On a gradient constraint problem for scalar conservation laws

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Problem statement

- $\Omega \subseteq [0, \infty\rangle \times \mathbf{R}$ open bounded domain
- boundary $\partial\Omega = \Gamma_N \dot{\cup} \Gamma_D$ of class $\mathrm{C}^{0,1}$, where $\Gamma_D \subset \{t = 0\}$
- consider the following mixed boundary problem:

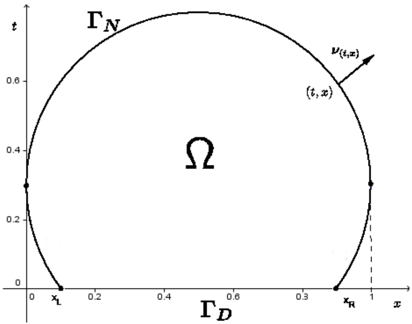
$$\partial_t u + \partial_x (f(t, x, u)) = 0 \text{ in } \Omega \tag{1}$$

$$\nabla_{(t,x)} u \cdot \nu = 0 \text{ on } \Gamma_N \tag{2}$$

$$u(0, .) = u^0(.) \in \mathrm{L}^\infty(\mathbf{R}) \text{ on } \Gamma_D, \tag{3}$$

- $f(t, x, \lambda)$ is a function of bounded variation with respect to the variables (t, x) and differentiable with respect to the third variable λ.

An example of domain $\Omega \subseteq [0, \infty\rangle \times \mathbf{R}$



Additional assumptions on f

Take $p \in \langle 2, \infty \rangle$ fixed. Assume that for all compact sets $\Lambda \subset \mathbf{R}$ and $K \subset \Omega$, the following holds:

A1: $(\exists C_1 = C_1(K, \Lambda) > 0)(\forall \xi \in \Lambda)$

$$\left\| \chi_K \int_0^\xi f(t, x, \lambda) d\lambda \right\|_{\mathrm{L}^p(\Omega)} < C_1,$$

A2: $(\exists C_2 = C_2(K, \Lambda) > 0)(\forall \xi \in \Lambda)$

$$\left\| \chi_K \int_0^\xi f'_x(t, x, \lambda) d\lambda \right\|_{\mathrm{L}^1(\Omega)} < C_2,$$

A3: $(\exists C_3 = C_3(K, \Lambda) > 0)(\forall \lambda \in \Lambda)$

$$\left\| \chi_K f(t, x, \lambda) \right\|_{\mathrm{L}^p(\Omega)} < C_3.$$

Assumptions A1 and A3, due to the boundedness of Ω, imply that for every $\Lambda \subset \mathbf{R}$ compact and every $\varphi \in \mathrm{C}_c(\Omega)$, the following holds for positive constants $C_{1,p,K,\Lambda}$ and $C_{3,p,K,\Lambda}$ with $K = \mathrm{supp}\varphi$:

C1: $(\forall \xi \in \Lambda) \qquad \left\| \varphi(t,x) \int_0^\xi f(t,x,\lambda) d\lambda \right\|_{\mathrm{L}^1(\Omega)} < C_{1,p,K,\Lambda} \|\varphi\|_{\mathrm{L}^\infty(\Omega)},$

C3: $(\forall \lambda \in \Lambda) \qquad \left\| \varphi(t,x) f(t,x,\lambda) \right\|_{\mathrm{L}^1(\Omega)} < C_{3,p,K,\Lambda} \|\varphi\|_{\mathrm{L}^\infty(\Omega)}.$

Approximation[1] of the problem

$$\begin{aligned} \partial_t u_n + \partial_x (f_n(t,x,u_n)) &= \frac{1}{n} \triangle_{(t,x)} u_n \text{ in } \Omega \\ \nabla_{(t,x)} u_n \cdot \nu &= 0 \text{ on } \Gamma_N \\ u_n(0,.) &= u_n^0(.) \text{ on } \Gamma_D, \end{aligned} \tag{4}$$

- $f_n(t,x,\lambda) = f(\cdot,\cdot,\lambda) \star n^2 \omega(nt,nx)$ is a regularization of the flux f via the standard non-negative mollifier $\omega \in \mathrm{C}_c^\infty((-1,1)^2)$,
- (u_n^0) is a bounded sequence of functions converging strongly in $\mathrm{L}^1_{loc}(\mathbf{R})$ toward u_0.

Problem: what is the appropriate solution concept?

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[1] Chapter 3 of J. L. Lions, E. Magenes: *Non-homogeneous Boundary value Problems and Applications I*, Springer–Verlag, 1972.

Concept of solution

Multiplying equation

$$\partial_t u_n + \partial_x(f_n(t,x,u_n)) = (1/n)\triangle_{(t,x)} u_n$$

by $\mathrm{sgn}(u_n(t,x)-\lambda)$, we get:

$$\begin{aligned}\partial_t|u_n-\lambda| + \partial_x\left(\mathrm{sgn}(u_n-\lambda)(f_n(u_n)-f_n(\lambda)\right) \leq&\\ \leq \frac{1}{n}\Delta_{(t,x)}|u_n-\lambda| - \mathrm{sgn}(u_n-\lambda)f'_{n,x}(t,x,\lambda) \quad &\text{in} \quad \Omega.\end{aligned}$$

Multiply by $\varphi \in \mathrm{C}^2(\Omega)$ supported away from $\{t=0\}$ and integrate over Ω. After taking into account (2), we get:

$$\begin{aligned}&-\int_\Omega \left(|u_n-\lambda|\partial_t\varphi + \mathrm{sgn}(u_n-\lambda)(f_n(u_n)-f_n(\lambda))\partial_x\varphi\right) dxdt + \\ &\quad + \int_{\partial\Omega}\Big(|u_n-\lambda|, \mathrm{sgn}(u_n-\lambda)(f_n(u_n)-f_n(\lambda))\Big)\cdot\nu\,\varphi\, ds \leq \\ &\quad \leq \frac{1}{n}\int_\Omega \nabla_{(t,x)}|u_n-\lambda|\cdot\nabla_{(t,x)}\varphi\, dxdt - \int_\Omega \varphi\,\mathrm{sgn}(u_n-\lambda)f'_{n,x}(t,x,\lambda)d\lambda\, dxdt.\end{aligned} \tag{5}$$

Concept of solution - continued

Using the main idea of the recent article by Andreianov & Mitrović[2], we introduce the following definition:

Definition

Function $u \in \mathrm{L}^2(\Omega)$ is called the solution to (1), (2), (3) if there exists a function $p \in \mathrm{L}^1(\Gamma_N)$ such that for every $\varphi \in \mathrm{C}_c(\overline{\Omega}\backslash\Gamma_D)$ the following holds:

- $$\begin{aligned} &\int_\Omega (|u-\lambda|\partial_t\varphi + \operatorname{sgn}(u-\lambda)(f(t,x,u)-f(t,x,\lambda))\partial_x\varphi)\,dxdt - \\ &- \int_{\partial\Omega} \Big(|p-\lambda|, \operatorname{sgn}(p-\lambda)(f(t,x,p)-f(t,x,\lambda))\Big)\cdot\nu\,\varphi\,ds \ \geq \\ &\geq \quad \int_\Omega \varphi \operatorname{sgn}(u-\lambda) f'_x(t,x,\lambda)\;d\lambda\;dxdt. \end{aligned} \tag{6}$$
- Initial data are satisfied in the strong sense i.e. for almost every $x \in \Gamma_D$ it holds $\lim\limits_{t\to 0} |u(t,x)-u_0(x)| = 0$.

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[2] Formula 7 of B. Andreianov, D. Mitrović: *Entropy conditions for scalar conservation laws with discontinuous flux revisited*, Annales Inst. Henry Poincare – Analyse Nonlineaire **32** (2015) 1307–1335

The main result

Theorem

Assume that the sequence (u_n) *of solutions to* (4) *is uniformly bounded by a constant* M*. If flux* f *satisfies the assumptions A1, A2 and A3, then a weak* $\mathrm{L}^2(\Omega)$*-limit of* (u_n) *along a subsequence satisfies the equation* (1) *in* Ω*.*

OUTLINE OF THE PROOF:

- $$\partial_t u_n + \partial_x \left(f(t,x,u_n) \right) \longrightarrow 0 \quad \text{in } \mathrm{H}^{-1}_{loc}(\Omega)$$

- for all entropy-entropy flux pairs $(\Phi(\lambda), \Psi_n(t,x,\lambda))$:

  $$\partial_t(\Phi(u_n)) + \partial_x(\Psi_n(t,x,u_n)) \text{ is precompact in } \mathrm{H}^{-1}_{loc}(\Omega)$$

- for all $k \in \mathbf{R}$:

  $$\partial_t |u_n - k| + \partial_x(\mathrm{sgn}(u_n - k)(f(t,x,u_n) - f(t,x,k))) \text{ is precompact in } \mathrm{H}^{-1}_{loc}(\Omega)$$

Case when $f \in \mathrm{C}^1$

A corollary of the proof of the theorem and Panov's result[3] in the case when the flux is continuously differentiable with respect to all variables is the fact that the limiting function u satisfies the Kruzhkov admissibility conditions. However, we do not have a working solution concept for (1), (3), (2) so we cannot say anything about uniqueness.

Corollary

Assume that the flux $f \in \mathrm{C}^1(\Omega \times (-M, M))$. The distributional limit u of the sequence (u_n) of solutions to (4) *satisfies for every entropy-entropy flux pair (Φ, Ψ)*

$$\partial_t(\Phi(u)) + \partial_x(\Psi(t,x,u)) \leq -\int_0^u f'_x(t,x,\lambda)\Phi''(\lambda)d\lambda \;\text{ in }\; \mathcal{D}'(\Omega).$$

[3] Remark 1 of E. Yu. Panov: *On weak completeness of the set of entropy solutions to a scalar conservation law*, SIAM J. Math. Anal. **41** (2009) 26–36

Lighthill-Whitham-Richards model for traffic flow

$$\partial_t \rho + \partial_x(\rho v(\rho)) = 0,$$

where the velocity is assumed to have linear dependence upon density of the cars

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right), \qquad 0 \leq \rho \leq \rho_{max}.$$

Let L and τ be a typical length and time, respectively, such that $v_{max} = L/\tau$. Introducing new variables

$$\bar{x} = \frac{x}{L}, \qquad \bar{t} = \frac{x}{L}, \qquad u = 1 - \frac{2\rho}{\rho_{max}},$$

we obtain the inviscid Burgers equation

$$\partial_t \rho + \partial_x\left[\rho\left(1 - \frac{\rho}{\rho_{max}}\right)\right] = -\frac{\rho_{max}}{2\tau}\partial_{\bar{t}} u - \frac{\rho_{max}}{2\tau}\partial_{\bar{x}}\left(\frac{u^2}{2}\right) = 0.$$

Examples

Let $\Omega = \{(t,x) \in \mathbf{R}^2 : 0 \leq x \leq 1,\, 0 \leq t \leq -4x(x-1)\}$.
We focus on solving the (regularized) Burgers equation

$$\begin{aligned} \partial_t u + \partial_x \left(u^2/2\right) &= \epsilon \Delta_{(t,x)} u \quad \text{in } \Omega, \\ \nabla_{(t,x)} u \cdot \nu &= 0 \quad \text{on } \Gamma_N, \\ u(0,x) &= u_D \quad \text{on } \Gamma_D, \end{aligned}$$

where $\Gamma_D = \{(t,x) \in \partial\Omega : t = 0\}$ and $\Gamma_N = \partial\Omega \setminus \Gamma_D$.
Let $V_D(\Omega) = \{v \in \mathrm{H}^1(\Omega) : v|_{\Gamma_D} = u_D\}$ and $\mathrm{H}^1_D(\Omega) = \{v \in \mathrm{H}^1(\Omega) : v|_{\Gamma_D} = 0\}$.

We use the following iterative scheme:

For given initial guess u_0, construct sequence $u_n \in V_D$, $n \geq 1$, that are solutions of

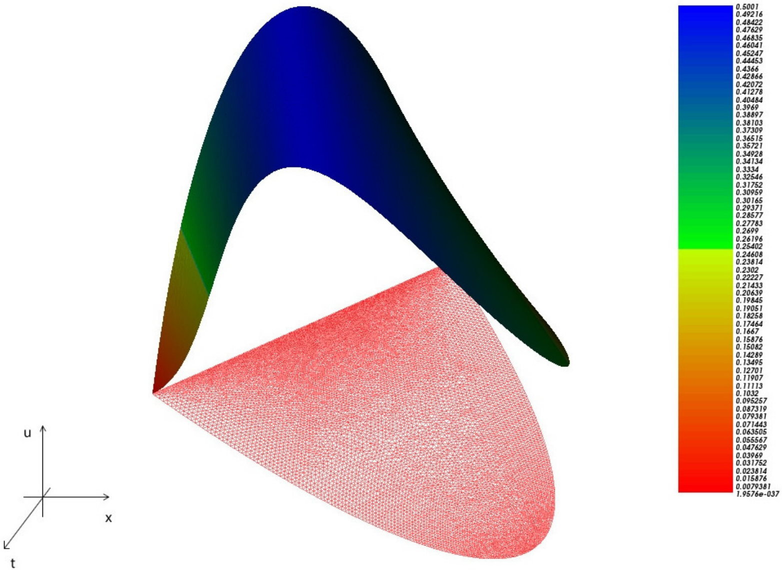
$$\int_\Omega (\partial_t u_n + u_{n-1}\partial_x u_n)\psi dtdx + \epsilon \int_\Omega \nabla_{(t,x)} u_n \cdot \nabla_{(t,x)} \psi dtdx = 0, \quad \forall \psi \in \mathrm{H}^1_D(\Omega). \tag{7}$$

Example 1

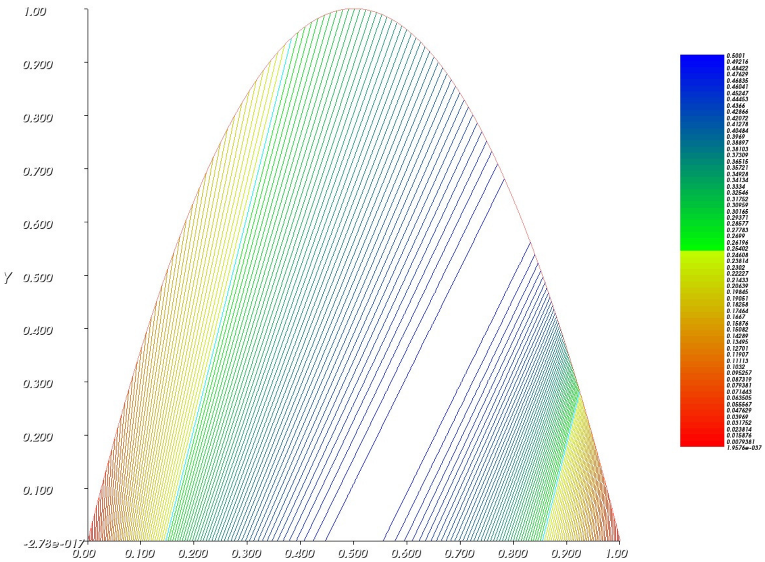
Two scenarios: in the first one $\epsilon = 1/N$ and in the second one $\epsilon = 1/N^2$ with $u_D = -2x(x-1)$ in both.
We performed two convergence tests, where referent solution u_R has been computed on $N \times N = 640^2$ grid.

$N = 1/\epsilon$	$\|u_N - u_R\|_2/\|u_R\|_2$	$N = 1/\sqrt{\epsilon}$	$\|u_N - u_R\|_2/\|u_R\|_2$
10	0.179448	10	0.0539613
20	0.130928	20	0.0137841
40	0.076787	40	0.0038117
80	0.038821	80	0.0010069
160	0.0167232	160	0.00029879
320	0.0054824	320	0.000093223

Example 1 - $N = 160$ and $\epsilon = 1/160^2$

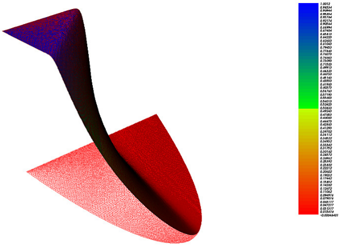


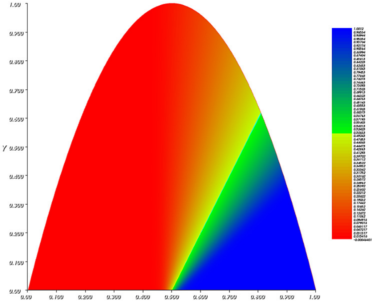
Example 1 - $N = 160$ and $\epsilon = 1/160^2$, iso-values of the solution



Example 2

$u_D = H(0.5 - x)$, where H is the Heaviside function





Example 3

$u_D = H(x - 0.5)$, where H is the Heaviside function