Friedrichs systems

(with complex coefficients)

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Joint work with Krešimir Burazin, Ivana Crnjac, Marko Erceg and Marko Vrdoljak

Friedrichs' system (KOF1958)

Assumptions:
$d, r \in \mathbf{N}$, $\Omega \subseteq \mathbf{R}^d$ open and bounded with Lipschitz boundary Γ;

$\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{C}))$

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The operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{C}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{C}^r)$

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k (\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

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$$\mathcal{L}\mathsf{u} = \mathsf{f}$$

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Symmetric hyperbolic systems (KOF1954)

$$\sum_{k=1}^{d} \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad .$$

In divergence form:

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}) + (\mathbf{B} - \partial_k \mathbf{A}^k)\mathsf{u} = \mathsf{f} \ .$$

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It is symmetric if all matrices $\mathbf{A}^k$ are real and symmetric; and uniformly hyperbolic if there is a $\boldsymbol{\xi} \in \mathbf{R}^d$ such that for any $\mathbf{x} \in \mathrm{Cl}\,\Omega$ the matrix $\xi_k \mathbf{A}^k(\mathbf{x})$ is positive definite.

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Such systems can easily be transformed into Friedrichs' systems.

It is known that the wave equation and the Maxwell system can be written as an equivalent hyperbolic system.

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The development of theory is nowadays mostly motivated by the needs in development of numerical methods.

An example – scalar elliptic equation

$\Omega \subseteq \mathbf{R}^2$, $\mu > 0$ and $f \in \mathrm{L}^2(\Omega)$ given.

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which is a Friedrichs system with the choice of

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Example – heat equation

… with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \mathrm{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ u = 0 \text{ on } \partial\Omega \times \langle 0, T\rangle \\ u(\cdot, 0) = 0 \text{ on } \Omega \end{cases}$$

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...as a Friedrichs system:

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \partial_t u_{d+1} + \mathrm{div}_{\mathbf{x}}\mathsf{u}_d + cu_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d = f \end{cases},$$

(note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$, where $\mathsf{u}_d = -\mathbf{A}\nabla u$, and $u_{d+1} = u$).

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The condition (F1) holds. The positivity condition $\mathbf{C} + \mathbf{C}^\top \geqslant 2\mu_0\mathbf{I}$ is fulfilled if and only if $c - \frac{1}{4}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b}$ is uniformly positive.

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Boundary condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0$$

allows the treatment of different types of usual boundary conditions.

Assumptions on the boundary matrix $\mathbf{M}$

We assume (for ae $\mathbf{x} \in \Gamma$) [KOF1958]

(FM1) $$(\forall\, \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

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The boundary problem: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{C}^r)$ find u such that

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases}.$$

Elliptic equation – different boundary conditions

$$\mathbf{M} \qquad\qquad \mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} \qquad\qquad (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix}_{|\Gamma} = 0$$

$$\begin{bmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 2\nu_1 \\ 0 & 0 & 2\nu_2 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_{|\Gamma} = 0$$

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All above matrices $\mathbf{M}$ satisfy (FM).

Different ways to enforce boundary conditions

Instead of

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u} = 0 \quad \text{on } \Gamma\,,$$

Lax proposed boundary conditions with

$$\mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma\,,$$

where $N = \{N(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ is a family of subspaces of $\mathbf{C}^r$.

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Assumptions on N

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$) [PDL]

(FX1) $N(\mathbf{x})$ is non-negative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$:

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,;$$

(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which contains $N(\mathbf{x})$;

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or [RSP&LS1966]

Let $N(\mathbf{x})$ and $\tilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ satisfy (for ae $\mathbf{x} \in \Gamma$)

(FV1)

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0$$

$$(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0$$

(FV2)

$$\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}\,.$$

Equivalence of different descriptions of boundary conditions

Theorem. *It holds*

$$\textit{(FM1)–(FM2)} \quad \iff \quad \textit{(FX1)–(FX2)} \quad \iff \quad \textit{(FV1)–(FV2)},$$

with

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In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].

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– uniqueness of the classical solution
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Contributions:
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, . . .
– the meaning of traces for functions in the graph space
– weak well-posedness results under additional assumptions (on $\mathbf{A}_{\boldsymbol{\nu}}$)
– regularity of solution
– numerical treatment

Classical theory

What are Friedrichs systems?

Examples

Boundary conditions for Friedrichs systems

Existence, uniqueness, well-posedness

Abstract formulation

Graph spaces

Cone formalism of Ern, Guermond and Caplain

Interdependence of different representations of boundary conditions

Kreĭn spaces

Equivalence of boundary conditions

What can we say for the Friedrichs operator now?

Some examples

Two-field theory

Concluding remarks

New approach...

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Diff. Eq.* **32** (2007) 317–341.

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. . . and new open questions.

They considered only the real case.

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L — real (complex) Hilbert space (L' is (anti)dual of L),
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The Friedrichs operator

Let $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{C}^r)$ and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ be defined by

$$T\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,,$$

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Prolongations

$(\mathcal{D}, \langle \cdot \mid \cdot \rangle_T)$ is an inner product space, where

$$\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle_L + \langle T\cdot \mid T\cdot \rangle_L \, .$$

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Therefore $T = \tilde{T}^*_{|W_0}$, and analogously $\tilde{T} = T^*_{|W_0}$.
Abusing notation: $T, \tilde{T} \in \mathcal{L}(L; W_0')$ ...(T1)–(T3)

Formulation of the problem

Lemma. *The graph space*

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\},$$

is a Hilbert space with respect to $\langle \cdot \mid \cdot \rangle_T$.

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Problem: for given $f \in L$ find $u \in W$ such that $Tu = f$.

Find sufficient conditions on $V \leqslant W$ such that $T_{|_V} : V \longrightarrow L$ is an isomorphism.

Boundary operator

Boundary operator $D \in \mathcal{L}(W; W')$:

$$ {}_{W'}\langle\, Du, v \,\rangle_W := \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,, \qquad u, v \in W \,. $$

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Lemma. D *is selfadjoint*

$$_{W'}\langle\, Du, v \,\rangle_W = \overline{_{W'}\langle\, Dv, u \,\rangle_W}$$

and satisfies

$$\begin{aligned} \ker D &= W_0 \\ \operatorname{im} D &= W_0^0 := \{g \in W' : (\forall\, u \in W_0) \quad {}_{W'}\langle\, g, u \,\rangle_W = 0\} \,. \end{aligned}$$

In particular, $\operatorname{im} D$ *is closed in* W'*.* ∎

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In particular, $\operatorname{im} D$ is closed in W'.

If T is the Friedrichs operator $\mathcal{L}$, then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ we have

$$ {}_{W'}\langle\, D\mathsf{u}, \mathsf{v} \,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) \,. $$

Well-posedness theorem

Let V and $\tilde{V}$ be subspaces of W that satisfy

(V1)
$$(\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u\,\rangle_W \geqslant 0$$
$$(\forall\, v \in \tilde{V}) \qquad {}_{W'}\langle\, Dv, v\,\rangle_W \leqslant 0$$

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Theorem. *Under assumptions* $(T1)-(T3)$ *and* $(V1)-(V2)$*, the operators* $T_{|_V} : V \longrightarrow L$ *and* $\tilde{T}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$ *are isomorphisms.* ∎

In the real case [AE&JLG&GC2007].

Correspondence with *classical* assumptions

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(FV1)

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant 0\,,$$
$$(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \leqslant 0\,,$$

(FV2)

$$\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}\,,$$

(for ae $\mathbf{x} \in \Gamma$)

Other sets of conditions in the classical setting (recall)

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$)

(FX1) $$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which contains $N(\mathbf{x})$,

admissible boundary conditions: there exists a matrix function $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{C})$ such that (for ae $\mathbf{x} \in \Gamma$)

(FM1) $$(\forall\, \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

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subspace V is maximal non-negative with respect to D:

(X1) V is non-negative with respect to D: $\quad (\forall\, v \in V) \quad {}_{W'}\langle\, Dv, v\,\rangle_W \geqslant 0\,,$

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admissible boundary condition: there exist $M \in \mathcal{L}(W; W')$ that satisfy

(M1) $$(\forall\, u \in W) \quad {}_{W'}\langle\, (M + M^*)u, u \,\rangle_W \geqslant 0\,,$$

(M2) $$W = \ker(D - M) + \ker(D + M)\,.$$

Equivalence of different descriptions of b.c.

Theorem. *(***classical***)* *It holds*

$$\text{(FM1)–(FM2)} \iff \text{(FV1)–(FV2)} \iff \text{(FX1)–(FX2)},$$

with

$$N(\mathbf{x}) := \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big).$$

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∎

Theorem. *(**A. Ern, J.-L. Guermond, G. Caplain**)* *It holds*

$$\textit{(M1)–(M2)} \quad \overset{\Longrightarrow}{\longleftarrow} \quad \textit{(V1)–(V2)} \quad \Longrightarrow \quad \textit{(X1)–(X2)}\,,$$

with

$$V := \ker(D - M)\,.$$

∎

This was obtained in the real case only.

(M1)–(M2) $\longleftarrow$ (V1)–(V2)

Theorem. *Let V and $\tilde{V}$ satisfy (V1)–(V2), and suppose that there exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$ such that*

$$\begin{aligned} (\forall\, v \in V) \quad & D(v - Pv) = 0\,, \\ (\forall\, v \in \tilde{V}) \quad & D(v - Qv) = 0\,, \\ & DPQ = DQP\,. \end{aligned}$$

Let us define $M \in \mathcal{L}(W;W')$ (for $u, v \in W$) with

$$\begin{aligned} {}_{W'}\langle\, Mu, v\,\rangle_W = {}& {}_{W'}\langle\, DPu, Pv\,\rangle_W - {}_{W'}\langle\, DQu, Qv\,\rangle_W \\ & + {}_{W'}\langle\, D(P+Q-PQ)u, v\,\rangle_W - {}_{W'}\langle\, Du, (P+Q-PQ)v\,\rangle_W\,. \end{aligned}$$

Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and M satisfies (M1)–(M2). ∎

New notation

$$[\, u \mid v \,] := {}_{W'}\langle\, Du, v \,\rangle_W = \langle\, Tu \mid v \,\rangle_L - \langle\, u \mid \tilde{T}v \,\rangle_L \,, \qquad u, v \in W$$

is an indefinite inner product on W.

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$$V = \tilde{V}^{[\perp]}\,, \qquad \tilde{V} = V^{[\perp]}\,.$$

($^{[\perp]}$ stands for $[\,\cdot \mid \cdot\,]$-orthogonal complement)

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subspace V is maximal non-negative in $(W, [\,\cdot \mid \cdot\,])$:

(X1) V is non-negative in $(W, [\,\cdot \mid \cdot\,])$: $\quad (\forall\, v \in V) \quad [\,v \mid v\,] \geqslant 0\,,$

(X2) there is no non-negative subspace in $(W, [\,\cdot \mid \cdot\,])$ containing V.

Kreĭn spaces

$(W, [\,\cdot \mid \cdot\,])$ is not a Kreĭn space – it is a degenerate space, because its Gramm operator $G := j \circ D$ $\quad (j : W' \longrightarrow W$ is the canonical isomorphism$)$ has large kernel:

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Theorem. *If G is the Gramm operator of the space W, then the quotient space $\hat{W} := W/\ker G$ is a Kreĭn space if and only if $\operatorname{im} G$ is closed.* ∎

$(W, [\,\cdot \mid \cdot\,])$ is not a Kreĭn space – it is a degenerate space, because its Gramm operator $G := j \circ D$ $\quad (j : W' \longrightarrow W$ is the canonical isomorphism$)$ $\quad$ has large kernel:

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Kreĭn spaces

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Important: $\operatorname{im} D$ is closed and $\ker D = W_0$.

Quotient Kreĭn space

Lemma. *Let $U \supseteq W_0$ and Y be subspaces of W. Then*

a) U is closed if and only if $\hat{U} := \{\hat{v} : v \in U\}$ is closed in $\hat{W}$;

b) $(\widehat{U+Y}) = \{u + v + W_0 : u \in U, v \in Y\} = \hat{U} + \hat{Y}$;

c) $U + Y$ is closed if and only if $\hat{U} + \hat{Y}$ is closed;

d) $(\hat{Y})^{[\perp]\hat{}} = \widehat{Y^{[\perp]}}$.

e) if Y is maximal non-negative (non-positive) in W, than $\hat{Y}$ is maximal non-negative (non-positive) in $\hat{W}$;

f) if $\hat{U}$ is maximal non-negative (non-positive) in $\hat{W}$, then U is maximal non-negative (non-positive) in W.

∎

(V1)–(V2) $\iff$ (X1)–(X2)

Theorem. *a) If subspaces V and $\tilde{V}$ satisfy (V1)–(V2), then V is maximal non-negative in W (satisfies (X1)–(X2)) and $\tilde{V}$ is maximal non-positive in W.*

b) If V is maximal non-negative in W, then V and $\tilde{V} := V^{[\perp]}$ satisfy (V1)–(V2). ∎

(M1)–(M2) $\Longrightarrow$ (V1)–(V2) (recall)

Theorem. *[EGC] (T1)–(T3) and $M \in \mathcal{L}(W; W')$ satisfy (M) imply*

$$V := \ker(D - M) \quad \textit{and} \quad \tilde{V} := \ker(D + M^*) \quad \textit{satisfy (V).}$$

∎

Corollary. *Under above assumptions*

$$T_{|_{\ker(D-M)}} : \ker(D - M) \longrightarrow L \qquad i \qquad \tilde{T}_{|_{\ker(D+M^*)}} : \ker(D + M^*) \longrightarrow L$$

are isomorphisms. ∎

(M1)–(M2) $\longleftarrow$ (V1)–(V2) (recall)

Theorem. *Let V and $\tilde{V}$ satisfy (V1)–(V2), and suppose that there exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$ such that*

$$\begin{aligned} (\forall\, v \in V) \quad & D(v - Pv) = 0\,, \\ (\forall\, v \in \tilde{V}) \quad & D(v - Qv) = 0\,, \\ & DPQ = DQP\,. \end{aligned}$$

Let us define $M \in \mathcal{L}(W;W')$ (for $u, v \in W$) with

$$\begin{aligned} {}_{W'}\langle\, Mu, v\,\rangle_W &= {}_{W'}\langle\, DPu, Pv\,\rangle_W - {}_{W'}\langle\, DQu, Qv\,\rangle_W \\ &\quad + {}_{W'}\langle\, D(P+Q-PQ)u, v\,\rangle_W - {}_{W'}\langle\, Du, (P+Q-PQ)v\,\rangle_W\,. \end{aligned}$$

Then $V := \ker(D - M)$, $\tilde{V} := \ker(D + M^*)$, and M satisfies (M1)–(M2). ∎

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (direct proof)

Theorem. *If $V, \tilde{V}$ are two closed subspaces of W that satisfy $W_0 \subseteq V \cap \tilde{V}$, then the following statements are equivalent:*
a) There exist operators $P \in \mathcal{L}(W;V)$ and $Q \in \mathcal{L}(W;\tilde{V})$, such that

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$$\begin{gathered} {P'}^2 = P' \quad \textit{and} \quad {Q'}^2 = Q'\,, \\ \operatorname{im} P' = V \quad \textit{and} \quad \operatorname{im} Q' = \tilde{V}\,, \\ P'Q' = Q'P'\,. \end{gathered}$$

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∎

(b) is equivalent to closedness of $V + \tilde{V}$.

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

Theorem.
a) $V, \tilde{V} \leqslant W$ satisfy (V), and exists a closed subspace $W_2 \subseteq C^-$ of W, $V \dot{+} W_2 = W$, then there exist an operator $M \in \mathcal{L}(W; W')$ satisfying (M) and $V = \ker(D - M)$.

If we define W_1 as orthogonal complement of W_0 in V, so that $W = W_1 \dot{+} W_0 \dot{+} W_2$, and denote by R_1, R_0, R_2 projectors that correspond to above direct sum, then one such operator is given with $M = D(R_1 - R_2)$.

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b) $M \in \mathcal{L}(W; W')$ *an operator satisfying (M1)–(M2),* $V := \ker(D - M)$*.*

For W_2*, the orthogonal complement of* W_0 *in* $\ker(D + M)$*,* $W_2 \subseteq C^-$ *is closed,* $V \dot{+} W_2 = W$*, and* M *coincide with the operator in (a).* ∎

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For W_2, the orthogonal complement of W_0 in $\ker(D + M)$, $W_2 \subseteq C^-$ is closed, $V \dot{+} W_2 = W$, and M coincide with the operator in (a). ∎

Lemma. *Let $W_2'' \leqslant W$ satisfies $W_2'' \subseteq C^-$ and $W_2'' + V = W$. Then there is a closed subspace W_2 of W, such that $W_2 \subseteq C^-$ and $W_2 \dot{+} V = W$.* ∎

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

Lemma. *If $U_1 + U_2 = W$ for some subspaces $U_1 \subseteq C^+$ and $U_2 \subseteq C^-$ of W, then $U_1 \cap U_2 \subseteq W_0$.*
If additionally U_1 is maximal nonnegative and U_2 maximal nonpositive, then $U_1 \cap U_2 = W_0$. ∎

(M1)–(M2) $\Longleftarrow$ (V1)–(V2) (cont.)

Lemma. *If* $U_1 + U_2 = W$ *for some subspaces* $U_1 \subseteq C^+$ *and* $U_2 \subseteq C^-$ *of* W*, then* $U_1 \cap U_2 \subseteq W_0$*.*
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Theorem. *For a maximal nonnegative subspace* V *of* W*, it is equivalent:*
a) There is a maximal nonpositive subspace W_2 *of* W*, such that* $W_2 + V = W$*;*
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Theorem. *For a maximal nonnegative subspace V of W, it is equivalent:*
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Corollary. *The conditions (V) and (M) are equivalent.* ∎

Scalar elliptic equation

Consider

$$-\mathsf{div}\,(\mathbf{A}\nabla u) + cu = f$$

in $\Omega \subseteq \mathbf{R}^d$,

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New unknown vector function taking values in $\mathbf{R}^{d+1}$:

$$\mathsf{u} = \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} -\mathbf{A}\nabla_{\mathbf{x}} u \\ u \end{bmatrix} .$$

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Then the starting equation can be written as a first-order system

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1} \mathsf{u}_d = 0 \\ \mathsf{div}\, \mathsf{u}_d + c u_{d+1} = f \end{cases} ,$$

Scalar elliptic equation (cont.)

which is a Friedrichs system with the choice of

$$\mathbf{A}_k = \mathbf{e}_k \otimes \mathbf{e}_{d+1} + \mathbf{e}_{d+1} \otimes \mathbf{e}_k \in \mathrm{M}_{d+1}(\mathbf{R}), \qquad \mathbf{C} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0 & c \end{bmatrix}.$$

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The graph space: $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$.

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The graph space: $W = \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1(\Omega)$.
Dirichlet, Neumann and Robin boundary conditions are imposed by the following choice of V and $\widetilde{V}$:

$$\begin{aligned} V_D = \widetilde{V}_D :=& \mathrm{L}^2_{\mathrm{div}}(\Omega) \times \mathrm{H}^1_0(\Omega), \\ V_N = \widetilde{V}_N :=& \{(\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = 0\}, \\ V_R :=& \{(\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = a {u_{d+1}}_{|\Gamma}\}, \\ \widetilde{V}_R :=& \{(\mathsf{u}_d, u_{d+1})^\top \in W : \boldsymbol{\nu} \cdot \mathsf{u}_d = -a {u_{d+1}}_{|\Gamma}\}. \end{aligned}$$

Heat equation

... with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \operatorname{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ u = 0 \text{ on } \partial\Omega \times \langle 0, T\rangle \\ u(\cdot, 0) = 0 \text{ on } \Omega \end{cases}$$

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...as a Friedrichs system:

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \partial_t u_{d+1} + \mathrm{div}_{\mathbf{x}}\mathsf{u}_d + cu_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d = f \end{cases},$$

(note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$).

Friedrichs operator and the graph space

The operator T is given by

$$T \begin{bmatrix} \mathsf{u}_d \\ u_{d+1} \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d \\ \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}} \mathsf{u}_d + c u_{d+1} - \mathbf{A}^{-1}\mathsf{b} \cdot \mathsf{u}_d \end{bmatrix},$$

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while the corresponding graph space is

$$\begin{aligned} W = \Big\{ & \mathsf{u} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^{d+1}) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \\ & \qquad \& \quad \partial_t u_{d+1} + \operatorname{\mathsf{div}}_{\mathbf{x}} \mathsf{u}_d \in \mathrm{L}^2(\Omega_T) \Big\} \\ = \Big\{ & \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : \nabla_{\mathbf{x}} u_{d+1} \in \mathrm{L}^2(\Omega_T; \mathbf{R}^d) \Big\} \\ = \Big\{ & \mathsf{u} \in \mathrm{L}^2_{\mathrm{div}}(\Omega_T) : u_{d+1} \in \mathrm{L}^2(0, T; \mathrm{H}^1(\Omega)) \Big\}. \end{aligned}$$

Properties of the last component

Lemma. *The projection* $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top \mapsto u_{d+1}$ *is a continuous linear operator from* W *to* $W(0,T)$*, which is continuously embedded to* $\mathrm{C}([0,T];\mathrm{L}^2(\Omega))$*.*

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The space

$$W(0,T) = \left\{ u \in \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) : \partial_t u \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega)) \right\},$$

is a Banach space when equipped by norm

$$\|\mathsf{u}\|_{W(0,T)} = \sqrt{\|u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))} + \|\partial_t u\|^2_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))}}\,.$$

Finally

Let

$$\begin{aligned} V &= \left\{ \mathsf{u} \in W : u_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad u_{d+1}(\cdot,0) = 0 \text{ a.e. on } \Omega \right\}, \\ \widetilde{V} &= \left\{ \mathsf{v} \in W : v_{d+1} \in \mathrm{L}^2(0,T;\mathrm{H}^1_0(\Omega)), \quad v_{d+1}(\cdot,T) = 0 \text{ a.e. on } \Omega \right\}. \end{aligned}$$

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Theorem

The above V and $\widetilde{V}$ satisfy (V1)–(V2), and therefore the operator $T_{|_V} : V \longrightarrow L$ is an isomorphism.

Two-field theory...

Heat equation with b $= 0$ and $c = 0$:

$$\begin{cases} \partial_t u - \mathrm{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) = f \text{ in } \Omega_T \\ u = 0 \text{ on } \Gamma \times \langle 0, T\rangle \\ u(\cdot, 0) = 0 \text{ on } \Omega \end{cases}$$

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matrices need to be of the form

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{0} & \mathsf{B}^k \\ (\mathsf{B}^k)^\top & a^k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix},$$

where $\mathsf{B}^k \in \mathbf{R}^d$ are constant vectors, $a^k \in \mathrm{W}^{1,\infty}(\Omega_T)$, $\mathbf{C}^d \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$ and $c^{d+1} \in \mathrm{L}^\infty(\Omega_T)$, $k \in 1..(d+1)$.

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developed by Ern and Guermond for elliptic problems

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$$\mathbf{A}^k = \begin{bmatrix} \mathbf{0} & \mathsf{B}^k \\ (\mathsf{B}^k)^\top & a^k \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}^d & 0 \\ 0^\top & c^{d+1} \end{bmatrix},$$

where $\mathsf{B}^k \in \mathbf{R}^d$ are constant vectors, $a^k \in \mathrm{W}^{1,\infty}(\Omega_T)$, $\mathbf{C}^d \in \mathrm{L}^\infty(\Omega_T; \mathrm{M}_d(\mathbf{R}))$ and $c^{d+1} \in \mathrm{L}^\infty(\Omega_T)$, $k \in 1..(d+1)$.

For the heat equation matrices have this form!

... with partial coercivity

Instead of coercivity (positivity) condition (F2), the following is required:

$$(\exists\, \mu_1 > 0)(\forall\, \boldsymbol{\xi} = (\boldsymbol{\xi}_d, \xi_{d+1}) \in \mathbf{R}^{d+1})$$
$$\Big(\mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^{d+1} \partial_k \mathbf{A}_k\Big)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 2\mu_1 |\boldsymbol{\xi}_d|^2 \qquad \text{(a.e. on } \Omega) \,,$$

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For our system both conditions are trivially fulfilled.

Therefore, we have the well-posedness result.

Classical theory

- What are Friedrichs systems?
- Examples
- Boundary conditions for Friedrichs systems
- Existence, uniqueness, well-posedness

Abstract formulation

- Graph spaces
- Cone formalism of Ern, Guermond and Caplain
- Interdependence of different representations of boundary conditions
- Kreĭn spaces
- Equivalence of boundary conditions

What can we say for the Friedrichs operator now?

- Some examples
- Two-field theory

Concluding remarks

Some further applications ...

Already known:

– non-stationary theory
  the Maxwell system, non-stationary div-grad problem, the wave equation
– homogenisation

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Work in progress:

– Dirac system
– time-harmonic Maxwell system
– what can be done for the Schrödinger equation?

Open problems . . .

- Find all representations of a particular equation in the form of a Friedrichs system.
- Application to other equations of practical importance (mixed-type problems).
- Compare the results to those already known in the classical setting.

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Some used properties

Theorem. *a)* $[\cdot \mid \cdot]$*-orthogonal complement of a maximal non-negative (non-positive) subspace is non-positive (non-negative).*

b) Each maximal semi-definite subspace contains all isotropic vectors in W.

c) If L *is a non-negative (non-positive) subspace of a Krein space, such that* $L^{[\perp]}$ *is non-positive (non-negative), then* $\mathsf{Cl}\, L$ *is maximal non-negative (non-positive).*

d) Each maximal semi-definite subspace of a Krein space is closed.

e) A subspace L *of a Krein space is closed if and only if* $L = L^{[\perp][\perp]}$.

f) For a subspace L *of a Krein space* W *it holds*

$$L \cap L^{[\perp]} = \{0\} \qquad \Longleftrightarrow \qquad \mathsf{Cl}\,(L + L^{[\perp]}) = W\,.$$

∎