H-measures and variants with a characteristic length

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Joint work with Marko Erceg and Martin Lazar

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_H \in \mathcal{M}_{\mathrm{b}}(\Omega\times\mathrm{S}^{d-1};\mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ such that for every $\varphi_1,\varphi_2 \in \mathrm{C}_0(\Omega)$ and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$

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Measure $\boldsymbol{\mu}_H$ we call the H-measure corresponding to the (sub)sequence (u_n).

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Above we use the notation

$$\mathsf{v} \cdot \mathsf{u} := \sum v_i \bar{u}_i \,, \quad (\mathsf{v} \otimes \mathsf{u})\mathsf{a} := (\mathsf{a} \cdot \mathsf{u})\mathsf{v} \,, \text{while} \quad (f \boxtimes g)(\mathbf{x}, \boldsymbol{\xi}) := f(\mathbf{x}) g(\boldsymbol{\xi}) \,.$$

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Theorem.

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_H = \mathbf{0} \ .$$

Example 1: Oscillation

Take a periodic function $v \in \mathrm{L}^2(\mathbf{R}^d/\mathbf{Z}^d)$, extend it to $\mathbf{R}^d$, and write

$$v(\mathbf{x}) = \sum_{\mathsf{k} \in \mathbf{Z}^d} \hat{v}_{\mathsf{k}} e^{2\pi i \mathsf{k} \cdot \mathbf{x}} \ .$$

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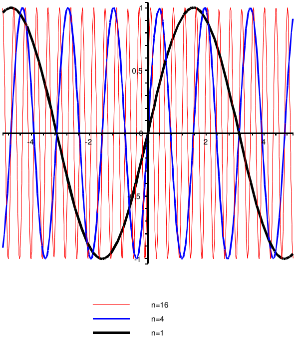

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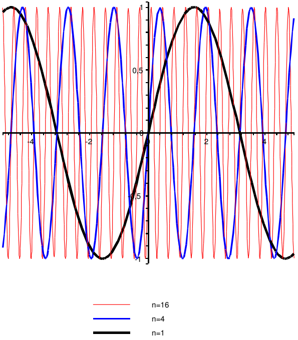


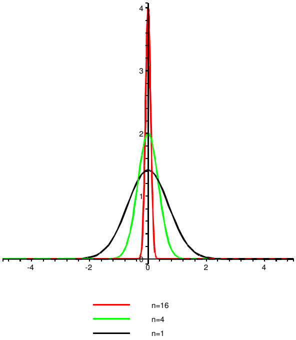
Associated H-measure

$$\mu_H = \sum_{\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}} |\hat{v}_{\mathsf{k}}|^2 \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) \lambda(\mathbf{x}) \,.$$

Example 2: Concentration

For $U \in \mathrm{L}^2(\mathbf{R}^d)$ define

$$u_n(x) = n^{\frac{d}{2}} U(nx) \ .$$



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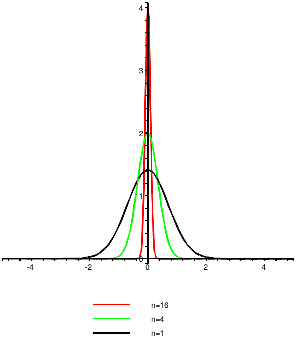

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$$\mu_H = \int_{\mathbf{R}^d} |\hat{U}(\mathbf{y})|^2 \delta_{\frac{\mathbf{y}}{|\mathbf{y}|}}(\boldsymbol{\xi}) \delta_0(\mathbf{x}) d\mathbf{y} \ .$$

Variants without a characteristic length

N. A., M. Lazar (2007–13): parabolic H-measures
E. Yu. Panov (2009): ultraparabolic H-measures
I. Ivec, D. Mitrović (2011): for fractional scalar conservation laws
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H-measures can be tailored to the equations, following the above ideas (and surmounting technical details, which do appear).
The objects are quadratic in nature, and are suited essentially to linear problems.

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Other dualities are also possible, like mixed-norm Lebesgue spaces by N.A. and I. Ivec (submitted), and Sobolev spaces by J. Aleksić, S. Pilipović and I. Vojnović.

Semiclassical measures

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A sample problem:
consider $T > 0$, $\Omega \subseteq \mathbf{R}^d$, $U := \langle 0, T\rangle \times \Omega$, (u_n) in $\mathrm{H}^1_{\mathrm{loc}}(U)$,
$u_n \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}(U)} 0$, $\mathbf{A} \in \mathrm{W}^{1,\infty}(U)$, $f_n \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}(U)} 0$, and $\varepsilon_n \searrow 0$

$$\partial_t u_n - \varepsilon_n \mathsf{div}\,(\mathbf{A}\nabla u_n) = f_n \ .$$

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What can we say about solutions on the limit $n \to \infty$?

Existence

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\omega_n \searrow 0$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_{sc} \in \mathcal{M}_{\mathrm{b}}(\Omega \times \mathbf{R}^d; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ such that for every $\varphi_1, \varphi_2 \in \mathrm{C}_0(\Omega)$ and $\psi \in \mathcal{S}(\mathbf{R}^d)$

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$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{sc} = \mathbf{0} \quad \& \quad (\mathsf{u}_n) \text{ is } (\omega_n) - \text{oscillatory} \, .$$

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(u_n) is (ω_n)-oscillatory if

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \lim_{R \to \infty} \limsup_n \int_{|\boldsymbol{\xi}| \geqslant \frac{R}{\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} = 0 \, .$$

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Example 1a: Oscillation — one characteristic length

$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{\mathsf{0}\}$, $\omega_n \searrow 0$:

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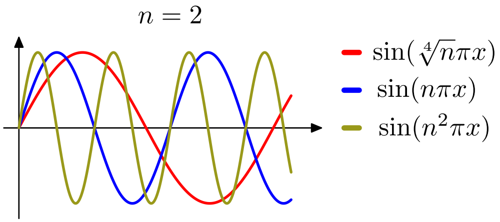

Example 1b: Oscillation — two characteristic lengths

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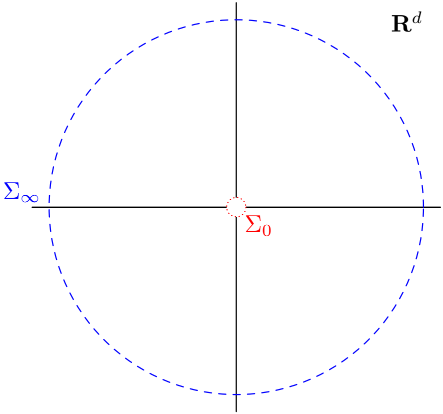
Further step would be to introduce multi-scale H-measures.
An attempt was made by Tartar (2014).

Compatification of $\mathbf{R}^d \setminus \{0\}$



$$\Sigma_0 := \{0^{\boldsymbol{\xi}_0} \ : \ \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}$$

$$\Sigma_\infty := \{\infty^{\boldsymbol{\xi}_0} \ : \ \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}$$

$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := (\mathbf{R}^d \setminus \{0\}) \cup \Sigma_0 \cup \Sigma_\infty$$

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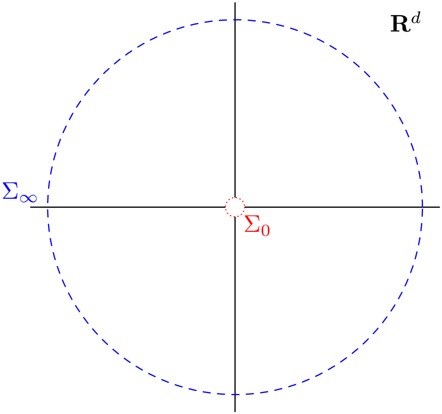

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We have:
a) $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \pi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, where $\pi(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

Existence and definition of one-scale H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, $\omega_n \searrow 0$, then there exist a subsequence $(\mathsf{u}_{n'})$ and $\boldsymbol{\mu}_{sc} \in \mathcal{M}_{\mathrm{b}}(\Omega\times\mathbf{R}^d;\mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ such that for every $\varphi_1,\varphi_2 \in \mathrm{C}_0(\Omega)$ and $\psi \in \mathcal{S}(\mathbf{R}^d)$

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Theorem.

a) $\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$

b) $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$

c) $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 \iff (\mathsf{u}_n)$ is (ω_n) – oscillatory

Example 1a revisited

$$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}},$$

$$\mu_H = \lambda(\mathbf{x}) \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi})$$

$$\mu_{sc} = \lambda(\mathbf{x}) \boxtimes \begin{cases} \delta_0(\boldsymbol{\xi}), & \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}}(\boldsymbol{\xi}), & \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ 0, & \lim_n n^\alpha \omega_n = \infty \end{cases}$$

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The corresponding measures of $u_n + v_n$ for:

$$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}} \quad , \qquad v_n(\mathbf{x}) = e^{2\pi i n^\beta \mathsf{s}\cdot\mathbf{x}} \; ,$$

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Other variants

One-scale parabolic H-measures

A similar construction can be carried out by starting with parabolic H-measures instead of classical H-measures.

The resulting objects will have two scales: one corresponding to t, and another to x.

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One-scale H-distributions

This construction requires much more work. The topological construction is not enough, as we also have to check the derivatives.

However, the construction is feasible, and we obtain the new objects.

Localisation principle

Most of the known applications of H-measures depend in one way or the other on the localisation principle, which gives the information on the support of H-measure.
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A similar statement holds for semiclassical measures as well.

Localisation principle for H-measures (symmetric systems)

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}) + \mathbf{B}\mathsf{u} = \mathsf{f} \ \ , \ \mathbf{A}^k \in \mathrm{C}_b(\Omega; \mathrm{M}_{r\times r}) \text{ Hermitian}$$

Assume:

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0 \ , \qquad \text{and defines } \boldsymbol{\mu}_H$$

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then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{k=1}^{d} \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

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Thus, the support of H-measure $\boldsymbol{\mu}$ is contaned in the set
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It contains a generalisation of compactness by compensation to variable coefficients.

Higher derivatives and parabolic variant

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$ and

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Then we have

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}_H^{\top} = \mathbf{0}\,,$$

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In the parabolic case the details become more involved.

One needs anisotropic Sobolev spaces and fractional derivatives in t.
However, similar results can be achieved.

Localisation principle for semiclassical measures

Let $\varepsilon \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\varepsilon; \mathrm{M_r}(\mathbf{C}))$, $\varepsilon_n \searrow 0$, $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\varepsilon; \mathbf{C}^r)$ and consider:

$$P_n \mathsf{u}_n = \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \varepsilon \, .$$

Furthermore, assume that $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\varepsilon; \mathbf{C}^r)$.

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Then we have

$$\mathbf{p}(\mathbf{x}, \boldsymbol\xi)\boldsymbol\mu_{sc}^{\top} = \mathbf{0}\,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol\xi) = \sum_{|\boldsymbol\alpha|\leqslant m} \boldsymbol\xi^{\boldsymbol\alpha}\mathbf{A}^{\boldsymbol\alpha}(\mathbf{x})$, and $\boldsymbol\mu_{sc}$ is the semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

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Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

One-scale H-measures

Let $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, $\varepsilon_n \searrow 0$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega;\mathrm{M}_r(\mathbf{C}))$

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

where $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ such that

$$(\forall\,\varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

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Lemma.
a) $(\mathrm{C}(\varepsilon_n))$ is equivalent to

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ $\quad \Longrightarrow \quad$ $(\varepsilon_n^{k-l} \mathsf{f}_n)$ satisfies $(\mathrm{C}(\varepsilon_n))$.

Localisation principle: Tartar's result

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem. [Tartar (2009)] Under previous assumptions and $l=1$, one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ε_n corresponding to (u_n) satisfies

$$\mathrm{supp}\,(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

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Theorem. Under previous assumptions, one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ε_n corresponding to (u_n) satisfies

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Localisation principle: a generalisation

Theorem. $\varepsilon_n \longrightarrow 0$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ satisfies $(\mathrm{C}(\varepsilon_n))$.
Then for $\omega_n \to 0$ such that $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in [0, \infty]$, the corresponding one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ω_n satisfies

$$\mathbf{p}\boldsymbol{\mu}^{\top}_{\mathrm{K}_{0,\infty}} = \mathbf{0} \,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

Sketch of the proof.

Suppose that we have already obtained the result for $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle$.

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In the case $\lim_n \frac{\omega_n}{\varepsilon_n} = \infty$ we rewrite equations in the form

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \omega_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{B}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

for $\mathbf{B}^{\boldsymbol{\alpha}} := \left(\frac{\varepsilon_n}{\omega_n}\right)^{|\boldsymbol{\alpha}|-l} \mathbf{A}^{\boldsymbol{\alpha}}$.

Sketch of the proof.

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In the case $\lim_n \frac{\omega_n}{\varepsilon_n} = \infty$ we rewrite equations in the form

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \omega_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{B}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

for $\mathbf{B}^{\boldsymbol{\alpha}} := \left(\frac{\varepsilon_n}{\omega_n}\right)^{|\boldsymbol{\alpha}|-l} \mathbf{A}^{\boldsymbol{\alpha}}$.
Similary for the case $\lim_n \frac{\omega_n}{\varepsilon_n} = 0$ we have

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \omega_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{B}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{g}_n \,,$$

where $\mathbf{B}^{\boldsymbol{\alpha}} := \left(\frac{\omega_n}{\varepsilon_n}\right)^{m-|\boldsymbol{\alpha}|} \mathbf{A}^{\boldsymbol{\alpha}}$, and $\mathsf{g}_n := \left(\frac{\omega_n}{\varepsilon_n}\right)^{m-l} \mathsf{f}_n$.

Proof (Step 1: inserting test function)

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n$$

Proof (Step 1: inserting test function)

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where $(\tilde{\mathsf{f}}_n)$ satisfies $(\mathrm{C}(\varepsilon_n))$.

Proof (Step 1: inserting test function)

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \;\; / \varphi \in \mathrm{C}_c^\infty(\Omega)$$

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$$\implies \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \sum_{0 \leqslant \boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} (-1)^{|\boldsymbol{\beta}|} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}-\boldsymbol{\beta}} \Big((\partial_{\boldsymbol{\beta}} \varphi) \mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n \Big) = \varphi \mathsf{f}_n$$

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$$\Longrightarrow \quad \partial_{\boldsymbol{\alpha}-\boldsymbol{\beta}} \Big((\partial_{\boldsymbol{\beta}} \varphi) \mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n \Big) \longrightarrow \mathsf{x} \text{ in } \mathrm{H}^{-|\boldsymbol{\alpha}|}(\Omega; \mathbf{C}^r) \,, \ 0 < \boldsymbol{\beta} \leqslant \boldsymbol{\alpha}$$

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$$\Longrightarrow \quad \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m}\sum_{0\leqslant\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} (-1)^{|\boldsymbol{\beta}|}\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}-\boldsymbol{\beta}}\Big((\partial_{\boldsymbol{\beta}}\varphi)\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n\Big) = \varphi\mathsf{f}_n$$

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We can rewrite

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} \Big(\mathbf{A}^{\boldsymbol{\alpha}} \varphi \mathsf{u}_n \Big) = \tilde{\mathsf{f}}_n$$

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Proof (Step 2: Fourier transform)

After applying Fourier transform and multiplying by $\frac{1}{1+|\boldsymbol{\xi}|^l+\varepsilon_n^{m-l}|\boldsymbol{\xi}|^m}$ we get:

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}(2\pi i)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}\,\widehat{\mathbf{A}^{\boldsymbol{\alpha}}\varphi \mathsf{u}_n}}{1+|\boldsymbol{\xi}|^l+\varepsilon_n^{m-l}|\boldsymbol{\xi}|^m} = \frac{\widehat{\tilde{\mathsf{f}}_n}}{1+|\boldsymbol{\xi}|^l+\varepsilon_n^{m-l}|\boldsymbol{\xi}|^m} \xrightarrow{\mathrm{L}^2} 0\,.$$

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Lemma. (f_n) mesurable (vector valued) on $\mathbf{R}^d$, $h_n \geqslant 0$ and

$$(\forall\, r>0)(\exists\, \tilde{C}>0)(\forall\, n\in\mathbf{N})(\forall\, \boldsymbol{\xi}\in\mathbf{R}^d\setminus \mathrm{K}(0,r)) \qquad h_n(\boldsymbol{\xi})\geqslant \tilde{C}\,,$$

(u_n) bounded in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\cap\mathrm{L}^1(\mathbf{R}^d;\mathbf{C}^r)$ and $\frac{\mathsf{f}_n}{1+h_n}\cdot\hat{\mathsf{u}}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d)$.
If $(h_n^{-2}|\mathsf{f}_n|^2)$ is equiintegrable then

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If $(h_n^{-2}|\mathsf{f}_n|^2)$ is equiintegrable then

$$\frac{\mathsf{f}_n}{h_n}\cdot\hat{\mathsf{u}}_n \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d)\,.$$

$$\Longrightarrow \quad \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m}(2\pi i)^{|\boldsymbol{\alpha}|}\frac{(\varepsilon_n\boldsymbol{\xi})^{\boldsymbol{\alpha}}}{|\varepsilon_n\boldsymbol{\xi}|^l+|\varepsilon_n\boldsymbol{\xi}|^m}\widehat{\mathbf{A}^{\boldsymbol{\alpha}}\varphi\mathsf{u}_n} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$$

The convergence is expressed in L^2.

Proof (Step 3: passing to the limit)

In order to apply the existence theorem, $\boldsymbol{\xi} \mapsto \frac{\varepsilon_n^{|\boldsymbol{\alpha}|-l}\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+\varepsilon_n^{m-l}|\boldsymbol{\xi}|^m}$ should be written as a function in variable $\omega_n\boldsymbol{\xi}$.

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Then, we need to prove (it is trivial for $l = m$)

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\psi_{\boldsymbol{\alpha}}(\omega_n\cdot)\widehat{\mathbf{A}^{\boldsymbol{\alpha}}\varphi \mathsf{u}_n} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,,$$

where $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) := \frac{c^{m-|\boldsymbol{\alpha}|}\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{c^{m-l}|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}$, defined for $\boldsymbol{\xi} \in \mathbf{R}^d_*$, can be understood as a function from $\mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

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This requires some calculations . . . (skipped)

Proof (Step 3: passing to the limit)

Multiplication by $\psi(\varepsilon_n\cdot)\widehat{\varphi_1 \mathsf{u}_n}$, with $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi_1 \in \mathrm{C}_c^\infty(\Omega)$, and integration

$$0 = \lim_n \int_{\mathbf{R}^d} \psi(\varepsilon_n \boldsymbol{\xi}) \left(\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|} \frac{(\varepsilon_n \boldsymbol{\xi})^{\boldsymbol{\alpha}}}{|\varepsilon_n \boldsymbol{\xi}|^l + |\varepsilon_n \boldsymbol{\xi}|^m} \widehat{\mathbf{A}^{\boldsymbol{\alpha}} \varphi \mathsf{u}_n} \right) \otimes \left(\widehat{\varphi_1 \mathsf{u}_n} \right) d\boldsymbol{\xi}$$

$$= \left\langle \overline{\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}}, \varphi \bar{\varphi}_1 \boxtimes \psi \right\rangle ,$$

where we have used $\boldsymbol{\xi} \mapsto \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $l \leqslant |\boldsymbol{\alpha}| \leqslant m$.
Taking $\varphi_1 = 1$ on $\operatorname{supp} \varphi$ and using $\overline{\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top$ we get the result.

Q.E.D.

Localisation principle - final generalisation

Theorem. $\varepsilon_n \longrightarrow 0$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ and

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n\,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega;\mathrm{M_r}(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ uniformly on compact sets, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ satisfies $(\mathrm{C}(\varepsilon_n))$.
Then for $\omega_n \to 0$ such that $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in [0,\infty]$, corresponding one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ω_n satisfies

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0,\infty\rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

Localisation principle - final generalisation

Theorem. $\varepsilon_n \longrightarrow 0$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ uniformly on compact sets, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ satisfies $(\mathrm{C}(\varepsilon_n))$.
Then for $\omega_n \rightarrow 0$ such that $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in [0, \infty]$, corresponding one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length ω_n satisfies

$$\mathbf{p} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0} \,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

Moreover, if there exists $\varepsilon_0 > 0$ such that $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, we can take

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

Localisation principle (H-measures)

- Using preceding theorem and $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_H$ on $\Omega \times \mathrm{S}^{d-1}$, we can obtained known localisation principle for H-measures:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\mu}_H^\top = \mathbf{0}\,,$$

where $\boldsymbol{\mu}_H$ is an H-measure associated to the sequence (u_n), while the symbol reads

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} (2\pi i)^m \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Localisation principle (semiclassical measures)

Under the assumptions of the preceding theorem, we have

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\mu}_{sc}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0,\infty\rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

Proof (only the case $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle$)

$$\psi \in \mathcal{S}(\mathbf{R}^d) \quad \Longrightarrow \quad \boldsymbol{\xi} \mapsto (|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|)\psi(\boldsymbol{\xi}) \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$$

Proof (only the case $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle$)

$$\psi \in \mathcal{S}(\mathbf{R}^d) \quad \Longrightarrow \quad \boldsymbol{\xi} \mapsto (|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|)\psi(\boldsymbol{\xi}) \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$$

$$\begin{aligned}
0 &= \left\langle \overline{\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \varphi \boxtimes (|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m)\psi \right\rangle \\
&= \left\langle \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \varphi \mathbf{A}^{\boldsymbol{\alpha}} \boxtimes \boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \right\rangle \\
&= \left\langle \boldsymbol{\mu}_{sc}, \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \varphi \mathbf{A}^{\boldsymbol{\alpha}} \boxtimes \boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \right\rangle = \left\langle \overline{\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}} \boldsymbol{\mu}_{sc}, \varphi \boxtimes \psi \right\rangle,
\end{aligned}$$

where we have used $\boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \in \mathcal{S}(\mathbf{R}^d)$ and that $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ and $\boldsymbol{\mu}_{sc}$ coincide on $\mathcal{S}(\mathbf{R}^d)$.

Summary

- H-measures do not catch frequency
- In some cases, semiclassical measures do not catch direction
- One-scale H-measures are a generalisation of H-measures and semiclassical measures and do not have the above anomalies

- Localisation principle for one-scale H-measures is obtained
- Localisation principles for H-measures and semiclassical measures is reproven via localisation principle for one-scale H-measures