Spaces of distributions of non-integer order and applications

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Motivation

∘ $T \in \mathcal{D}'(X)$ is of order smaller or equal to $r \in \mathbf{N}$ if the restriction of T to $\mathrm{C}_c^\infty(X;K)$ is continuous with respect to the topology induced by that of $\mathrm{C}_c^r(X;K)$, for every compact $K \subset X$.

▷ $\mathcal{D}'_r(X) = (\mathrm{C}_c^r(X))'$

∘ possibility of extending the order to positive real numbers was already mentioned in a book by L. Tartar: *An Introduction to Sobolev spaces and Interpolation Spaces*[1]

[1] Footnote 1 on p.18

Preliminaries

$\Omega \subseteq \mathbf{R}^d$ an open set; (K_n) a sequence of compact subsets of Ω such that

$$\Omega = \cup_{n\in\mathbf{N}} K_n \text{ and } K_n \subset \operatorname{Int} K_{n+1}.$$

For $\alpha \in \langle 0, 1]$, and a compact set $K \subset \Omega$ denote by $\mathrm{C}^{0,\alpha}_K(\Omega)$ the set of all α-Hölder continuous functions on Ω, whose support is contained in K. Since K is compact, $\mathrm{C}^{0,\alpha}_K(\Omega)$ is a Banach space with norm given by

$$\begin{aligned}\|f\|_{\mathrm{C}^{0,\alpha}_K(\Omega)} &= \|f\|_{\mathrm{L}^\infty(\Omega)} + \sup_{\mathbf{x},\mathbf{y}\in\Omega,\, \mathbf{x}\neq\mathbf{y}} \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} \\ &= \|f\|_{\mathrm{L}^\infty(\Omega)} + [f]_{\mathrm{C}^{0,\alpha}_K(\Omega)}\end{aligned}$$

and thus it is a locally convex space.

Preliminaries II

∘ given two compact sets K and L such that $\mathrm{Int}\, K \subset L$, it holds

$$\mathrm{C}_K^{0,\alpha+\varepsilon}(\Omega) \subset \mathrm{C}_L^{0,\alpha}(\Omega)$$

for $0 < \varepsilon < 1 - \alpha$, the embedding being continuous and compact.

∘ $f \in \mathrm{C}_K^{0,\alpha+\varepsilon}$ can be approximated by C_L^{∞} functions in the norm of $\mathrm{C}_K^{0,\alpha}$.

∘ the obstruction: for $g \in \mathrm{C}_K^{0,\alpha+\varepsilon}(\Omega)$

$$\lim_{|\mathbf{x}-\mathbf{y}|\to 0,\, \mathbf{x}\neq\mathbf{y}} \frac{|g(\mathbf{x}) - g(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{\alpha}} = 0,$$

and hence any limit of smooth functions in the norm of $\mathrm{C}_K^{0,\alpha}(\Omega)$ also has this property.

Spaces of Hölder test functions – $\mathrm{C}_c^{0,\alpha}$

- $\mathrm{C}_c^{0,\alpha}$ — $\mathrm{C}^{0,\alpha}$ functions with compact support
- equipped with *strict inductive limit* topology generated by inclusions

$$\imath_n : \mathrm{C}_{K_n}^{0,\alpha} \to \mathrm{C}_c^{0,\alpha}$$

- sequence of seminorms:

$$p_{K_n}^{0,\alpha}(f) = \|f\|_{\mathrm{C}_{K_n}^{0,\alpha}}$$

- C_c^{∞} embeds continuously, but not densly into $\mathrm{C}_c^{0,\alpha}$
  - ▷ problem: its dual are not distributions!

Spaces of Hölder test functions – $\mathrm{c}_c^{0,\alpha}$

- $\mathrm{c}_c^{0,\alpha}$ — closure of C_c^∞ in $\mathrm{C}^{0,\alpha}$
- $f \in \mathrm{c}_c^{0,\alpha} \qquad \Longleftrightarrow \qquad \lim_{|\mathbf{x}-\mathbf{y}|\to 0,\, \mathbf{x}\neq\mathbf{y}} \frac{|f(\mathbf{x})-f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} = 0$
- strict inductive limit of Banach spaces $\mathrm{c}_{K_n}^{0,\alpha}$

Properties of $\mathrm{C}_c^{0,\alpha}$ and $\mathrm{c}_c^{0,\alpha}$

- Hausdorff, complete
- barrelled[2] – Banach-Steinhaus holds
- bornological – for linear operators: continuity $=$ boundedness
- webbed – open mapping and closed graph theorems holds
- Dieudonné-Schwartz theorem holds:

$$B \subset \mathrm{C}_c^{0,\alpha} \text{ bounded } \iff (\exists n \in \mathbf{N})\, B \subset \mathrm{C}_{K_n}^{0,\alpha} \text{ and bounded there}$$

- NOT Montel – closed and bounded sets are not compact
- NOT semi-reflexive

- $\mathrm{c}_c^{0,\alpha}$ is separable
- $\mathrm{C}_{K_n}^{0,\alpha}$ is not separable;
  - ▷ Is $\mathrm{C}_c^{0,\alpha}$ separable? NO!

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[2] fr. espace tonnelé

Spaces of Hölder test functions – $\mathrm{C}_c^{0,\alpha+}$

- $\mathrm{C}_c^{0,\alpha+}$ – inductive limit (NOT strict) of $\mathrm{C}_{K_n}^{0,\alpha+1/n}$

- all $\mathrm{C}_c^{0,\alpha+\varepsilon}$ functions with the finest locally convex topology such the following natural inclusions are continuous:

$$\imath_n : \mathrm{C}_{K_n}^{0,\alpha+1/n} \to \mathrm{C}_c^{0,\alpha+}$$

- $\operatorname{ind}\lim \mathrm{c}_{K_n}^{0,\alpha+1/n} \equiv \operatorname{ind}\lim \mathrm{C}_{K_n}^{0,\alpha+1/n}$

- C_c^{∞} embeds continuously and densly into $\mathrm{C}_c^{0,\alpha+}$

Properties of $\mathrm{C}_c^{0,\alpha+}$

Embeddings:

$$\imath_{n,n+1} : \mathrm{C}_{K_n}^{0,\alpha+1/n} \to \mathrm{C}_{K_{n+1}}^{0,\alpha+1/(n+1)}$$

are compact! (Arzelà-Ascoli)

It follows:[3]

∘ separable Hausdorff complete bornological (DF) Montel space.
In particular: reflexive, barrelled, webbed.

∘ Dieudonné-Schwartz theorem holds

∘ (f_n) converges in $\mathrm{C}_c^{0,\alpha+}$ if and only if it is contained in $\mathrm{C}_{K_n}^{0,\alpha+1/n}$ for some $n \in \mathbf{N}$, and converges in its Banach space topology

[3] Theores 6'& 7' of H. Komatsu: *Projective and injective limits of weakly compact sequences of locally convex spaces*, J. Math. Soc. Japan **19** (1967)

Hölder distributions

Definition

A *distribution of order smaller or equal to* α, $0 < \alpha < 1$, is any continuous linear functional T on $\mathrm{C}_c^\infty(\Omega)$ satisfying the estimates of the form

$$|\langle T, \varphi\rangle| \leq C_K \|\varphi\|_{\mathrm{C}_K^{0,\alpha}(\Omega)},$$

for all $K \subseteq \Omega$ compact and $\varphi \in \mathrm{C}_K^\infty(\Omega)$.
We denote the linear space of all such functionals by $\mathcal{D}'_\alpha(\Omega)$.

∘ $\mathcal{D}'_\alpha(\Omega) = \left(\mathrm{c}_c^{0,\alpha}(\Omega)\right)'$ is a Fréchet space.

Definition

A *distribution of order smaller or equal to* $\alpha+$, $0 \leq \alpha < 1$, is any continuous linear functional on $\mathrm{C}_c^{0,\alpha+}(\Omega)$.
We denote the space of all such functionals by $\mathcal{D}'_{\alpha+}(\Omega)$.

Theorem

The space $\mathcal{D}'_{\alpha+}(\Omega)$ *with the strong topology is a Fréchet-Schwartz space, it is the projective limit of spaces* $\left(\mathrm{C}^{0,\alpha+1/n}_{K_n}(\Omega)\right)'$*, and its topology is generated by the increasing sequence of seminorms*

$$p_n(T) = \sup_{\substack{\varphi \in \mathrm{C}^{0,\alpha+1/n}_{K_n}(\Omega) \\ \|\varphi\|_{\mathrm{C}^{0,\alpha+1/n}_{K_n}(\Omega)} \leq 1}} |T(\varphi)| \quad = \quad \|T\|_{\left(\mathrm{C}^{0,\alpha+1/n}_{K_n}(\Omega)\right)'}.$$

Francis Bonahon's work[4]

- *Hölder distributions* on metric spaces
  - ▷ corresponds to our space $\mathcal{D}'_{0+}$
  - ▷ investigation of geodesic laminations
  - ▷ did not consider order nor properties of the space of such distributions
- nice support property of Hölder distributions (similar to Radon measures)

[4] F. Bonahon: *Transverse Hölder distributions for geodesic laminations*, Topology **36** (1997) 103–122.

Examples

∘[5] $\mathrm{vp.}\left(\frac{1}{x}\right)$ on $\mathbf{R}$ whose action on $\varphi \in \mathrm{C}_c^\infty(\mathbf{R})$ can be defined equivalently by

$$\left\langle \mathrm{vp.}\left(\frac{1}{\mathrm{x}}\right), \varphi \right\rangle = \lim_{\varepsilon \to 0+} \int_{\mathbf{R} \setminus [-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x}\, dx = \int_0^{+\infty} \frac{\varphi(x) - \varphi(-x)}{x}\, dx\,.$$

- ▷ classical order: 1
- ▷ non-integer order $0+$

∘ *Partie finie* of $\frac{1}{x^2}$, defined for $\varphi \in \mathrm{C}_c^\infty(\Omega)$ by

$$\left\langle \mathrm{Pf.}\frac{1}{x^2}, \varphi \right\rangle = \lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x^2} dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x^2} dx - \frac{2\varphi(0)}{\varepsilon} \right)$$

- ▷ classical order: 2
- ▷ non-integer order $1+$

[5] Footnote 1 on p.18 of L. Tartar: *An Introduction to Sobolev spaces and Interpolation Spaces*, Springer, 2007.

Non-nuclearity

- counterexample?
- metric entropy of Kolmogorov[6][7]?
- Köthe spaces (Grothendieck-Pietsch theorem)

X Fréchet space with basis (e_n), and $\|.\|_k$ increasing sequence of seminorms

Define $a_n^k = \|e_n\|_k$, and $K(a) = \left\{\boldsymbol{\xi} = (\xi_n)_{n\in\mathbf{Z}} : |\boldsymbol{\xi}|_k = \sum_{n\in\mathbf{Z}} |\xi_n| a_n^k < \infty\right\}$

▷ Fréchet space with seminorms $|.|_k$

If X is nuclear, then $K(a)$ is nuclear, and

$$(\forall k \in \mathbf{N})(\exists j \in \mathbf{N}) \qquad \sum_{n\in\mathbf{Z}} \frac{a_n^k}{a_n^j} < \infty\,.$$

[6] A.N. Kolmogorov: *New Metric Invariant of Transitive Dynamical Systems and Endomorphisms of Lebesgue Spaces*, Doklady of Russian Academy of Sciences **119**(1958) 861–864.

[7] B.S. Mitiagin: *Approximate dimension and bases in nuclear spaces*, Uspekhi Mat. Nauk. **16** (1961) 63–132.

Non-nuclearity – continued

Lemma

The sequence $(x \mapsto e^{2\pi i n x})_{n\in\mathbf{Z}}$ *is a Schauder basis for* $\mathrm{C}^{0,\alpha+}(\mathbb{T})$*.*

Calculate:

$$a_n^k = \|e_n\|_k = \sup_{\substack{\varphi\in \mathrm{C}^{0,\alpha+1/k}(\Omega)\\ \|\varphi\|_{\mathrm{C}^{0,\alpha+1/k}(\Omega)}\le 1}} |e_n(\varphi)| = \sup_{\substack{\varphi\in \mathrm{C}^{0,\alpha+1/k}(\Omega)\\ \|\varphi\|_{\mathrm{C}^{0,\alpha+1/k}(\Omega)}\le 1}} |\hat{\varphi}(n)|\,.$$

This implies:

$$\frac{c}{|n|^{\alpha+1/k}} \le a_n^k \le \frac{C}{|n|}\,.$$

It follows:

$$(\forall k,j\in\mathbf{N}) \qquad \sum_{n\in\mathbf{Z}} \frac{a_n^k}{a_n^j} = \infty\,.$$

Fourier transform of L^p-functions, $p > 2$

- Chapter 7 of Hörmander's book[8]:

$$f \in \mathrm{L}^p(\mathbf{R}^d), p > 2, \text{ then } \mathcal{F}f \in \mathcal{D}' \text{ is of order } \left\lceil d\left(\frac{1}{2} - \frac{1}{p}\right)\right\rceil$$

- it is of order $d\left(\frac{1}{2} - \frac{1}{p}\right)+$

Theorem
$\mathrm{C}_c^{\left[\frac{d}{2}\right],\left(\frac{d}{2}-\left[\frac{d}{2}\right]\right)+}(\mathbf{R}^d)$ *is continuously embedded into the Wiener algebra and each distribution having Fourier transform in* $\mathrm{L}^\infty(\mathbf{R}^d)$ *is of order at most* $\frac{d}{2}+$.

[8] L. Hörmander: *The Analysis of Linear Partial Differential Operators I*, Springer, 1983.

$d = 1$ case

Theorem

There exists a $\mathrm{C}_c^{0,\frac{1}{2}}(\mathbf{R})$ *function which is not in the Wiener algebra, i.e. it doesn't have an absolutely integrable Fourier transform.*

Theorem

The dual of the Wiener algebra does not embed into $\mathcal{D}'_{1/2-\varepsilon}(\mathbf{R})$ *for any* ε*, where both spaces are considered as subsets of the space of distributions.*

- ▷ There exist a distribution T satisfying

$$|\langle T, \varphi \rangle| \leq C \|\mathcal{F}\varphi\|_{\mathrm{L}^1(\mathbf{R})}$$

which is not of order at most $1/2 - \varepsilon$.

  - ▷ Its order is strictly larger than $1/2 - \varepsilon$.

Further work

- Paley-Wiener-Schwartz theorem
- fractional derivatives of Radon measures
- Ornstein's result[9]
- kernel theorem

[9] D. Ornstein: *A Non-Inequality for Differential Operators in the* L^1*-Norm*, Arch. Rational Mech. Anal. **11** (1962) 40–49.

Reference

- M. Mišur, Lj. Pallé: *A note on the order of distributions*, in preparation, 24pp.