Generalised compactness by compensation

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L^2 theory

Div-rot lemma and Quadratic theorem

Lemma. *Assume that Ω is open and bounded subset of $\mathbf{R}^3$, and that it holds:*

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } \mathrm{L}^2(\Omega;\mathbf{R}^3),$$

$$\mathbf{v}_n \rightharpoonup \mathbf{v} \text{ in } \mathrm{L}^2(\Omega;\mathbf{R}^3),$$

$$\operatorname{rot}\mathbf{u}_n \text{ bounded in } \mathrm{L}^2(\Omega;\mathbf{R}^3),\ \operatorname{div}\mathbf{v}_n \text{ bounded in } \mathrm{L}^2(\Omega).$$

Then

$$\mathbf{u}_n\cdot\mathbf{v}_n \rightharpoonup \mathbf{u}\cdot\mathbf{v}$$

in the sense of distributions. ∎

Theorem. *Assume that $\Omega \subseteq \mathbf{R}^d$ is open and that $\Lambda \subseteq \mathbf{R}^r$ is defined by*

$$\Lambda := \left\{\boldsymbol\lambda\in\mathbf{R}^r : (\exists\boldsymbol\xi\in\mathbf{R}^d\setminus\{0\}) \quad \sum_{k=1}^d \xi_k\mathbf{A}^k\boldsymbol\lambda = 0\right\},$$

where Q is a real quadratic form on $\mathbf{R}^r$, which is nonnegative on Λ, i.e.

$$(\forall\boldsymbol\lambda\in\Lambda)\quad Q(\boldsymbol\lambda)\geqslant 0\,.$$

Furthermore, assume that the sequence of functions $(\mathbf{u}_n)$ satisfies

$$\mathbf{u}_n \longrightarrow \mathbf{u} \quad \text{weakly in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{R}^r)\,,$$

$$\Big(\sum_k \mathbf{A}^k\partial_k\mathbf{u}_n\Big) \quad \text{relatively compact in} \quad \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega;\mathbf{R}^q)\,.$$

Then every subsequence of $(Q\circ\mathbf{u}_n)$ which converges in distributions to it's limit L, satisfies

$$L\geqslant Q\circ\mathbf{u}$$

in the sense of distributions. ∎

Panov's result

The most general version of the classical L^2 results has recently been proved by E. Yu. Panov (2011):

Assume that the sequence $(\mathbf{u}_n)$ is bounded in $L^p(\mathbf{R}^d;\mathbf{R}^r)$, $2\le p<\infty$, and converges weakly in $\mathcal{D}'(\mathbf{R}^d)$ to a vector function $\mathbf{u}$.
Let $q=p'$ if $p<\infty$, and $q>1$ if $p=\infty$. Assume that the sequence

$$\sum_{k=1}^{\nu}\partial_k(\mathbf{A}^k\mathbf{u}_n)+\sum_{k,l=\nu+1}^{d}\partial_{kl}(\mathbf{B}^{kl}\mathbf{u}_n)$$

is precompact in the anisotropic Sobolev space $W^{-1,-2;q}_{loc}(\mathbf{R}^d;\mathbf{R}^m)$, where $m\times r$ matrices $\mathbf{A}^k$ and $\mathbf{B}^{kl}$ have variable coefficients belonging to $L^{2\bar q}(\mathbf{R}^d)$, $\bar q=\frac{p}{p-2}$ if $p>2$, and to the space $C(\mathbf{R}^d)$ if $p=2$.

We introduce the set $\Lambda(\mathbf{x})$

$$\Lambda(\mathbf{x})=\left\{\boldsymbol\lambda\in\mathbf{C}^r\,\middle|\,(\exists\boldsymbol\xi\in\mathbf{R}^d\setminus\{0\}):\Big(i\sum_{k=1}^{\nu}\xi_k\mathbf{A}^k(\mathbf{x})-2\pi\sum_{k,l=\nu+1}^{d}\xi_k\xi_l\mathbf{B}^{kl}(\mathbf{x})\Big)\boldsymbol\lambda=\mathbf{0}_m\right\},$$

and consider the bilinear form on $\mathbf{C}^r$

$$q(\mathbf{x},\boldsymbol\lambda,\boldsymbol\eta)=\mathbf{Q}(\mathbf{x})\boldsymbol\lambda\cdot\boldsymbol\eta, \tag{1}$$

where $\mathbf{Q}\in L^{\bar q}_{loc}(\mathbf{R}^d;\mathrm{Sym}_r)$ if $p>2$ and $\mathbf{Q}\in C(\mathbf{R}^d;\mathrm{Sym}_r)$ if $p=2$.
Finally, let $q(\mathbf{x},\mathbf{u}_n,\mathbf{u}_n)\rightharpoonup\omega$ weakly in the space of distributions.

The following theorem holds

Theorem. *[P, 2011] Assume that $(\forall\boldsymbol\lambda\in\Lambda(\mathbf{x}))\ q(\mathbf{x},\boldsymbol\lambda,\boldsymbol\lambda)\ge 0$ (a.e. $\mathbf{x}\in\mathbf{R}^d$) and $\mathbf{u}_n\rightharpoonup\mathbf{u}$, then $q(\mathbf{x},\mathbf{u}(\mathbf{x}),\mathbf{u}(\mathbf{x}))\le\omega$.* ∎

The connection between q and Λ given in the previous theorem, we shall call *the consistency condition*.

Goal: to formulate and extend the results from the preceeding theorem to the L^p-L^q framework for appropriate (greater than one) indices p and q where $p<2$.

Generalisation: L^p-L^q setting, $1/p+1/q<1$

H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović (2011) as an extension of H-measures to the L^p-L^q context.
M. Lazar and D. Mitrović (2012) extended and applied them on a velocity averaging problem.

We need multiplier operators with symbols defined on a manifold P determined by an d-tuple $\boldsymbol\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbf{R}^d_+$ where $\alpha_k\in\mathbf{N}$ or $\alpha_k\ge d$:

$$\mathrm{P}=\Big\{\boldsymbol\xi\in\mathbf{R}^d:\ \sum_{k=1}^d|\xi_k|^{2\alpha_k}=1\Big\},$$

In order to associate an L^p Fourier multiplier to a function defined on P, we extend it to $\mathbf{R}^d\backslash\{0\}$ by means of the projection

$$\big(\pi_{\mathrm{P}}(\boldsymbol\xi)\big)_j=\xi_j\left(|\xi_1|^{2\alpha_1}+\dots+|\xi_d|^{2\alpha_d}\right)^{-1/2\alpha_j},\quad j=1,\dots,d.$$

We need the following variant of H-distributions.

Theorem. *Let (u_n) be a bounded sequence in $L^p(\mathbf{R}^d)$, $p>1$, and let (v_n) be a bounded sequence of uniformly compactly supported functions in $L^q(\mathbf{R}^d)$, $1/q+1/p<1$, weakly converging to 0 in the sense of distributions. Then, after passing to a subsequence (not relabelled), for any $\bar s\in(1,\frac{pq}{p+q})$ there exists a continuous bilinear functional B on $L^{\bar s'}(\mathbf{R}^d)\otimes C^d(\mathrm{P})$ such that for every $\varphi\in L^{\bar s'}(\mathbf{R}^d)$ and $\psi\in C^d(\mathrm{P})$, it holds*

$$B(\varphi,\psi)=\lim_{n\to\infty}\int_{\mathbf{R}^d}\varphi(\mathbf{x})u_n(\mathbf{x})\big(\mathcal{A}_{\psi_{\mathrm{P}}}v_n\big)(\mathbf{x})d\mathbf{x}\,,$$

*where $\mathcal{A}_{\psi_{\mathrm{P}}}$ is the Fourier multiplier operator on $\mathbf{R}^d$ associated to $\psi\circ\pi_{\mathrm{P}}$.
The bilinear functional B can be continuously extended as a linear functional on $L^{\bar s'}(\mathbf{R}^d;C^d(\mathrm{P}))$.* ∎

Localisation principle

For $\alpha\in\mathbf{R}^+$, we define $\partial^\alpha_{x_k}$ to be a pseudodifferential operator with a polyhomogeneous symbol $(2\pi i\xi_k)^\alpha$, i.e.

$$\partial^\alpha_{x_k}u=((2\pi i\xi_k)^\alpha\hat u(\boldsymbol\xi))^{\check{}}.$$

In the sequel, we shall assume that sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are uniformly compactly supported. This assumption can be removed if the orders of derivatives $(\alpha_1,\dots,\alpha_d)$ are natural numbers.

Lemma. *Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $L^p(\mathbf{R}^d;\mathbf{R}^r)$ and $L^q(\mathbf{R}^d;\mathbf{R}^r)$, respectively, and converge toward $\mathbf{0}$ and $\mathbf{v}$ in the sense of distributions. Furthermore, assume that sequence $(\mathbf{u}_n)$ satisfies:*

$$\mathbf{G}_n:=\sum_{k=1}^d\partial_k^{\alpha_k}(\mathbf{A}^k\mathbf{u}_n)\ \to\mathbf{0}\text{ in }W^{-1,p}(\Omega;\mathbf{R}^m), \tag{2}$$

*where either $\alpha_k\in\mathbf{N}$, $k=1,\dots,d$ or $\alpha_k>d$, $k=1,\dots,d$, and elements of matrices $\mathbf{A}^k$ belong to $L^{\bar s'}(\mathbf{R}^d)$, $\bar s\in(1,\frac{pq}{p+q})$.
Finally, by $\boldsymbol\mu$ denote a matrix H-distribution corresponding to subsequences of $(\mathbf{u}_n)$ and $(\mathbf{v}_n-\mathbf{v})$. Then the following relation holds*

$$\Big(\sum_{k=1}^d(2\pi i\xi_k)^{\alpha_k}\mathbf{A}^k\Big)\boldsymbol\mu=\mathbf{0}.$$

∎

Compactness by compensation result

Introduce the set

$$\Lambda_{\mathcal{D}}=\Big\{\boldsymbol\mu\in\ L^{\bar s}(\mathbf{R}^d;(C^d(\mathrm{P}))')^r:\Big(\sum_{k=1}^n(2\pi i\xi_k)^{\alpha_k}\mathbf{A}^k\Big)\boldsymbol\mu=\mathbf{0}_m\Big\},$$

where the given equality is understood in the sense of $L^{\bar s}(\mathbf{R}^d;(C^d(\mathrm{P}))')^m$.

Let us assume that coefficients of the bilinear form q from (1) belong to space $L^t_{loc}(\mathbf{R}^d)$, where $1/t+1/p+1/q<1$.

Definition. We say that set $\Lambda_{\mathcal{D}}$, bilinear form q from (1) and matrix $\boldsymbol\mu=[\boldsymbol\mu_1,\dots,\boldsymbol\mu_r]$, $\boldsymbol\mu_j\in L^{\bar s}(\mathbf{R}^d;(C^d(\mathrm{P}))')^r$ satisfy the strong consistency condition if $(\forall j\in\{1,\dots,r\})$ $\boldsymbol\mu_j\in\Lambda_{\mathcal{D}}$, and it holds

$$\langle\phi\mathbf{Q}\otimes 1,\boldsymbol\mu\rangle\ge\mathbf{0},\quad\phi\in L^{\bar s}(\mathbf{R}^d;\mathbf{R}^+_0).$$

Theorem. *Assume that sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$ are bounded in $L^p(\mathbf{R}^d;\mathbf{R}^r)$ and $L^q(\mathbf{R}^d;\mathbf{R}^r)$, respectively, and converge toward $\mathbf{u}$ and $\mathbf{v}$ in the sense of distributions. Assume that (2) holds and that*

$$q(\mathbf{x};\mathbf{u}_n,\mathbf{v}_n)\rightharpoonup\omega\ \text{ in }\ \mathcal{D}'(\mathbf{R}^d).$$

If the set $\Lambda_{\mathcal{D}}$, the bilinear form (1), and matrix H-distribution $\boldsymbol\mu$, corresponding to subsequences of $(\mathbf{u}_n-\mathbf{u})$ and $(\mathbf{v}_n-\mathbf{v})$, satisfy the strong consistency condition, then

$$q(\mathbf{x};\mathbf{u},\mathbf{v})\le\omega\ \text{ in }\ \mathcal{D}'(\mathbf{R}^d).$$

∎

Case $L^p-L^{p'}$, $p>1$

In the case $1/p+1/q=1$, applying the same proof gives us continuous bilinear functional on $C(\mathbf{R}^d)\otimes C^d(\mathrm{P})$. Using Schwartz's kernel theorem, we can only extend it to a distribution from $\mathcal{D}'(\mathbf{R}^d\times\mathrm{P})$.
Introduce the truncation operator $T_l(v)=v$ if $|v|\le l$ and $T_l(v)=0$ if $|v|\ge l$, for $l\in\mathbf{N}$.

Theorem. *Assume that*

- *sequences $(\mathbf{u}_r)$ and $(\mathbf{v}_r)$ are bounded in $L^p(\mathbf{R}^d;\mathbf{R}^N)$ and $L^{p'}(\mathbf{R}^d;\mathbf{R}^N)$, where $1/p+1/p'=1$, and converge toward $\mathbf{u}$ and $\mathbf{v}$ in the sense of distributions;*
- *for every $l\in\mathbf{N}$, the sequences $(T_l(\mathbf{v}_r))$ converge weakly in $L^{p'}(\mathbf{R}^d;\mathbf{R}^N)$ toward $\mathbf{h}^l$, where the truncation operator T_l is understood coordinatewise;*
- *there exists a vector valued function $\mathbf{V}\in L^{p'}(\mathbf{R}^d;\mathbf{R}^N)$ such that $\mathbf{v}_r\le\mathbf{V}$ holds coordinatewise for every $r\in\mathbf{N}$;*
- *(2) holds with $a_{skl}\in C_0(\mathbf{R}^d)$ and that $q_{jm}\in C(\mathbf{R}^d)$.*

Assume that

$$q(\mathbf{x};\mathbf{u}_r,\mathbf{v}_r)\rightharpoonup\omega\ \text{ in }\ \mathcal{D}'(\mathbf{R}^d).$$

If for every $l\in\mathbf{N}$, the set $\Lambda_{\mathcal{D}}$, the bilinear form (1), and the (matrix of) H-distributions $\boldsymbol\mu_l$ corresponding to the sequences $(\mathbf{u}_r-\mathbf{u})$ and $(T_l(\mathbf{v}_r)-\mathbf{h}^l)_r$ satisfy the strong consistency condition, then it holds

$$q(\mathbf{x};\mathbf{u},\mathbf{v})\le\omega\ \text{ in }\ \mathcal{D}'(\mathbf{R}^d). \tag{3}$$

∎

Application

Now, let us consider the non-linear parabolic type equation

$$L(u)=\partial_t u-\operatorname{div}\operatorname{div}(g(t,\mathbf{x},u)\mathbf{A}(t,\mathbf{x})),$$

on $(0,\infty)\times\Omega$, where Ω is an open subset of $\mathbf{R}^d$. We assume that

$$u\in L^p((0,\infty)\times\Omega),\ \ g(t,\mathbf{x},u(t,\mathbf{x}))\in L^q((0,\infty)\times\Omega),\ \ 1<p,q,$$
$$\mathbf{A}\in L^s_{loc}((0,\infty)\times\Omega)^{d\times d},\ \ \text{where }\ 1/p+1/q+1/s<1,$$

and that the matrix $\mathbf{A}$ is strictly positive definite, i.e.

$$\mathbf{A}\boldsymbol\xi\cdot\boldsymbol\xi>0,\ \ \boldsymbol\xi\in\mathbf{R}^d\setminus\{\mathbf{0}\},\ \ (a.e.(t,\mathbf{x})\in(0,\infty)\times\Omega).$$

Furthermore, assume that g is a Carathèodory function and non-decreasing with respect to the third variable.

Theorem. *Assume that sequences*

- *(u_r) and $g(\cdot,u_r)$ are such that $u_r,g(u_r)\in L^2(\mathbf{R}^+\times\mathbf{R}^d)$ for every $r\in\mathbf{N}$;*
- *that they are bounded in $L^p(\mathbf{R}^+\times\mathbf{R}^d)$, $p\in(1,2]$, and $L^q(\mathbf{R}^+\times\mathbf{R}^d)$, $q>2$, respectively, where $1/p+1/q<1$;*
- *$u_r\rightharpoonup u$ and, for some, $f\in W^{-1,-2;p}(\mathbf{R}^+\times\mathbf{R}^d)$, the sequence*

$$L(u_r)=f_r\to f\quad\text{strongly in }W^{-1,-2;p}(\mathbf{R}^+\times\mathbf{R}^d).$$

Under the assumptions given above, it holds

$$L(u)=f\ \text{ in }\ \mathcal{D}'(\mathbf{R}^+\times\mathbf{R}^d).$$

∎

Comment

Question: What is the connection between the standard consistency condition and the strong consistency condition?

We can rewrite the consistency condition in the following form (we shall omit the second order derivatives since they have no influence on the reasoning below):

$$\Lambda_{\mathcal{F}}=\{\boldsymbol\lambda:\mathbf{R}^d\times S^{d-1}\to\mathbf{R}^N:\ \sum_{k=1}^{\nu}\xi_k\mathbf{A}^k(\mathbf{x})\boldsymbol\lambda(\mathbf{x},\boldsymbol\xi)=\mathbf{0}_m\}$$

and

$$q(\mathbf{x};\boldsymbol\lambda(\mathbf{x},\boldsymbol\xi),\boldsymbol\lambda(\mathbf{x},\boldsymbol\xi))\ge 0\quad\text{for all }\boldsymbol\lambda\in\Lambda_{\mathcal{F}}\text{ and all }(\mathbf{x},\boldsymbol\xi)\in\mathbf{R}^d\times S^{d-1}.$$

Having such a representation of the consistency condition, it seems reasonable to ask whether $\Lambda_{\mathcal{D}}$ is a closure of $\Lambda_{\mathcal{F}}$ in the sense of distributions. If this is the case, the generalisation presented here holds under the standard consistency condition. At this moment, we do not have any answer to this question.

References

[1] N. Antonić, D. Mitrović: *H-distributions: An Extension of H-Measures to an L^p-L^q Setting*, Abstr. Appl. Anal. (2011), Article ID 901084, 12 pages.

[2] P. Gérard: *Microlocal Defect Measures*, Comm. Partial Differential Equations **16** (1991) 1761–1794.

[3] M. Lazar, D. Mitrović: *Velocity averaging – a general framework*, Dyn. Partial Differ. Equ. **9** (2012) 239–260.

[4] M. Lazar, D. Mitrović: *On an extension of a bilinear functional on $L^p(\mathbf{R}^n)\times E$ to Bôchner spaces with an application to velocity averaging*, C. R. Math. Acad. Sci. Paris 351 (2013) 261–264.

[5] F. Murat: *A survey on compensated compactness* in Contributions to modern calculus of variations (Bologna, 1985), 145–183, Pitman Res. Notes Math. Ser. 148, Longman Sci. Tech., Harlow, 1987.

[6] E. Yu. Panov: *Ultraparabolic H-measures and compensated compactness*, Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011) 47–62.

[7] L. Tartar: *Compensated compactness and applications to partial differential equations*, in Nonlinear Analysis and Mechanics: Heriot Watt Symposium, Vol IV, Res. Notes in Math. 39, Pitman, Boston (1979) 136–212.

[8] L. Tartar: *H-measures, a new approach for studying homogenisation, oscillation and concentration effects in PDEs*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990) 193–230.