p\,\partial_m Q \Big\rangle = 0\,.$$

This covers both the classical and the parabolic case.

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For $\mathbf{p} \in [1,\infty\rangle^d$, by $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ denote the space of f on $\mathbf{R}^d$ with finite norm

$$\|f\|_{\mathbf{p}} = \left(\int_{\mathbf{R}} \cdots \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(x_1,\dots,x_d)|^{p_1} dx_1\right)^{p_2/p_1} dx_2\right)^{p_3/p_2} \cdots dx_d\right)^{1/p_d},$$

and analogously for $p_i = \infty$.

These Banach spaces can be seen as vector-valued Lebesgue spaces in the sense

$$\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) = \mathrm{L}^{p_d}_{x_d}(\mathbf{R}; \mathrm{L}^{(p_1,\dots,p_{d-1})}_{x_1,\dots,x_{d-1}}(\mathbf{R}^{d-1})) .$$

$$\mathbf{p}' = (p_1',\dots,p_d'), \quad \frac{1}{p_i} + \frac{1}{p_i'} = 1$$

Some facts:

(a) $\mathcal{S} \hookrightarrow \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$,
(b) $\mathcal{S}$ is dense in $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1,\infty\rangle^d$,
(c) $\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)$ is topological dual of $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1,\infty\rangle^d$,
(d) $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}'$.

Basic results

Some generalisations of classical results are still valid:

dominated convergence for $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ spaces, $\mathbf{p} \in [1, \infty\rangle^d$
Let (f_n) be sequence of measurable functions. If $f_n \longrightarrow f$ (ss), and if there exists $G \in \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ such that $|f_n| \leqslant G$ (ss), for $n \in \mathbf{N}$, then $\|f_n - f\|_{\mathbf{p}} \longrightarrow 0$.

Minkowski ineaquality for integrals
For $\mathbf{p} \in [1, \infty]^{d_1}$ and $f \in \mathrm{L}^{(\mathbf{p},1,\dots,1)}(\mathbf{R}^{d_1+d_2})$ we have

$$\Big\| \int_{\mathbf{R}^{d_2}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \Big\|_{\mathbf{p}} \leqslant \int_{\mathbf{R}^{d_2}} \|f(\cdot, \mathbf{y})\|_{\mathbf{p}} d\mathbf{y}.$$

Hölder's ineaquality and its converse
For $\mathbf{p} \in [1, \infty]^d$ we have

$$\Big| \int_{\mathbf{R}^d} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} \Big| \leqslant \|f\|_{\mathbf{p}} \|g\|_{\mathbf{p}'}.$$

and

$$\|f\|_{\mathbf{p}} = \sup_{g \in \mathrm{S}_{\mathbf{p}'}} \Big| \int f \bar{g} d\mathbf{x} \Big| = \sup_{g \in \mathrm{S}_{\mathbf{p}'} \cap \mathcal{S}} \Big| \int f \bar{g} d\mathbf{x} \Big|,$$

where $\mathrm{S}_{\mathbf{p}'}$ is a unit sphere in $\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)$.

Boundedness of pseudodifferential operators on classical spaces

(ρ, δ)-*symbol* of order $m \in \mathbf{N}$ $\quad (\lambda(\boldsymbol{\xi}) = \sqrt{1 + 4\pi^2 |\boldsymbol{\xi}|^2})$

$$|\partial_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} a(\mathbf{x}, \boldsymbol{\xi})| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \lambda^{m-\rho|\boldsymbol{\beta}|+\delta|\boldsymbol{\alpha}|}(\boldsymbol{\xi}),$$

and the associated operator $a(\cdot, D) : \mathcal{S} \longrightarrow \mathcal{S}$

$$(a(\mathbf{x}, D)\varphi)(\mathbf{x}) = \int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} a(\mathbf{x}, \boldsymbol{\xi}) \hat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

Using the adjoint operator, it can be extended to an operator on $\mathcal{S}'$.

Classical boundedness results on Lebesgue spaces:

- L. Hörmander: for $0 \leqslant \delta \leqslant \rho \leqslant 1$ and $\delta < 1$ the necessary condition is

$$m \leqslant -d(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right|.$$

- C. Fefferman: for $1 < p < \infty$ this condition is also sufficient.

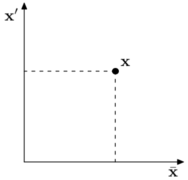
The strongest results are for $\rho = 1$, valid even for $m = 0$, which easily leads to generalisations for Sobolev spaces ($|\boldsymbol{\gamma}| \leqslant k - m$) via

$$(\partial^{\boldsymbol{\gamma}} a)(\cdot, D) = \sum_{|\boldsymbol{\alpha}| \leqslant k} a_{\boldsymbol{\alpha}}(\cdot, D) \partial^{\boldsymbol{\alpha}}\ .$$

Notation

$$\mathbf{x} = (\bar{\mathbf{x}}, \mathbf{x}'),\ \bar{\mathbf{x}} = (x_1, \dots, x_r),\ \mathbf{x}' = (x_{r+1}, \dots, x_d),\ 0 \leqslant r \leqslant d-1,$$

$$\mathrm{L}^{\bar{\mathbf{p}},\, p}(\mathbf{R}^d) = \mathrm{L}^{(\bar{\mathbf{p}}, p, \dots, p)}(\mathbf{R}^d),\ \|f\|_{\bar{\mathbf{p}},\, p} = \|f\|_{(\bar{\mathbf{p}},\, p, \dots,\, p)},\ \bar{\mathbf{p}} = (p_1, \dots, p_r).$$



If $r = 0$: $\quad \|f(\cdot, \mathbf{x}')\|_{\bar{\mathbf{p}}} = |f(\mathbf{x}')|, \quad \|f\|_{\bar{\mathbf{p}},\, p} = \|f\|_{\mathrm{L}^p}.$

Distribution function:

$$\lambda_f(\alpha) = \lambda(f; \alpha) = \mathrm{vol}\{\mathbf{x} \in \mathbf{R}^d : |f(\mathbf{x})| > \alpha\}.$$

(a) λ_f is non-increasing and right continuous.
(b) If $|f| \leqslant |g|$, then $\lambda_f \leqslant \lambda_g$.
(c) If $|f_n| \nearrow |f|$, then $\lambda_{f_n} \nearrow \lambda_f$.
(d) If $f = g + h$, it follows $\lambda(f; \alpha) \leqslant \lambda(g; \frac{\alpha}{2}) + \lambda(h; \frac{\alpha}{2})$.

General framework

Theorem. *Assume:*

1) $A, A^* : \mathrm{L}^\infty_c(\mathbf{R}^d) \to \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$ *are formally adjoint linear operators.*
2) *For both* $T = A$ *and* $T = A^*$ *there exist constants* $N > 1$ *and* $c_1 > 0$ *satisfying*

$$(\forall\, r \in 0..(d{-}1))(\forall\, \mathbf{x}'_0 \in \mathbf{R}^{d-r})(\forall\, t > 0) \quad \int\limits_{|\mathbf{x}'-\mathbf{x}'_0|_\infty > Nt} \|Tf(\cdot,\mathbf{x}')\|_{\bar{\mathbf{p}}}\, d\mathbf{x}' \leqslant c_1 \|f\|_{\bar{\mathbf{p}},1} \, ,$$

for any function f *in a subspace of* $\mathrm{L}^\infty_c(\mathbf{R}^d)$ *determined by properties:*
(a) $\operatorname{supp} f \subseteq \mathbf{R}^r \times \{\mathbf{x}' : |\mathbf{x}' - \mathbf{x}'_0|_\infty \leqslant t\}$,
(b) $\int_{\mathbf{R}^{d-r}} f(\bar{\mathbf{x}},\mathbf{x}')\, d\mathbf{x}' = 0 \quad (a.e.\ \bar{\mathbf{x}} \in \mathbf{R}^r)$.
3) *For some* $q \in \langle 1, \infty\rangle$ A *has a continuous extension to an operator from* $\mathrm{L}^q(\mathbf{R}^d)$ *to itself with norm* c_q.

Then A *has a continuous extension to an operator from* $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ *to itself for any* $\mathbf{p} \in \langle 1, \infty\rangle^d$, *with the norm*

$$\|A\|_{\mathrm{L}^{\mathbf{p}} \to \mathrm{L}^{\mathbf{p}}} \leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(c_1 + c_q)$$
$$\leqslant c' \prod_{j=0}^{d-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(c_1 + c_q),$$

where c *and* c' *are constants depending only on* N *and* d. ∎

A few words about the proof

Note that we are using $\mathrm{L}^\infty_c(\mathbf{R}^d)$ as a dense subspace, and not $\mathrm{C}^\infty_c(\mathbf{R}^d)$, as we have to use the Calderón-Zygmund decomposition.

The proof follows by repeated application of

Lemma. *Assume that $A, A^* : \mathrm{L}^\infty_c(\mathbf{R}^d) \to \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$ are linear operators satisfying assumptions of the theorem.*
If operator A has a continuous extension from $\mathrm{L}^{\bar{\mathbf{p}},\,q}(\mathbf{R}^d)$ to itself with norm c_q, for some $\bar{\mathbf{p}} \in \langle 1, \infty\rangle^r$ and $q \in \langle 1, \infty\rangle$, then A has a continuous extension from $\mathrm{L}^{\bar{\mathbf{p}},\,p}(\mathbf{R}^d)$ to itself for all $p \in \langle 1, \infty\rangle$, with norm

$$\|A\| \leqslant c \cdot \max(p, (p-1)^{-1/p})(c_1 + c_q),$$

where c is a constant depending only on N and d. ∎

If some of the consecutive p_i-s are equal, we can get a bit better estimate.

The boundedness

Teorem. *Pseudodifferential operators of class* $S^0_{1,\delta}$, $\delta \in [0,1\rangle$ *are bounded on* $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in \langle 1, \infty\rangle^d$, *with an estimate as in the previous theorem.* ∎

We have considered several venues for the proof:

- Using the techniques from N.A.& I. Ivec (2016) ... work only for compactly supported operators.
- Modifying the apporach in M. W. Wong's book (1999), as it was done in J. Aleksić, S. Pilipović & I. Vojnović (2017) ... in the mixed norm case some calculations did not work out.
- We followed Stein (1993): the representation of pseudodifferential operator

$$(a(\mathbf{x}, D)\varphi)(\mathbf{x}) = k(\mathbf{x}, \cdot) * \varphi \,,$$

where the kernel $k(\mathbf{x}, \cdot)$ is a tempered distrubution such that $\widehat{k(\mathbf{x}, \cdot)} = a(\mathbf{x}, \cdot)$. Outside the origin kernel $k(\mathbf{x}, \cdot)$ is in fact a smooth function decreasing at infinity; more precisely, the following theorem holds:

Teorem. *If* $a \in S^m_{1,0}$, *then* $k \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d \setminus \{\mathbf{0}\})$

$$|\partial_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} k(\mathbf{x}, \mathbf{z})| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta},N} |\mathbf{z}|^{-d-m-|\boldsymbol{\beta}|-N}, \qquad \mathbf{z} \neq \mathbf{0},$$

for multiindices $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$, *and* $N \in \mathbf{N}_0$ *such that* $d + m + |\boldsymbol{\beta}| + N > 0$. ∎

The key lemma

The theorem is true also for $0 \leqslant \delta < 1$, if $\boldsymbol{\alpha} = 0$!

Lemma. *For each* $N > d$ *there is a* $c > 0$ *such that for* $\bar{\mathbf{p}} \in \langle 1, \infty \rangle^r$ *and any* $f \in \bigcup_{1<p< \ infty} \mathrm{L}^p(\mathbf{R}^d)$ *satisfying:*

$\operatorname{supp} f \subseteq \mathbf{R}^r \times \{\mathbf{x}' : |\mathbf{x}' - \mathbf{x}'_0|_\infty \leqslant t\},\ \mathbf{x}'_0 \in \mathbf{R}^{d-r},\ t > 0,$

$\int f(\bar{\mathbf{x}}, \mathbf{x}') d\mathbf{x}' = 0$ *a.e.* $\bar{\mathbf{x}}$,

it holds

$$\int_{|\mathbf{x}' - \mathbf{x}'_0|_\infty > Nt} \|a(\cdot, D) f\|_{\bar{\mathbf{p}}} \mathbf{x}' \leqslant c \|f\|_{\bar{\mathbf{p}},1}.$$

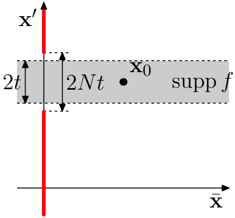


Figure 1. The support of function and the area of integration are disjoint ∎

Corollary. $a \in S^m_{1,\delta}$*,* $k \geqslant m$*, then* $a(\cdot, D) : \mathrm{W}^{k,\mathbf{p}}(\mathbf{R}^d) \longrightarrow \mathrm{W}^{k-m,\mathbf{p}}(\mathbf{R}^d)$ *is bounded.* ∎

Boundedness of integral operators

Another application of the general theorem on

$$Af(\mathbf{x}) = \int_{\mathbf{R}^d} K(\mathbf{x},\mathbf{y})f(\mathbf{y})\,d\mathbf{y},$$

and it is a known fact that they are bounded on $\mathrm{L}^p(\mathbf{R}^d)$ for $p \in [1,\infty]$ (the Schur test) if the following sufficient conditions are satisfied:

$$(\exists\, C_1, C_2 > 0)\ \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\,d\mathbf{x} < C_1\ \ \text{(a.e. }\mathbf{y}\text{)},\quad \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\,d\mathbf{y} < C_2\ \ \text{(a.e. }\mathbf{x}\text{)}.$$

Theorem. *If kernel K of the integral operator satisfies*

$$C_1 := \int_{\mathbf{R}^d} \|K(\cdot,\cdot-\mathbf{y})\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\,d\mathbf{y} < \infty, \qquad C_2 := \int_{\mathbf{R}^d} \|K(\cdot-\mathbf{y},\cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\,d\mathbf{y} < \infty$$

then it is bounded on $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in \langle 1,\infty\rangle^d$. ∎

Compactness

We consider

$$\mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d) = \left\{ u \in \mathcal{S}' : \mathcal{F}^{-1}((1+4\pi^2|\boldsymbol{\xi}|^2)^{\frac{s}{2}}\hat{u}) \in \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \right\}.$$

For two Banach spaces $A_0, A_1 \leqslant X$, we can define a space $(A_0, A_1)_{[\theta]}$ for $\theta \in [0,1]$ by complex interpolation.

First define a vector space $\mathcal{F}(A_0, A_1)$ consisting of all functions of complex variable with values in $A_0 + A_1$, which are bounded and continuous on the closed strip

$$S = \{z \in \mathbf{C} : 0 \leqslant \mathrm{Re}\, z \leqslant 1\},$$

and analytic on the open strip.

Moreover, the functions $t \mapsto f(j+it)$ are continuous from $\mathbf{R}$ into A_j, and tend to zero as $|t| \longrightarrow \infty$. The norm is

$$\|f\|_{\mathcal{F}} = \max\left\{ \sup_{t\in\mathbf{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbf{R}} \|f(1+it)\|_{A_1} \right\}.$$

Then we define

$$(A_0, A_1)_{[\theta]} = \left\{ a \in A_0 + A_1 \ : \ a = f(\theta) \text{ for some } f \in \mathcal{F}(A_0, A_1) \right\},$$

with the norm

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} \ : \ f(\theta) = a, \ f \in \mathcal{F}(A_0, A_1)\}.$$

Main theorem

Theorem. *Let* $s_0, s_1 \in \mathbf{R}$*,* $0 < \theta < 1$*, and* $s = (1-\theta)s_0 + \theta s_1$*. Then*

$$\left(\mathrm{H}^{s_0,\mathbf{p}_0}(\mathbf{R}^d), \mathrm{H}^{s_1,\mathbf{p}_1}(\mathbf{R}^d)\right)_{[\theta]} = \mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d) \ ,$$

for any $\mathbf{1} < \mathbf{p}_0, \mathbf{p}_1 < \boldsymbol{\infty}$*, where* $1/\mathbf{p} = (1-\theta)/\mathbf{p}_0 + \theta/\mathbf{p}_1$*.* ∎

This, in turn, leads to a form of the Rellich-Kondrašov theorem for mixed-norm spaces

Theorem. *Let* $\mathbf{p} \in \langle 1, \infty \rangle^d$*,* $t < s$*,* $\varphi \in C_c^\infty(\mathbf{R}^d)$*. Assume that* (u_n) *is a bounded sequence in* $H^{s,\mathbf{p}}(\mathbf{R}^d)$*. Then there exists a subsequence of the given sequence (not relabelled) such that* (φu_n) *converges in* $H^{t,\mathbf{p}}(\mathbf{R}^d)$*.* ∎

Microlocal energy density for hyperbolic systems

- Existence of H-measures
- Propagation property

Lebesgue spaces with mixed norm

- Mixed-norm Lebesgue spaces
- Boundedness of pseudodifferential operators of class $S^0_{1,\delta}$
- A compactness result

H-distributions

- Existence
- Localisation principle

Good bounds: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)\ , \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

can be extended to a continuous mapping $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$.

Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|}\ ,$$

then for any $p \in \langle 1, \infty\rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty)\ .$$

∎

For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

Hörmander-Mihlin's theorem for mixed-norm spaces

Theorem. *Let* $m \in \mathrm{L}^\infty(\mathbf{R}^d \setminus \{0\})$ *for some* $A > 0$ *and any* $|\boldsymbol{\alpha}| \leqslant [\frac{d}{2}] + 1$
(a) either Mihlin's condition $\quad |\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})| \leqslant A|\boldsymbol{\xi}|^{-|\boldsymbol{\alpha}|}$ *or*
(b) Hörmander's condition

$$\sup_{R>0} R^{-d+2|\boldsymbol{\alpha}|} \int_{R<|\boldsymbol{\xi}|<2R} |\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leqslant A^2 < \infty \, .$$

Then m *lies in* $\mathcal{M}_{\mathbf{p}}$, *for any* $\mathbf{p} \in \langle 1, \infty\rangle^d$, *and we have the estimate*

$$\begin{aligned} \|m\|_{\mathcal{M}_{\mathbf{p}}} &\leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max\{p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}}\}(A + \|m\|_{\mathrm{L}^\infty}) \\ &\leqslant c' \prod_{j=0}^{d-1} \max\{p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}}\}(A + \|m\|_{\mathrm{L}^\infty}) \, , \end{aligned}$$

where c *and* c' *are constants that depend only on* d. ∎

[N.A. & I. Ivec (2016)]

First commutation lemma on mixed-norm Lebesgue spaces

Lemma. *Let* (v_n) *be bounded both in* $\mathrm{L}^2(\mathbf{R}^d)$ *and in* $\mathrm{L}^{\mathbf{r}}(\mathbf{R}^d)$*, for some* $\mathbf{r} \in [2,\infty]^d$*, and such that* $v_n \longrightarrow 0$ *in* $\mathcal{D}'$*. Then* (Cv_n)*, where the commutator is defined by* $C := \mathcal{A}_\psi M_\varphi - M_\varphi \mathcal{A}_\psi$*, strongly converges to zero in* $\mathrm{L}^{\mathbf{q}}(\mathbf{R}^d)$*, for any* $\mathbf{q} \in [2,\infty\rangle^d$ *such that there exists* $\lambda \in \langle 0,1\rangle$ *for which it holds*

$$\frac{1}{q_i} = \frac{\lambda}{2} + \frac{1-\lambda}{r_i}, \qquad i \in 1..d\,.$$

∎

H-distributions on mixed-norm Lebesgue spaces

Theorem. *Let* $\kappa = [d/2] + 1$ *and* $\mathbf{p} \in \langle 1, \infty\rangle^d$*. If* $u_n \rightharpoonup 0$ *weakly in* $\mathrm{L}^{\mathbf{p}}_{\mathrm{loc}}(\mathbf{R}^d)$*,* $v_n \overset{*}{\rightharpoonup} v$ *in* $\mathrm{L}^{\mathbf{q}}_{\mathrm{loc}}(\mathbf{R}^d)$*, for some* $\mathbf{q} \in [2, \infty]^d$ *such that* $\mathbf{q} > \mathbf{p}'$*, then there exist subsequences* $(u_{n'})$ *and* $(v_{n'})$ *and a complex distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times S^{d-1})$*, such that for* $\phi_1, \phi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(S^{d-1})$ *one has*

$$\begin{aligned}\lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\Big\langle \mathcal{A}_\psi(\phi_1 u_{n'}), \phi_2 v_{n'} \Big\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)} &= \lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\Big\langle \phi_1 u_{n'}, \mathcal{A}_{\overline{\psi}}(\phi_2 v_{n'}) \Big\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)} \\ &= \langle \mu, \overline{\phi}_1 \phi_2 \boxtimes \overline{\psi}\rangle \,,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \longrightarrow \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ *is the Fourier multiplier operator.* ∎

μ is the *H-distribution* corresponding to (a subsequence of) (u_n) and (v_n).

If (u_n), (v_n) are defined on $\Omega \subseteq \mathbf{R}^d$, extension by zero to $\mathbf{R}^d$ preserves the convergence, and we can apply the Theorem. μ is supported on $\mathrm{Cl}\,\Omega \times \mathrm{S}^{d-1}$.

We distinguish $u_n \in \mathrm{L}^p(\mathbf{R}^d)$ and $v_n \in \mathrm{L}^q(\mathbf{R}^d)$. For $p \geqslant 2$, $p' \leqslant 2$ and we can take $q \geqslant 2$; this covers the L^2 case (including $u_n = v_n$).

The assumptions imply $u_n, v_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, resulting in a distribution μ of order zero (an unbounded Radon measure, not a general distribution).

The novelty in Theorem is for $p < 2$.

For vector-valued $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

The H-distribution would correspond to a non-diagonal block for an H-measure.

The proof is based on First commutation lemma

If $q < r$, we can apply the classical interpolation inequality:

$$\|Cv_n\|_q \leqslant \|Cv_n\|_2^{\alpha} \|Cv_n\|_r^{1-\alpha} \ ,$$

for $\alpha \in \langle 0, 1\rangle$ such that $1/q = \alpha/2 + (1-\alpha)/r$. As C is compact on $\mathrm{L}^2(\mathbf{R}^d)$ by Tartar's First commutation lemma, while it is bounded on $\mathrm{L}^r(\mathbf{R}^d)$, we get the claim.

For the most interesting case, where $q = r$, we need a better result: the Krasnosel'skij theorem (a variant of Riesz-Thorin theorem).

In fact, the commutator C is compact on all $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$.
For that we need an extension of the Krasnosel'skij's result to unbounded domains [N.A., M. Mišur, D. Mitrović (2018)]

Lemma. *Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle \setminus \{2\}$. Then A is also compact on any $\mathrm{L}^p(\Omega)$, where $1/p = \theta/2 + (1-\theta)/r$, for a $\theta \in \langle 0, 1\rangle$.* ∎

We still need a lemma on *compactness* of uniformly bounded bilinear forms, and an application of the Schwartz kernel theorem.

By carefully revisiting the proof, we know that H-distributions are actually of order 0 in $\mathbf{x}$, and of order not more than $d(\kappa + 2)$ in $\boldsymbol{\xi}$.
[M. Mišur's talk at ISAAC17 congress in Vaxjö last summer]

However, the specific examples of H-distributions that we have are all of order 0 in both variables.

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d \rangle$, *such that*

$$\mathsf{div}\,(\mathsf{a}(\mathbf{x}) u_n(\mathbf{x})) = f_n(\mathbf{x}) \; .$$

Take an arbitrary (v_n) *bounded in* $\mathrm{L}^\infty(\mathbf{R}^d)$, *and by* μ *denote the* H*-distribution corresponding to a subsequence of* (u_n) *and* (v_n)*. Then*

$$(\mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}) \mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}$ *being the symbol of the linear PDO with* C_0^κ *coefficients.*

∎

In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\boldsymbol{\xi}|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\frac{\xi_j}{i|\boldsymbol{\xi}|}}$.

Note that

$$\int I_1(\phi) \partial_j g = \int (R_j \phi) g, \quad g \in \mathcal{S}(\mathbf{R}^d).$$

Using the density argument and that R_j is bounded from $\mathrm{L}^p(\mathbf{R}^d)$ to itself, we conclude $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in \mathrm{L}^p(\mathbf{R}^d)$.

Compactness by compensation: L^2 case

It is well known that weak convergences are ill behaved under nonlinear transformations. Only in some particular cases of compensation it is even possible to pass to the limit in a product of two weakly converging sequences.

The prototype of this compensation effect is Murat-Tartar's div-rot lemma.

For simplicity consider 2D case, (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^2(\mathbf{R}^2)$, such that $(\partial_x u_n^1 + \partial_y u_n^2)$ and $(\partial_y v_n^1 - \partial_x v_n^2)$ are both contained in a compact set of $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$ (which then implies that they converge to zero strongly in $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$).

We can define $\mathsf{U}_n := \begin{bmatrix} \mathsf{u}_n \\ \mathsf{v}_n \end{bmatrix}$, which (on a subsequence) defines a 4×4 H-measure $\boldsymbol{\mu}$. By the localisation principle, as the above relations can be written in the form ($\mathbf{A}^1, \mathbf{A}^2$ are 4×4 constant matrices with all entries zero except $A^1_{11} = A^2_{12} = A^2_{33} = 1$ and $A^1_{34} = -1$)

$$\mathbf{A}^1 \partial_1 \mathsf{U}_n + \mathbf{A}^2 \partial_2 \mathsf{U}_n \to 0 \text{ strongly in } \mathrm{H}^{-1}_{loc}(\mathbf{R}^2)^4 ,$$

the corresponding H-measure satisfies $(\xi_1 \mathbf{A}^1 + \xi_2 \mathbf{A}^2)\boldsymbol{\mu} = \mathbf{0}$. After straightforward calculations this shows that $u_n^1 v_n^1 + u_n^2 v_n^2 \longrightarrow 0$ weak $*$ in the sense of Radon measures (and therefore in the sense of distributions as well).

What for sequences in L^p?

For the above we have used only the non-diagonal blocks $\boldsymbol{\mu}_{12} = \boldsymbol{\mu}_{21}^*$ of

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix} ,$$

corresponding to products of u_n^i and v_n^j; in fact, the calculation shows that $\mu_{12}^{11} + \mu_{12}^{22} = 0$, which gives the above result.

Assume now (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^p(\mathbf{R}^2)$ and $\mathrm{L}^{p'}(\mathbf{R}^2)$, and $(\partial_1 u_n^1 + \partial_2 u_n^2)$ bounded in $\mathrm{L}^p(\mathbf{R}^2)$, while $(\partial_2 v_n^1 - \partial_1 v_n^2)$ in $\mathrm{L}^{p'}(\mathbf{R}^2)$ (thus precompact in $\mathrm{W}^{-1,p}_{loc}(\mathbf{R}^2)$, and $\mathrm{W}^{-1,p'}_{loc}(\mathbf{R}^2)$).

Then $(u_n^1 v_n^1 + u_n^2 v_n^2)$ is bounded in $\mathrm{L}^1(\mathbf{R}^2)$, so also in $\mathcal{M}_b$ (Radon measures), and by weak $*$ compactness it has a weakly converging subsequence. However, we can say more—the whole sequence converges to zero.

Denote by μ^{ij} the H-distribution corresponding to (some sub)sequences (of) (u_n^1, u_n^2) and (v_n^1, v_n^2).
Since $(\partial_1 u_n^1 + \partial_2 u_n^2)$ is bounded in $\mathrm{L}^p(\mathbf{R}^2)$, and $(\partial_2 v_n^1 - \partial_1 v_n^2)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^2)$, they are weakly precompact, while the only possible limit is zero, so

$$\partial_1 u_n^1 + \partial_2 u_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^p , \qquad \text{and}$$
$$\partial_2 v_n^1 - \partial_1 v_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^{p'} .$$

From the compactness of the Riesz potential I_1 mentioned above, we conclude that for $\varphi \in \mathrm{C}_c(\mathbf{R}^2)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ the following limit holds in $\mathrm{L}^p(\mathbf{R}^2)$:

$$\mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_1}{|\boldsymbol{\xi}|}}(\varphi u_n^1) + \mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_2}{|\boldsymbol{\xi}|}}(\varphi u_n^2) = \mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\partial_1(\varphi u_n^1) + \partial_2(\varphi u_n^2)) \to 0\,.$$

Multiplying it first by φv_n^1 and then by φv_n^2, integrating over $\mathbf{R}^2$ and passing to the limit, we conclude from the existence theorem that:

$$\xi_1\mu^{11} + \xi_2\mu^{21} = 0, \qquad \text{and} \qquad \xi_1\mu^{12} + \xi_2\mu^{22} = 0\,.$$

Next, take

$$w_n^j = \varphi\mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\varphi u_n^j) \in \mathrm{W}^{1,p'}(\mathbf{R}^d), \quad j = 1, 2.$$

From the last limits on the preceeding slide we get

$$\langle(\varphi v_n^1, -\varphi v_n^2), \nabla w_n^j\rangle = -\langle \mathsf{rot}\,(\varphi v_n^1, \varphi v_n^2), w_n^j\rangle \to 0 \quad \text{as} \quad n \to \infty,$$

for $j = 1, 2$. Rewriting it in the integral formulation, we obtain again from the existence theorem:

$$\xi_2\mu^{11} - \xi_1\mu^{12} = 0, \quad \xi_2\mu^{21} - \xi_1\mu^{22} = 0.$$

From the algebraic relations above, we can easily conclude

$$\xi_1\left(\mu^{11} + \mu^{22}\right) = 0 \quad \text{and} \quad \xi_2\left(\mu^{11} + \mu^{22}\right) = 0,$$

implying that the distribution $\mu^{11} + \mu^{22}$ is supported on the set $\{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap P = \emptyset$, which implies $\mu^{11} + \mu^{22} \equiv 0$.
After inserting $\psi \equiv 1$ in the definition of H-distribution, we immediately reach the conclusion.

This proof is similar to the L^2 case, but it should be noted that we had used only a non-diagonal block of 4×4 H-measure, which corresponds to the only available 2×2 H-distribution.

There is no reason to limit oneself to two dimensions; take (u_n) and (v_n) converging weakly to zero in $\mathrm{L}^p(\mathbf{R}^d)^d$ and $\mathrm{L}^{p'}(\mathbf{R}^d)^d$, and by $\boldsymbol{\mu}$ denote $d \times d$ matrix H-distribution corresponding to some chosen subsequences of (u_n) and (v_n).

Theorem. *Let (u_n) and (v_n) be vector valued sequences converging to zero weakly in $\mathrm{L}^p(\mathbf{R}^d)^d$ and $\mathrm{L}^{p'}(\mathbf{R}^d)^d$, respectively. Assume the sequence $(\operatorname{div} \mathsf{u}_n)$ is bounded in $\mathrm{L}^p(\mathbf{R}^d)$, and the sequence $(\operatorname{rot} \mathsf{v}_n)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^d)^{d\times d}$. Then the sequence $(\mathsf{u}_n \cdot \mathsf{v}_n)$ converges to zero in the sense of distributions (or vaguely in the sense of Radon measures).* ∎

The results carry on to loc spaces as well.

Thank you for your attention!