H-distributions, distributions of anisotropic order and Schwartz kernel theorem

Marin Mišur

email: mmisur@math.hr
University of Zagreb

joint work with Nenad Antonić and Marko Erceg, University of Zagreb

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What are H-measures?

Mathematical objects introduced (1989/90) by:

- Luc Tartar[1], who was motivated by possible applications in homogenisation, and independently by
- Patrick Gérard[2], whose motivation were problems in kinetic theory.

Theorem 1. *If* $u_n \rightharpoonup 0$ *and* $v_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)$*, then there exist their subsequences and a complex valued Radon measure* μ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$*, such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} (\psi \circ \pi) d\boldsymbol{\xi} = \langle \mu, \varphi_1 \overline{\varphi_2} \boxtimes \psi \rangle,$$

where $\pi : \mathbf{R}^d \setminus \{0\} \longrightarrow \mathrm{S}^{d-1}$ *is the projection along rays.* ∎

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[1] L. Tartar, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh **115A** (1990) 193–230.

[2] P. Gérard, *Microlocal defect measures*, Comm. Partial Diff. Eq. **16** (1991) 1761–1794.

Question: How to replace L^2 with L^p?

Notice: if we denote by $\mathcal{A}_\psi$ the Fourier multiplier operator with symbol $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$:

$$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee,$$

we can rewrite the equality from the theorem as

$$\langle \mu, \varphi_1 \overline{\varphi_2} \boxtimes \psi \rangle = \lim_{n'} \int_{\mathbf{R}^d} \varphi_1 u_{n'}(\mathbf{x}) \overline{\mathcal{A}_{\overline{\psi \circ \pi}}(\varphi_2 u_{n'})(\mathbf{x})} d\mathbf{x} \ .$$

Hörmander-Mihlin Theorem

Theorem 2. *Let* $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ *have partial derivatives of order less than or equal to* $\kappa = [d/2] + 1$*. If for some* $k > 0$

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \quad |\boldsymbol{\alpha}| \le \kappa \Longrightarrow \int_{r/2 \le |\boldsymbol{\xi}| \le r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \le k^2 r^{d-2|\boldsymbol{\alpha}|},$$

then for any $p \in \langle 1, \infty \rangle$ *and the associated multiplier operator* $\mathcal{A}_\psi$ *there exists a constant* C_d *such that*

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \le C_d \max\{p, 1/(p-1)\}(k + \|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}).$$

∎

For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d \setminus \{0\}$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}$.

Y. Heo, F. Nazarov, A. Seeger, *Radial Fourier multipliers in high dimensions*, Acta Mathematica **206** (2011) 55-92.

H-distributions

H-distributions were introduced by N. Antonić and D. Mitrović[3] as an extension of H-measures to the $\mathrm{L}^p - \mathrm{L}^q$ context.

Existing applications are related to the velocity averaging[4] and $\mathrm{L}^p - \mathrm{L}^q$ compactness by compensation[5].

[3] N. Antonić, D. Mitrović, *H-distributions: An Extension of H-Measures to an $\mathrm{L}^p - \mathrm{L}^q$ Setting*, Abs. Appl. Analysis Volume 2011, Article ID 901084, 12 pages.

[4] M. Lazar, D. Mitrović, *On an extension of a bilinear functional on $\mathrm{L}^p(\mathbf{R}^d) \times \mathrm{E}$ to Bochner spaces with an application to velocity averaging*, C. R. Math. Acad. Sci. paris **351** (2013) 261–264.

[5] M. Mišur, D. Mitrović, *On a generalization of compensated compactness in the $\mathrm{L}^p - \mathrm{L}^q$ setting*, Journal of Functional Analysis **268** (2015) 1904–1927.

Existence of H-distributions

Theorem 3. *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ *and* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$ *for some* $p \in \langle 1, \infty \rangle$ *and* $q \geq p'$*, then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$*, such that, for every* $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*, for* $\kappa = [d/2] + 1$*, one has:*

$$\begin{aligned}
\lim_{n' \to \infty} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x}) \overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} &= \lim_{n' \to \infty} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} \\
&= \langle \mu, \varphi_1 \overline{\varphi}_2 \boxtimes \psi \rangle,
\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ *is the Fourier multiplier operator with symbol* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*.* ∎

Distributions of anisotropic order

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds of dimenions d and r) and $\Omega \subseteq X \times Y$ an open set. By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \leq l$ and $|\boldsymbol{\beta}| \leq m$, $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega)$.

$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}| \leq l, |\boldsymbol{\beta}| \leq m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} ,$$

where $K_n \subseteq \Omega$ are compacts, such that $\Omega = \cup_{n \in \mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int} K_{n+1}$, Consider the space

$$\mathrm{C}_c^{l,m}(\Omega) := \bigcup_{n \in \mathbf{N}} \mathrm{C}_{K_n}^{l,m}(\Omega) ,$$

and equip it by the topology of *strict inductive limit*.

Conjecture

Definition. A *distribution of order* l *in* $\mathbf{x}$ *and order* m *in* $\mathbf{y}$ is any linear functional on $\mathrm{C}_c^{l,m}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Conjecture. *Let* X, Y *be* C^∞ *manifolds and let* u *be a linear functional on* $\mathrm{C}_c^{l,m}(X \times Y)$*. If* $u \in \mathcal{D}'(X \times Y)$ *and satisfies*
$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(X))(\forall \psi \in \mathrm{C}_L^\infty(Y))$

$$|\langle u, \varphi \boxtimes \psi \rangle| \leq C p_K^l(\varphi) p_L^m(\psi),$$

then u *can be uniquely extended to a continuous functional on* $\mathrm{C}_c^{l,m}(X \times Y)$ *(i.e. it can be considered as an element of* $\mathcal{D}'_{l,m}(X \times Y)$*).* ■

If the conjecture were true, then the H-distribution μ from the preceeding theorem belongs to the space $\mathcal{D}'_{0,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than κ in $\boldsymbol{\xi}$.

Indeed, from the proof of the existence theorem, we already have $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ and the following bound with $\varphi := \varphi_1 \overline{\varphi_2}$:

$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leq C \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} ,$$

where C does not depend on φ and ψ.

Schwartz kernel theorem

Let X and Y be two C^∞ manifolds. Then the following statements hold:

a) Let $K \in \mathcal{D}'(X \times Y)$. Then for every $\varphi \in \mathcal{D}(X)$, the linear form K_φ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathcal{D}(X)$ to $\mathcal{D}'(Y)$ is linear and continuous.

b) Let $A : \mathcal{D}(X) \to \mathcal{D}'(Y)$ be a continous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Schwartz kernel theorem for anisotropic distributions

Let X and Y be two C^∞ manifolds of dimensions d and r, respectively. Then the following statements hold:

a) Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for every $\varphi \in \mathrm{C}^l_c(X)$, the linear form K_φ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.

b) Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y)$.

How to prove it?

Attempts:

- regularisation? (Schwartz)
- constructive proof? (Simanca, Gask, Ehrenpreis)
- nuclear spaces? (Treves)

- structure theorem of distributions (Dieudonne)

Two steps:

Step I: assume the range of A is $\mathrm{C}(Y)$
Step II: use structure theorem and go back to Step I

Consequence: H-distributions are of order 0 in $\mathbf{x}$ and of finite order not greater than $d(\kappa+2)$ with respect to $\boldsymbol{\xi}$.

References

- N. Antonić, M. Erceg, M. Mišur, *Distributions of anisotropic order and applications*, in preparation, 24 pages