Extension of Cordes’ and Tartar’s results on compactness of commutator

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Joint work with Nenad Antonić and Darko Mitrović.

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What is this talk about?

Compactness of the commutator of the following operators:

- $A_\psi u := (\psi \hat{u})^\vee$
- $M_b u := bu$

$$[A_\psi, M_b] := A_\psi M_b - M_b A_\psi$$

- also known as *The First commutation lemma*
Compactness on $L^2$ - Cordes’ result$^1$

**Theorem**

**If bounded continuous functions** $b$ **and** $\psi$ **satisfy**

$$\lim_{|\xi| \to \infty} \sup_{|h| \leq 1} \{ | \psi(\xi+h) - \psi(\xi) | \} = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \sup_{|h| \leq 1} \{ | b(x+h) - b(x) | \} = 0,$$

**then the commutator** $[A_\psi, M_b]$ **is a compact operator on** $L^2(\mathbb{R}^d)$.

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Compactness on $L^2$ - Tartar’s version

For given $M, \varrho \in \mathbb{R}^+$ we denote the set

$$Y(M, \varrho) = \{(\xi, \eta) \in \mathbb{R}^{2d} : |\xi|, |\eta| \geq M \& |\xi - \eta| \leq \varrho\}.$$
Lemma (general form of the First commutation lemma)

If $b \in C_0(\mathbb{R}^d)$, while $\psi \in L^\infty(\mathbb{R}^d)$ satisfies the condition

$$(\forall \varrho, \varepsilon \in \mathbb{R}^+)(\exists M \in \mathbb{R}^+) \ |\psi(\xi) - \psi(\eta)| \leq \varepsilon \ (\text{s.s. } (\xi, \eta) \in Y(M, \varrho)) \ , \ (1)$$

then $[A_\psi, M_b]$ is a compact operator on $L^2(\mathbb{R}^d)$.

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Lemma
Let $\pi : \mathbb{R}^d \to \Sigma$ be a smooth projection to a smooth compact hypersurface $\Sigma$, such that $\|\nabla \pi(\xi)\| \to 0$ for $|\xi| \to \infty$, and let $\psi \in C(\Sigma)$. Then $\psi \circ \pi$ ($\psi$ extended by homogeneity of order 0) satisfies (1).

In the special case of the sphere, one has $\|\nabla \pi(\xi)\| \leq 1/|\xi|$. 
Where is it used?


Boundedness on $L^p$ - the Hörmander-Mihlin theorem

**Theorem**

Let $\psi \in L^\infty(\mathbb{R}^d)$ have partial derivatives of order less than or equal to $\kappa = \lfloor d/2 \rfloor + 1$. If for some $k > 0$

$$\forall r > 0 \, (\forall \alpha \in \mathbb{N}^d_0) \quad |\alpha| \leq \kappa \implies \int_{r/2 \leq |\xi| \leq r} |\partial^\alpha \psi(\xi)|^2 d\xi \leq k^2 r^{d-2|\alpha|},$$

then for any $p \in \langle 1, \infty \rangle$ and the associated multiplier operator $A_\psi$ there exists a constant $C_d$ such that

$$\|A_\psi\|_{L^p \to L^p} \leq C_d \max\{p, 1/(p-1)\}(k + \|\psi\|_{L^\infty(\mathbb{R}^d)}).$$
What about the $L^p$ variant of the First commutation lemma?

One variant can be found in the article by Cordes - complicated proof and higher regularity assumptions. Namely, the symbol is required to satisfy:

- $\psi \in C^{2\kappa}(\mathbb{R}^d)$,
- for every $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq 2\kappa$:

$$ (1 + |\xi|)^{|\alpha|} D^\alpha \psi(\xi) \quad \text{is bounded.} $$

A different variant was given by Antonić and Mitrović$^4$:

**Lemma**

Assume $\psi \in C^\kappa(S^{d-1})$ and $b \in C_0(\mathbb{R}^d)$. Let $(v_n)$ be a bounded sequence, both in $L^2(\mathbb{R}^d)$ and in $L^r(\mathbb{R}^d)$, for some $r \in [2, \infty]$, and such that $v_n \rightharpoonup 0$ in the sense of distributions.

Then $[A_\psi, M_b]v_n \rightarrow 0$ strongly in $L^q(\mathbb{R}^d)$, for any $q \in [2, r]$.

The proof was based on a simple interpolation inequality of $L^p$ spaces:

$$ \|f\|_{L^q} \leq \|f\|_{L^2}^{\theta} \|f\|_{L^r}^{1-\theta}, \text{ where } 1/q = \theta/2 + (1 - \theta)/r. $$

A variant of Krasnoselskij’s type of result\textsuperscript{5}

Lemma
Assume that linear operator $A$ is compact on $L^2(\mathbb{R}^d)$ and bounded on $L^r(\mathbb{R}^d)$, for some $r \in \langle 1, \infty \rangle \setminus \{2\}$. Then $A$ is also compact on $L^p(\mathbb{R}^d)$, for any $p$ between 2 and $r$ (i.e. such that $1/p = \theta/2 + (1 - \theta)/r$, for some $\theta \in \langle 0, 1 \rangle$).

Corollary
If $b \in C_0(\mathbb{R}^d)$, while $\psi \in C^\kappa(\mathbb{R}^d)$ satisfies the conditions of the Hörmander-Mihlin theorem and condition from the general form of the First commutation lemma, then the commutator $[A\psi, M_b]$ is a compact operator on $L^p(\mathbb{R}^d)$, for any $p \in \langle 1, \infty \rangle$.

Theorem
Let $\psi \in C^\kappa(\mathbb{R}^d \setminus \{0\})$ be bounded and satisfy Hörmander's condition, while $b \in C_c(\mathbb{R}^d)$. Then for any $u_n \rightharpoonup 0$ in $L^\infty(\mathbb{R}^d)$ and $p \in \langle 1, \infty \rangle$ one has:

$$(\forall \varphi, \phi \in C_c^\infty(\mathbb{R}^d)) \quad \phi[A_\psi, M_b](\varphi u_n) \longrightarrow 0 \quad \text{in} \quad L^p(\mathbb{R}^d).$$

Corollary
Let $(u_n)$ be a bounded, uniformly compactly supported sequence in $L^\infty(\mathbb{R}^d)$, converging to 0 in the sense of distributions. Assume that $\psi \in C^\kappa(\mathbb{R}^d)$ satisfies Hörmander’s condition and condition from the general form of the First commutation lemma.

Then for any $b \in L^s(\mathbb{R}^d)$, $s > 1$ arbitrary, it holds

$$\lim_{n \to \infty} \|bA_\psi(u_n) - A_\psi(bu_n)\|_{L^r(\mathbb{R}^d)} = 0, \quad r \in \langle 1, s \rangle.$$
A locally integrable function \( f \) is said to belong to \( \text{BMO}(\mathbb{R}^d) \) if there exists a constant \( A > 0 \) such that the following inequality holds for all balls \( B \subseteq \mathbb{R}^d \):

\[
\frac{1}{|B|} \int_B |f - f_B| \, dx \leq A,
\]

where \( f_B \) is the mean value of \( f \) over the ball \( B \).

\( \text{VMO}(\mathbb{R}^d) \) is the closure of \( C_c(\mathbb{R}^d) \) functions in the \( \text{BMO}(\mathbb{R}^d) \) norm.
Uchiyama's result\textsuperscript{6}

Denote: $R_j := A_i \xi_j / |\xi|$, for $j \in \{1, \cdots, d\}$

**Theorem**

Let $b \in \bigcup_{q>1} L^q_{\text{loc}}(\mathbb{R}^d)$. Then the commutator $[M_b, R_j]$ is a compact operator on $L^p(\mathbb{R}^d)$, for any $p \in \langle 1, \infty \rangle$, if and only if $b \in \text{VMO}(\mathbb{R}^d)$.

**Lemma**

Let $a$ be a function which is a polynomial in $\xi / |\xi|$ and $b \in \text{VMO}(\mathbb{R}^d)$. Then the commutator $[M_b, A_a]$ is a compact operator on $L^p(\mathbb{R}^d)$, for any $p \in \langle 1, \infty \rangle$.

Corollary

Let $b \in L^\infty(\mathbb{R}^d) \cap \text{VMO}(\mathbb{R}^d)$ and $\psi \in C(S^{d-1})$. Then the commutator $[M_b, A_\psi]$ is compact on $L^2(\mathbb{R}^d)$.\footnote{L. Tartar, The general theory of homogenization: A personalized introduction, Springer, 2009.}

Corollary

Let $b \in L^\infty(\mathbb{R}^d) \cap \text{VMO}(\mathbb{R}^d)$ and $\psi \in C^{\kappa}(S^{d-1})$. Then the commutator $[M_b, A_\psi]$ is compact on $L^p(\mathbb{R}^d)$, $p \in \langle 1, \infty \rangle$. 
Further comments...

- results of I.-L. Hwang and A. Stefanov...