Localisation principles for variant H-measures

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Joint work with Marko Erceg, Ivan Ivec, Martin Lazar, Marin Mišur and Darko Mitrović

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Start from $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$, and take the Fourier transform:

$$\widehat{\varphi u_n}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} (\varphi u_n)(\mathbf{x}) d\mathbf{x}$$

As φu_n is supported on a fixed compact set K, so $|\widehat{\varphi u_n}(\boldsymbol{\xi})| \leqslant C$.
Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\boldsymbol{\xi}) \longrightarrow 0$ pointwise.
By the Lebesgue dominated convergence theorem on bounded sets, we get $\widehat{\varphi u_n} \longrightarrow 0$ strong, i.e. strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$.

On the other hand, by the Plancherel theorem: $\|\widehat{\varphi u_n}\|_{\mathrm{L}^2(\mathbf{R}^d)} = \|\varphi u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}$.

If $\varphi u_n \not\longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then $\widehat{\varphi u_n} \not\longrightarrow 0$; some information must go to infinity.

How does it go to infinity in various directions? Take $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, and consider:

$$\lim_n \int_{\mathbf{R}^d} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|) |\widehat{\varphi u_n}|^2 d\boldsymbol{\xi} = \int_{\mathrm{S}^{d-1}} \psi(\boldsymbol{\xi}) d\nu_\varphi(\boldsymbol{\xi}) \ .$$

The limit is a linear functional in ψ, thus an integral over the sphere of some nonegativne Radon measure (a bounded sequence of Radon measures has an accumulation point), which depends on φ. How does it depent on φ?

H-measures: Rough geometric idea

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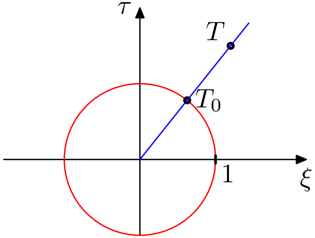

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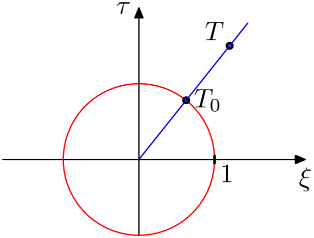

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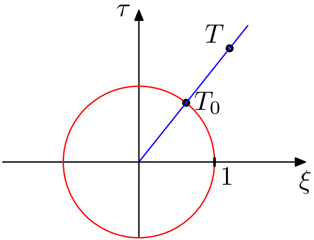

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and projection $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ onto the curve (surface):

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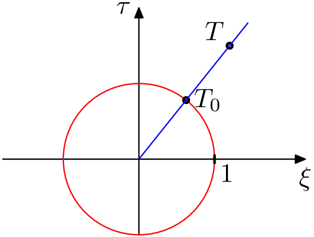

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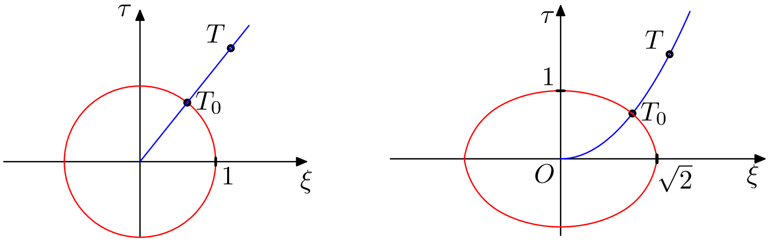

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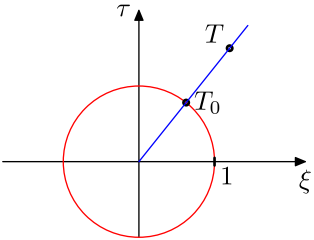

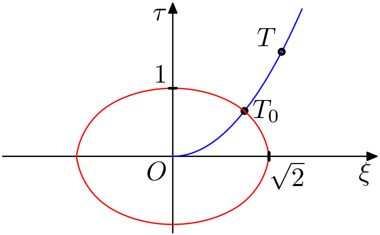

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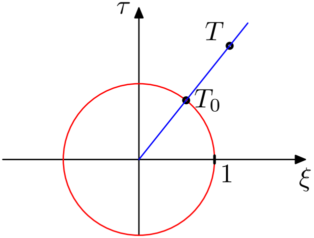

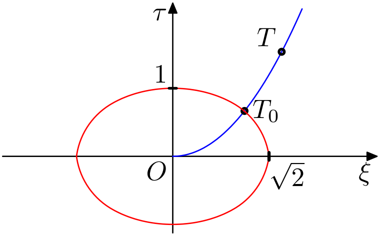

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Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
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Now we can state the main theorem, where we use the notation

$$\mathsf{v}\cdot\mathsf{u} := \sum v_i \bar{u}_i\ , \quad (\mathsf{v}\otimes\mathsf{u})\mathsf{a} := (\mathsf{a}\cdot\mathsf{u})\mathsf{v}\ , \text{while} \quad (f \boxtimes g)(\mathbf{x},\boldsymbol{\xi}) := f(\mathbf{x})g(\boldsymbol{\xi})\ .$$

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, then there exists a subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times \mathrm{S}^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and

$$\psi \in \mathrm{C}(\mathrm{S}^{d-1})$$

one has

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ p\ )\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

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Taking sequences in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, one gets unbounded Radon measures (i.e. distributions of order zero) as H-measures.
It holds: $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^r)$ if and only if $\boldsymbol{\mu} = \mathbf{0}$.

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There are some other variants (E. Ju. Panov, D. Mitrović & I. Ivec, M. Erceg & I. Ivec, ...).

Important lemma

Lemma. (first commutation — Luc Tartar) *If* $b \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfy the condition*

$$(\forall\, \rho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\text{a.e. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \rho)) \; ,$$

then $C := [\mathcal{A}_a, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$. ∎

For given $M, \rho \in \mathbf{R}^+$ denote the set

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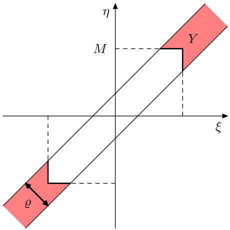

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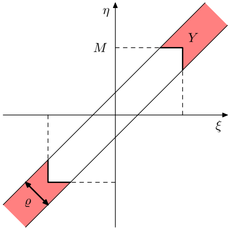

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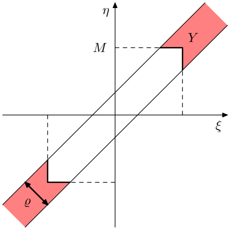

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Similar results were obtained and used earlier in the theory of pseudodifferential operators.

Localisation principle for classical H-measures

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}\ ) + \mathbf{B}\mathsf{u} \ = \mathsf{f} \quad , \qquad \mathbf{A}^k \in \mathrm{C}_b(\mathbf{R}^d; \mathrm{M}_{l\times r})$$

Assume:

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0\ , \qquad \text{and defines } \boldsymbol{\mu}$$

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Theorem. (localisation principle) If u_n satisfies:

$$\sum_{k=1}^{d} \partial_k\big(\mathbf{A}^k \mathsf{u}_n\big) \longrightarrow 0 \qquad \text{in } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{C}^r) \,,$$

then for $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{k=1}^{d} \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}^\top = \mathbf{0} \,.$$

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Thus, if $l = r$, the support of H-measure $\boldsymbol{\mu}$ is contaned in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{p}$ is a singular matrix.

The localisation principle is behind most of the known applications (e.g. to the small-amplitude homogenisation). It contains a generalisation of compactness by compensation to variable coefficients.

Localisation principle for parabolic H-measures

In the parabolic case the details become more involved.

Anisotropic Sobolev spaces ($s \in \mathbf{R}$; $k_p(\tau, \boldsymbol{\xi}) := \sqrt[4]{1 + (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4}$)

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{S}' : k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d}) \right\} .$$

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Theorem. **(localisation principle)** Let $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$, uniformly compactly supported in t, satisfy ($s \in \mathbf{N}$)

$$\sqrt{\partial_t}^{\,s}(\mathbf{A}^0 \mathsf{u}_n) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow 0 \quad \text{strongly in} \quad \mathrm{H}_{\mathrm{loc}}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d}) ,$$

where $\mathbf{A}^0, \mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}_b(\mathbf{R}^{1+d}; \mathrm{M}_{l\times r}(\mathbf{C}))$, for some $l \in \mathbf{N}$, while $\sqrt{\partial_t}$ is a pseudodifferential operator with symbol $\sqrt{2\pi i\tau}$, i.e.

$$\sqrt{\partial}_t u = \overline{\mathcal{F}}\left(\sqrt{2\pi i\tau}\,\hat{u}(\tau)\right) .$$

Then for a parabolic H-measure $\boldsymbol{\mu}$ associated to (a sub)sequence (of) (u_n) one has

$$\left((\sqrt{2\pi i\tau})^s \mathbf{A}^0 + \sum_{|\boldsymbol{\alpha}|=s} (2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}} \right) \boldsymbol{\mu}^\top = \mathbf{0}.$$

Good bounds in the L^p case: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d) \,, \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

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Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|} \,,$$

then for any $p \in \langle 1, \infty \rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty) \,.$$

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For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d_*$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

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Theorem. [N.A. & D. Mitrović (2011)] *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$ *and* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^q(\mathbf{R}^d)$ *for some* $q \geqslant \max\{p', 2\}$, *then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ *of order not more than* $\kappa = [d/2] + 1$ *in* $\boldsymbol{\xi}$, *such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ *we have:*

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For vector-valued $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

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The H-distribution would correspond to a non-diagonal block for an H-measure.

The proof is based on First commutation lemma

$\psi \in \mathrm{C}^{\kappa}(\mathrm{S}^{d-1})$ satisfies the conditions of the Hörmander-Mihlin theorem.
Therefore, $\mathcal{A}_\psi$ and M_φ are bounded operators on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty \rangle$.
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$$\|Cv_n\|_q \leqslant \|Cv_n\|_2^\alpha \|Cv_n\|_r^{1-\alpha} ,$$

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We still need a lemma on *compactness* of uniformly bounded bilinear forms, and an application of the Schwartz kernel theorem.

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d \rangle$, *such that*

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Take an arbitrary (v_n) *bounded in* $\mathrm{L}^\infty(\mathbf{R}^d)$*, and by* μ *denote the* H*-distribution corresponding to a subsequence of* (u_n) *and* (v_n)*. Then*

$$(\mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi})\mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$*,* $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}$ *being the symbol of the linear PDO with* C^κ_0 *coefficients.*

∎

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d \rangle$, *such that*

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In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\boldsymbol{\xi}|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\frac{\xi_j}{i|\boldsymbol{\xi}|}}$.

Note that

$$\int I_1(\phi)\partial_j g = \int (R_j \phi) g, \quad g \in \mathcal{S}(\mathbf{R}^d).$$

Using the density argument and that R_j is bounded from $\mathrm{L}^p(\mathbf{R}^d)$ to itself, we conclude $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in \mathrm{L}^p(\mathbf{R}^d)$.

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(an application suggested by Darko Mitrović) For scalar conservation law with discontinuous flux, the most up to date existence result for the equation

$$u_t + \mathsf{div}\,\mathsf{f}(t, \mathbf{x}, u) = 0$$

is obtained under the assumptions

$$\max_{\lambda \in \mathbf{R}} |\mathsf{f}(t, \mathbf{x}, \lambda)| \in L^{2+\varepsilon}(\mathbf{R}^d_+) \ .$$

Using the H-distributions, it is poossible to prove an existence result for the given equation under the assumption

$$\max_{\lambda \in \mathbf{R}} |\mathsf{f}(t, \mathbf{x}, \lambda)| \in L^{1+\varepsilon}(\mathbf{R}^d_+) \ .$$

Further variants

N.A. & I. Ivec: extension to Lebesgue spaces with mixed norm

M. Lazar & D. Mitrović: applications to velocity averaging

M. Mišur & D. Mitrović: a form of compactness by compensation

J. Aleksić, S. Pilipović, I. Vojnović (preprint): in $\mathcal{S} - \mathcal{S}'$ setting

F. Rindler (ARMA, 2015): microlocal compactness forms

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_\mathrm{b}(\Omega\times\mathbf{R}^d;\mathrm{M_r}(\mathbf{C}))$ *such that for any* $\varphi_1,\varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi\in\mathcal{S}(\mathbf{R}^d)$

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Measure $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ *we call the semiclassical measure with characteristic length* (ω_n) *corresponding to the (sub)sequence* (u_n)*.* ∎

Semiclassical measures

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Theorem.

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{sc}^{(\omega_n)} = \mathbf{0} \quad \& \quad (\mathsf{u}_n) \text{ is } (\omega_n)-\text{oscillatory} .$$

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Definition (u_n) is (ω_n)-oscillatory if
$(\forall\,\varphi \in \mathrm{C}^\infty_c(\Omega)) \quad \lim_{R\to\infty} \limsup_n \int_{|\boldsymbol{\xi}|\geqslant \frac{R}{\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} = 0\,.$

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Localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\mathbf{P}_n \mathsf{u}_n := \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

Then we have

$$\mathbf{p} \boldsymbol{\mu}_{sc}^{\top} = \mathbf{0} \,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

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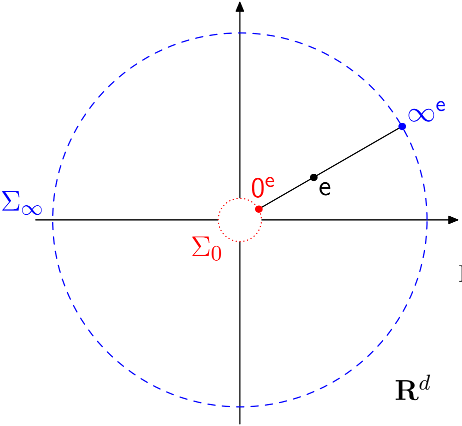
Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

Compatification of $\mathbf{R}^d \setminus \{0\}$



$$\Sigma_0 := \{0^{\mathrm{e}} \ : \ \mathrm{e} \in \mathrm{S}^{d-1}\}$$
$$\Sigma_\infty := \{\infty^{\mathrm{e}} \ : \ \mathrm{e} \in \mathrm{S}^{d-1}\}$$
$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Corollary. *a)* $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$. ∎

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exists a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_\mathrm{b}(\Omega\times\mathbf{R}^d;\mathrm{M}_\mathrm{r}(\mathbf{C}))$ *such that for any* $\varphi_1,\varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi\in\mathcal{S}(\mathbf{R}^d)$

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Luc Tartar: *The general theory of homogenization: A personalized introduction, Springer*, 2009.
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N. A., Marko Erceg, Martin Lazar: *Localisation principle for one-scale H-measures*, submitted (`arXiv`).

Idea of the proof

Tartar's approach:

- $\mathsf{v}_n(\mathbf{x}, x^{d+1}) := \mathsf{u}_n(\mathbf{x}) e^{\frac{2\pi i x^{d+1}}{\omega_n}} \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega \times \mathbf{R}; \mathbf{C}^r)$
- $\boldsymbol{\nu}_H \in \mathcal{M}(\Omega \times \mathbf{R} \times \mathrm{S}^d; \mathrm{M_r}(\mathbf{C}))$
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Our approach:

- First commutation lemma:

Lemma. *Let $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, and denote $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. Then the commutator can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K \,,$$

where K is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$, while $\tilde{C}_n \longrightarrow 0$ in the operator norm on $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$. ∎

- standard procedure: (a variant of) the kernel theorem, separability, ...

Some properties of $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$

Theorem.

a) $\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$, $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0}$

b) $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} \mathsf{0} \iff \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$

c) $\mathsf{tr}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 \iff (\mathsf{u}_n)$ *is* $(\omega_n) -$ *oscillatory*

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Theorem. $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^d)$, $\tilde\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \to 0^+$,

a) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle = \langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle\ ,$

b) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi\circ\boldsymbol{\pi}\rangle = \langle \boldsymbol{\mu}_H, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi\rangle\ ,$

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

∎

Localisation principle

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $l \in 0..m$
- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

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Lemma. *a) $(\mathrm{C}(\varepsilon_n))$ is equivalent to*

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \quad \Longrightarrow \quad (\varepsilon_n^{k-l} \mathsf{f}_n)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$. ∎

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem. [Tartar (2009)] *Under previous assumptions and $l=1$, one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) corresponding to (u_n) satisfies*

$$\operatorname{supp}\,(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

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Theorem. [N.A., Erceg, Lazar (2015)] *Under previous assumptions, one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ε_n) *corresponding to* (u_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

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Localisation principle - final generalisation

Theorem. *Take* $\varepsilon_n > 0$ *bounded,* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *and*

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$*,* $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *uniformly on compact sets, and* $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$.
Then for $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]$*, the corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0} \,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = 0 \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = \infty \end{cases}$$

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$*,* $n \in \mathbf{N}$*, we can take*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

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Theorem. *Take* $\varepsilon_n > 0$ *bounded,* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *and*

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Then for $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]$*, the corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

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As a corollary from the previous theorem we can derive localisation principles for H-measures and semiclassical measures.

Thank you for your attention.