Adaptive micro-local defect functionals with application on degenerate equations

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WeConMApp

Degenerate parabolic equation

∘ effects of nonlinear convection and degenerate diffusion

$$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathfrak{f}(t, \mathbf{x}, u) = D^2 \cdot A(u)$$

∘ matrix A is such that the mapping

$$\mathbf{R} \ni \lambda \mapsto \langle A(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle \text{ is non-decreasing}$$

i.e. that the diffusion matrix $A'(\lambda)$ is merely non-negative definite.

∘ How to approach this type of problems?

▷ write the kinetic formulation (put $f = \mathfrak{f}'_\lambda$ and $a = A'$):

$$\begin{aligned}&\partial_t h(t, \mathbf{x}, \lambda) + \mathrm{div}(f(t, \mathbf{x}, \lambda) h(t, \mathbf{x}, \lambda))\\ &= \mathrm{div}\big(\mathrm{div}\,(a(\lambda) h(t, \mathbf{x}, \lambda))\big) + \partial_\lambda G(t, \mathbf{x}, \lambda) + \mathrm{div} P(t, \mathbf{x}, \lambda),\end{aligned}$$

The problem statement

$$
\begin{aligned}
&\partial_t u_n(t,\mathbf{x},\lambda) + \operatorname{div}(f(t,\mathbf{x},\lambda)u_n(t,\mathbf{x},\lambda)) \\
&= \operatorname{div}\big(\operatorname{div}(a(\lambda)u_n(t,\mathbf{x},\lambda))\big) + \partial_\lambda G_n(t,\mathbf{x},\lambda) + \operatorname{div} P_n(t,\mathbf{x},\lambda),
\end{aligned} \tag{1}
$$

The goal: show that for every $\rho \in \mathrm{C}_c^1(\mathbf{R})$, the sequence $\big(\int_{\mathbf{R}} \rho(\lambda)u_n(t,\mathbf{x},\lambda)d\lambda\big)$ is strongly precompact in $\mathrm{L}^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^d)$.

For the coefficients, we assume:

a) (u_n) weakly converges to zero in $\mathrm{L}^q(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$, $q \geq 2$;

b) $a \in \mathrm{C}^{0,1}(\mathbf{R};\mathbf{R}^{d\times d})$ is such that there exists a representation $a(\lambda) = \sigma(\lambda)^T \sigma(\lambda)$;

c) $f \in \mathrm{L}^p(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R};\mathbf{R}^d)$, $p > 1$ such that $1/p + 1/q < 1$;

d) $G_n \to 0$ strongly in $\mathrm{W}^{-1/2,r_0}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$ for some $r_0 \in \langle 1,\infty\rangle$;

e) $P_n \to 0$ strongly in $\mathrm{L}^{p_0}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R};\mathbf{R}^d)$ for some $p_0 \in \langle 1,\infty\rangle$.

Velocity averaging

- ▷ hyperbolic situations: $a \equiv 0$
- ▷ flux independent of space or time[1][2]
- ▷ ultra-parabolic equations[3]

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[1] Tadmor, Tao: *Velocity averaging, kinetic formulations, and regularizing effects in quasi-linear PDEs*, Communications on Pure and Applied Mathematics **60** (2007) 1488–1521.

[2] Lazar, Mitrović: *Velocity averaging – a general framework*, Dynamics of PDEs **9** (2012) 239–260.

[3] Holden, Karlsen, Mitrović, Panov: *Strong Compactness of Approximate Solutions to Degenerate Elliptic-Hyperbolic Equations with Discontinuous Flux Functions*, Acta Mathematica Scientia **29B** (2009) 1573–1612.

Tao-Tadmor result[4]

- degenerate parabolic equation: a changes rank with respect to λ
- flux is homogeneous (does not depend on $(t, \mathbf{x})$)

Recall that we want to consider:

$$\partial_t u_n(t,\mathbf{x},\lambda) + \operatorname{div}(f(t,\mathbf{x},\lambda)u_n(t,\mathbf{x},\lambda))$$
$$= \operatorname{div}\big(\operatorname{div}(a(\lambda)u_n(t,\mathbf{x},\lambda))\big) + \partial_\lambda G_n(t,\mathbf{x},\lambda) + \operatorname{div} P_n(t,\mathbf{x},\lambda),$$

with
d) $G_n \to 0$ strongly in $\mathrm{W}^{-1/2,r_0}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$ for some $r_0 \in \langle 1, \infty\rangle$;
e) $P_n \to 0$ strongly in $\mathrm{L}^{p_0}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}; \mathbf{R}^d)$ for some $p_0 \in \langle 1, \infty\rangle$.

In TT: $P_n \equiv 0$ and $G_n \in \mathrm{L}^q(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$, for $1 < q < 2$ and $G \in \mathcal{M}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$, for $q = 1$.

[4] Tadmor, Tao: *Velocity averaging, kinetic formulations, and regularizing effects in quasi-linear PDEs*, Communications on Pure and Applied Mathematics **60** (2007) 1488–1521.

H-measures[5][6]

Theorem 1. *If* $(u_n)_{n\in\mathbf{N}}$ *is a sequence in* $\mathrm{L}^2_{loc}(\Omega;\mathbf{R}^r)$, $\Omega\subset\mathbf{R}^d$, *such that* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{loc}(\Omega)$, *then there exists subsequence* $(u_{n'})_{n'}\subset(u_n)_n$ *and positive complex bounded measure* $\mu=\{\mu^{jk}\}_{j,k=1,\dots,r}$ *on* $\mathbf{R}^d\times\mathrm{S}^d$ *such that for all* $\varphi_1,\varphi_2\in\mathrm{C}_0(\Omega)$ *and* $\psi\in\mathrm{C}(\mathrm{S}^d)$,

$$\begin{aligned}\lim_{n'\to\infty}\int_\Omega(\varphi_1 u^j_{n'})(\boldsymbol{\xi})\overline{\mathcal{A}_{\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}(\varphi_2 u^k_{n'})(\boldsymbol{\xi})}d\mathbf{x} &= \langle\mu^{jk},\varphi_1\bar{\varphi}_2\psi\rangle\\ &=\int_{\mathbf{R}^d\times\mathrm{S}^d}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}\psi(\boldsymbol{\xi})d\mu^{jk}(\mathbf{x},\boldsymbol{\xi})\end{aligned}$$

where $\mathcal{A}_{\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}$ *is the multiplier operator with the symbol* $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$. ∎

[5] Tartar: *H-measures, a new approach for studying homogenisation, oscillation and concentration effects in PDEs*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990) 193–230.

[6] Gérard: *Microlocal Defect Measures*, Comm. Partial Differential Equations **16** (1991), 1761–1794.

H-measures - some applications

- velocity averaging results

  ◦ Gérard: *Microlocal Defect Measures*, Comm. Partial Differential Equations **16** (1991), 1761–1794.
  ◦ Lazar, Mitrović: *Velocity averaging – a general framework*, Dynamics of PDEs **9** (2012) 239–260.

- existence of traces and solutions to nonlinear evolution equations

  ◦ Panov: *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws.* J. Hyperbolic Differ. Equ. **2** (2005) 885–908.
  ◦ Holden, Karlsen, Mitrović, Panov: *Strong Compactness of Approximate Solutions to Degenerate Elliptic-Hyperbolic Equations with Discontinuous Flux Functions*, Acta Mathematica Scientia **29B** (2009) 1573–1612.
  ◦ Aleksić, Mitrović: *On the compactness for two dimensional scalar conservation law with discontinuous flux*, Comm. Math. Sci. **7** (2009) 963–971.

- control theory

  ◦ Dehman, Léautaud, Le Rousseau: *Controllability of two coupled wave equations on a compact manifold*, Arch. Rational Mech. Anal. **211** (2014) 113–187.
  ◦ Lazar, Zuazua: *Averaged control and observation of parameter-depending wave equations*, C. R. Acad. Sci. Paris, Ser. I **352** (2014) 497–502.

H-measure *sees* only derivatives of the same highest order

Instead of $\boldsymbol{\xi}/|\boldsymbol{\xi}|$, put

$$\frac{\boldsymbol{\xi}}{|(\xi_1, \dots, \xi_k)| + |(\xi_{k+1}, \dots, \xi_d)|^2}.$$

H-measure will be able to *see* the first order derivatives with respect to $(x_1, \dots, x_k)$, and second order derivatives with respect to $(x_{i+1}, \dots, x_d)$.

$\Longrightarrow$ No changing of the highest order of the equation is permitted!

We need to consider symbols of the form

$$\psi\left(\frac{(\tau, \boldsymbol{\xi})}{|(\tau, \boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi}\rangle}\right), \quad \psi \in \mathrm{C}(\mathbf{R}^d),$$

where the matrix a represents the diffusion matrix in the degenerate parabolic equation.

Matrix analysis

Let matrix $a(\lambda)$ have $k(\lambda)$ eigenvalues strictly greater than 0.

Write $a = \sigma^T \sigma$, where

$$\sigma = \begin{bmatrix} [\sigma_{11}] & [\sigma_{12}] \\ O & O \end{bmatrix},$$

and $[\sigma_{11}]$ is regular $k \times k$ matrix.

Later we will use a change of variables

$$\boldsymbol{\eta} = M\boldsymbol{\xi} \quad \text{where} \quad M = \begin{bmatrix} [\sigma_{11}] & [\sigma_{12}] \\ O & I \end{bmatrix}.$$

We will assume the following uniform bounds:

$$0 < c \leq \|M^{-1}\|_2 \leq \widehat{C} < \infty \,, \qquad \|M\|_2 \leq \widetilde{C} \,, \qquad \|\sigma'\|_2 \leq \bar{C} \,.$$

We have $\widetilde{C} = \max\{1, \|a\|_2\} + \|a\|_2$ and $c = 1/\widetilde{C}$. For $\widehat{C}$ we do not have a uniform bound, so this together with assumption on $\bar{C}$ are the only new assumptions here.

Matrix example

- $A(u) = \begin{bmatrix} u & -\frac{u^2}{2} \\ -\frac{u^2}{2} & \frac{u^3}{3} \end{bmatrix}$

▷ $a(\lambda) = \begin{bmatrix} 1 & -\lambda \\ -\lambda & \lambda^2 \end{bmatrix}, \qquad \sigma(\lambda) = \begin{bmatrix} -1 & \lambda \\ 0 & 0 \end{bmatrix}$

For $\boldsymbol{\xi} = (x, y)$, we have $\langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle = (x - \lambda y)^2$.

▷ $M = \begin{bmatrix} -1 & \lambda \\ 0 & 1 \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} -1 & \lambda \\ 0 & 1 \end{bmatrix}$

  ▷ $\|M^{-1}(\lambda)\|_2 = \frac{1}{2} \max\{\lambda^2 \pm \sqrt{\lambda^2 + 1}\lambda + 2\}$

  ▷ $\|a(\lambda)\|_2 = 1 + \lambda^2$

Fourier multipliers I

Let $a : \mathbf{R} \to M^{d\times d}$ be a non-negative definite matrix. Define:

$$\pi_P(\tau, \boldsymbol{\xi}, \lambda) = \frac{(\tau, \boldsymbol{\xi})}{|(\tau, \boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi}\rangle}$$

$$\Pi_\lambda = \mathrm{Cl}\{\pi_P(\tau, \boldsymbol{\xi}, \lambda) : \, (\tau, \boldsymbol{\xi}) \in \mathbf{R}^{d+1}\}$$

where Cl denotes the closure of a set.

For $\psi \in \mathrm{C}^{d+1}(\Pi_\lambda)$ the composition $\psi(\pi_P)$ is a symbol of an $\mathrm{L}^p(\mathbf{R}^{d+1})$ multiplier (here, we consider λ to be fixed).

Lema 1. *Under the conditions stated above, for any $\psi \in \mathrm{C}^{d+1}(\mathbf{R}^{d+1})$, the function $\psi(\pi_P)$ is an L^p multiplier.* ∎

Fourier multipliers II

We will show that a Fourier multiplier with the symbol

$$\partial_j^{1/2} \circ \partial_\lambda \left(\frac{1}{|(\tau, \boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle} \right)$$

satisfies conditions of Marcinkiewicz's multiplier theorem.
The symbol of $\partial_\lambda \left(\mathcal{A}_{\frac{1}{|(\tau,\boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}} \right)$ is:

$$\partial_\lambda \left(\frac{1}{|(\tau, \boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle} \right) = \frac{-\langle a'(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle}{\left(|(\tau, \boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle \right)^2}.$$

Using a representation $a(\lambda) = \sigma(\lambda)^T \sigma(\lambda)$ and the change of variables $\boldsymbol{\eta} = M\boldsymbol{\xi}$, the symbol becomes:

$$\frac{-2(2\pi i \eta_j)^{1/2} \left\langle \sigma'(\lambda) M^{-1} \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}} \right\rangle}{\left(\, |(\tau, M^{-1}\boldsymbol{\eta})| + |\tilde{\boldsymbol{\eta}}|^2 \, \right)^2}.$$

Corollary 1.

Let $p \in \langle 1, \infty \rangle$. *Then* $\partial_\lambda \left(\mathcal{A}_{\frac{1}{|(\tau, \boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle}} \right)$ *continuously maps* $\mathrm{L}^p(\mathbf{R} \times \mathbf{R}^d)$ *to* $\mathrm{W}^{1/2,p}(\mathbf{R} \times \mathbf{R}^d)$.

Let $r > 2(d+1)$. *Then* $\partial_\lambda \left(\mathcal{A}_{\frac{1}{|(\tau, \boldsymbol{\xi})| + \langle a(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle}} \right)$ *continuously maps* $\mathrm{L}^r(\mathbf{R} \times \mathbf{R}^d)$ *to* $\mathrm{C}^0(\mathbf{R} \times \mathbf{R}^d)$.

∎

The function space $W^p_\Pi(\Omega)$

$$\widetilde{W}^p_\Pi(\Omega) = \Big\{ \sum_{j=1}^k \varphi_j(t,\mathbf{x})\psi_j(\lambda, \pi_P(\tau,\boldsymbol{\xi},\lambda)) : (t,\mathbf{x}) \in \Omega,\ (\lambda,\tau,\boldsymbol{\xi}) \in \mathbf{R}^{d+2}$$

$$\varphi_j \in \mathrm{L}^p(\Omega),\ \psi_j \in \mathrm{C}^d(\mathbf{R} \times [-1,1]^{d+1}) \Big\}.$$

$$\|\Psi\|_{W^p_\Pi} = \left(\int_\Omega \left[\sup_{(\tau,\boldsymbol{\xi})\in\mathbf{R}^{d+1}} \left(\int_{\mathbf{R}} |\Psi(t,\mathbf{x},\lambda,\pi_P(\tau,\boldsymbol{\xi},\lambda))|^2 \, d\lambda \right)^{1/2} \right]^p dt d\mathbf{x} \right)^{1/p}.$$

$W^p_\Pi(\Omega)$ is the closure of $\widetilde{W}^p_\Pi(\Omega)$ in $\mathrm{L}^p(\Omega; \mathrm{C}_0([-1,1]^{d+1}; \mathrm{L}^2(\mathbf{R})))$, with respect to the norm $\|\cdot\|_{W^p_\Pi}$:

$$W^p_\Pi(S) = \mathrm{Cl}_{\|\cdot\|_{W^p_\Pi}} \left(\widetilde{W}^p_\Pi(\Omega) \subset \mathrm{L}^p(\Omega; \mathrm{C}_0([-1,1]^{d+1}; \mathrm{L}^2(\mathbf{R}))) \right).$$

Existence

Theorem 2. *Let*

- $(u_n(t,\mathbf{x},\lambda))$ *be an uniformly compactly supported on* $S_u \subset\subset \mathbf{R}^+ \times \mathbf{R}^{d+1}$ *sequence weakly converging to zero in* $\mathrm{L}^p(\mathbf{R}^+ \times \mathbf{R}^{d+1})$, $p > 2$.
- $(v_n(t,\mathbf{x}))$ *be an uniformly compactly supported on* $S_v \subset\subset \mathbf{R}^+ \times \mathbf{R}^d$ *sequence bounded in* $\mathrm{L}^\infty(\mathbf{R}^+ \times \mathbf{R}^d)$.

Then for $\varepsilon > 0$ *such that* $p' + \varepsilon \geq \frac{2p}{p-2}$ *there exists a subsequence and a continuous functional* μ *on* $\widetilde{W}_{\Pi}^{p'+\varepsilon}(\Omega)$ *such that for every* $\varphi \in \mathrm{L}^{p'+\varepsilon}(\Omega)$ *and* $\psi \in \mathrm{C}^{d+1}(\mathbf{R} \times [-1,1]^{d+1})$ *it holds*

$$\mu(\varphi\psi) = \lim_{n\to\infty} \int_{\Omega\times\mathbf{R}} \varphi(t,\mathbf{x}) u_n(t,\mathbf{x},\lambda) \overline{\mathcal{A}_{\psi(\lambda,\pi_P(\tau,\boldsymbol{\xi},\lambda))}(v_n)(t,\mathbf{x})}\, dt d\mathbf{x} d\lambda. \quad (2)$$

∎

Corollary 2. *Under the conditions of the previous theorem, representation* (2) *holds for* $\varphi \in \mathrm{L}^{p'+\varepsilon}(\Omega \times \mathbf{R})$ *and* $\psi \in \mathrm{C}^{d+1}([-1,1]^{d+1})$.

■

Lema 2. *Let* $\mu \in \left(W_{\Pi}^{p'+\varepsilon}(\Omega)\right)'$ *be the functional defined in the previous theorem. Let* $K_\lambda \subset \mathbf{R}$ *be a fixed arbitrary compact set.*
If the function $F \in W_{\Pi}^{p'+\varepsilon}(\Omega)$ *is such that for some* $\alpha > 0$

$$\operatorname*{ess\,sup}_{(t,\mathbf{x})\in\mathbf{R}^+\times\mathbf{R}^d} \sup_{(\tau,\boldsymbol{\xi})\mathbf{R}^{d+1}} \operatorname{meas}\{\lambda \in K_\lambda : \ |F\,(t,\mathbf{x},\lambda,\pi_P(\tau,\boldsymbol{\xi},\lambda))\,| \leq \sigma\} \ \leq \ \sigma^\alpha \tag{3}$$

and

$$F\mu \equiv 0\,,$$

then

$$\mu \equiv 0.$$

■

Idea of the proof:

∘ multiply $F\mu \equiv 0$ by $\phi \frac{\overline{F}}{|F|^2+\sigma}$

∘

$$0 = \left\langle \mu,\, \phi \frac{|F|^2}{|F|^2+\sigma} \right\rangle = \langle \mu,\, \phi \rangle - \left\langle \mu,\, \phi \frac{\sigma}{|F|^2+\sigma} \right\rangle$$

Assumptions

$$
\begin{aligned}
&\partial_t u_n(t,\mathbf{x},\lambda) + \operatorname{div}(f(t,\mathbf{x},\lambda)u_n(t,\mathbf{x},\lambda)) \\
&= \operatorname{div}\big(\operatorname{div}(a(\lambda)u_n(t,\mathbf{x},\lambda))\big) + \partial_\lambda G_n(t,\mathbf{x},\lambda) + \operatorname{div} P_n(t,\mathbf{x},\lambda),
\end{aligned}
$$

a) (u_n) weakly converges to zero in $\mathrm{L}^q(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$, $q \geq 2$;
b) $a \in \mathrm{C}^{0,1}(\mathbf{R}; \mathbf{R}^{d\times d})$ is such that there exists a representation $a(\lambda) = \sigma(\lambda)^T \sigma(\lambda)$;
c) $f \in \mathrm{L}^p(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}; \mathbf{R}^d)$, $p > 1$ such that $1/p + 1/q < 1$;
d) $G_n \to 0$ strongly in $\mathrm{W}^{-1/2,r_0}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R})$ for some $r_0 \in \langle 1, \infty\rangle$;
e) $P_n \to 0$ strongly in $\mathrm{L}^{p_0}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}; \mathbf{R}^d)$ for some $p_0 \in \langle 1, \infty\rangle$.

Theorem 3. *Assume that the function*

$$F(t,\mathbf{x},\lambda,\pi_P(\tau,\boldsymbol{\xi},\lambda)) = i\frac{\tau+\langle\boldsymbol{\xi}, f(t,\mathbf{x},\lambda)\rangle}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle} + \frac{\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}$$

satisfies non-degeneracy condition (3)*.*

Then, for any $\rho\in \mathrm{C}^1_c(\mathbf{R})$, the sequence $(\int_{\mathbf{R}}\rho(\lambda)u_n(t,\mathbf{x},\lambda)d\lambda)$ is strongly precompact in $\mathrm{L}^1_{loc}(\mathbf{R}^+\times\mathbf{R}^d)$. ∎

Idea of the proof:

∘ special test functions:

$$\overline{\theta_n(t,\mathbf{x},\lambda)} = \varphi(t,\mathbf{x})\rho(\lambda)\mathcal{A}_{\frac{1}{|(\tau,\boldsymbol{\xi})|+\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}}(v_n(\cdot,\cdot))(t,\mathbf{x})$$

∘ take:

$$v_n(t,\mathbf{x}) = \varphi(t,\mathbf{x})\left(\operatorname{sgn}\left(\int_{\mathbf{R}}\rho(\eta)u_n(t,\mathbf{x},\eta)d\eta\right) - V(t,\mathbf{x})\right)$$

∘ conclude:

$$\lim_{n\to\infty}\int_{\mathbf{R}^d_+}\varphi^2(t,\mathbf{x})\left|\int_{\mathbf{R}}\rho(\lambda)u_n(t,\mathbf{x},\lambda)d\lambda\right| dtd\mathbf{x} = \langle\mu,\rho\varphi\otimes 1\rangle = 0$$

Cauchy problem for an advection-diffusion equation

$$\partial_t u + \mathrm{div}_{\mathbf{x}} \mathfrak{f}(t, \mathbf{x}, u) = D^2 \cdot A(u) \tag{4}$$
$$u|_{t=0} = u_0(\mathbf{x}) \in \mathrm{L}^1(\mathbf{R}^d) \cap \mathrm{L}^\infty(\mathbf{R}^d).$$

The equation describes a flow governed by

- the convection effects (bulk motion of particles) which are represented by the first order terms;

- diffusion effects which are represented by the second order term and the matrix $A(\lambda)$ describes direction and intensity of the diffusion;

Degeneracy in the sense that the derivative of the diffusion matrix A' can be equal to zero in some direction.
Roughly speaking, if this is the case (i.e. for some vector $\boldsymbol{\xi} \in \mathbf{R}^d$: $\langle A'(\lambda)\boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 0$), then diffusion effects do not exist at the point $\mathbf{x}$ for the state λ in the direction $\boldsymbol{\xi}$.

Assumptions on coefficients of (4)

- The initial data are bounded between $\tilde{a}$ and $\tilde{b}$ and the flux function annuls at $\lambda = \tilde{a}$ and $\lambda = \tilde{b}$:

$$\tilde{a} \leq u_0(\mathbf{x}) \leq \tilde{b} \quad \text{and} \quad \mathfrak{f}(t, \mathbf{x}, \tilde{a}) = \mathfrak{f}(t, \mathbf{x}, \tilde{b}) = 0 \ \ \text{a.e.} \ \ (t, \mathbf{x}) \in \mathbf{R}^+ \times \mathbf{R}^d.$$

- The convective term $\mathfrak{f}(t, \mathbf{x}, \lambda)$ is continuously differentiable with respect to $\lambda \in \mathbf{R}$, and it belongs to $\mathrm{L}^r(\mathbf{R}^+ \times \mathbf{R}^d \times [\tilde{a}, \tilde{b}])$, $r > 1$
We also assume:

$$\mathrm{div}_{\mathbf{x}} \mathfrak{f}(t, \mathbf{x}, \lambda) \in \mathcal{M}(\mathbf{R}^+ \times \mathbf{R}^d \times [\tilde{a}, \tilde{b}]).$$

- The matrix $A(\lambda) = (A_{ij}(\lambda))_{i,j=1,\dots,d} \in \mathrm{C}^{1,1}(\mathbf{R}; \mathbf{R}^{d \times d})$, is non-decreasing with respect to $\lambda \in \mathbf{R}$, i.e. the (diffusion) matrix $a(\lambda) = A'(\lambda)$ satisfies

$$\langle a(\lambda) \boldsymbol{\xi}, \boldsymbol{\xi} \rangle \geq 0$$

and there exists a representation $a(\lambda) = \sigma(\lambda)^T \sigma(\lambda)$.

Quasi-solution

Definition

A measurable function u defined on $\mathbf{R}^+ \times \mathbf{R}$ is called a quasi-solution to (4) if $\mathfrak{f}_k(t,\mathbf{x},u), A_{kj}(u) \in \mathrm{L}^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^d)$, $k,j = 1,\dots,d$, and for a.e. $\lambda \in \mathbf{R}$ the Kruzhkov type entropy equality holds

$$\begin{aligned} &\partial_t |u-\lambda| + \operatorname{div}\left[\operatorname{sgn}(u-\lambda)(\mathfrak{f}(t,\mathbf{x},u) - \mathfrak{f}(t,\mathbf{x},\lambda))\right] \\ &- D^2 \cdot \left[\operatorname{sgn}(u-\lambda)(A(u) - A(\lambda))\right] = -\zeta(t,\mathbf{x},\lambda), \end{aligned}$$

where $\zeta \in \mathrm{C}(\mathbf{R}_\lambda; w\star\!-\!\mathcal{M}(\mathbf{R}^+ \times \mathbf{R}^d))$ we call the quasi-entropy defect measure.

Remark. For a regular flux $\mathfrak{f}$, the measure $\zeta(t,\mathbf{x},\lambda)$ can be rewritten in the form $\zeta(t,\mathbf{x},\lambda) = \bar\zeta(t,\mathbf{x},\lambda) + \operatorname{sgn}(u-\lambda)\operatorname{div}_{\mathbf{x}}\mathfrak{f}(t,\mathbf{x},\lambda)$, for a measure $\bar\zeta$.
If $\bar\zeta$ is non-negative, then the quasi-solution u is an entropy solution to (4). For the uniqueness of such entropy solution, we additionally need the chain rule[7] [8].

∎

[7] Chen, Perthame: *Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations*, Ann. Inst. H. Poincare Anal. Non Lineaire **4** (2002) 645–668.

[8] Chen, Karlsen: *Quasilinear Anisotropic Degenerate Parabolic Equations with Time-Space Dependent Diffusion Coefficients*, Comm. Pure and Applied Analysis **4** (2005) 241–266.

Kinetic formulation

Theorem 4. *If function u is a quasi-solution to (4), then the function*

$$h(t,\mathbf{x},\lambda) = \operatorname{sgn}(u(t,\mathbf{x})-\lambda) = -\partial_\lambda |u(t,\mathbf{x})-\lambda|$$

is a weak solution to the following linear equation:

$$\partial_t h + \operatorname{div}\left(\mathfrak{F}(t,\mathbf{x},\lambda)h\right) - D^2\cdot[a(\lambda)h] = \partial_\lambda \zeta(t,\mathbf{x},\lambda)\,,$$

where $\mathfrak{F} = \mathfrak{f}'_\lambda$ and $a = A'_\lambda$. ∎

Theorem 5. *Assume that $\mathfrak{F} = \mathfrak{f}'_\lambda$ and $a = A'_\lambda$ are such that the function*

$$F(t,\mathbf{x},\pi_P(\tau,\boldsymbol{\xi},\lambda)) = i\frac{\tau + \langle \boldsymbol{\xi}, \mathfrak{F}(t,\mathbf{x},\lambda)\rangle}{|(\tau,\boldsymbol{\xi})| + \langle A(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle} + \frac{\langle a(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}{|(\tau,\boldsymbol{\xi})| + \langle A(\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle}$$

satisfies (3).

Then, there exists a solution to (4) augmented with the initial conditions $u|_{t=0} = u_0(\mathbf{x})$, $\tilde{a} \le u_0 \le \tilde{b}$. ∎

Reference

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