On L^p compactness of commutator of multiplication and Fourier multiplier operator

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31^{st} of October 2016.

What is this talk about?

Compactness of the commutator of the following operators:

- $\mathcal{A}_\psi u := (\psi \hat{u})^\vee$
- $M_b u := bu$

$$[\mathcal{A}_\psi, M_b] := \mathcal{A}_\psi M_b - M_b \mathcal{A}_\psi$$

- also known as *The First commutation lemma*

Compactness on L^2 - Cordes' result[1]

Theorem

If bounded continuous functions b and ψ satisfy

$$\lim_{|\boldsymbol{\xi}|\to\infty} \sup_{|\mathbf{h}|\leq 1} \{|\psi(\boldsymbol{\xi}+\mathbf{h})-\psi(\boldsymbol{\xi})|\} = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{|\mathbf{h}|\leq 1} \{|b(\mathbf{x}+\mathbf{h})-b(\mathbf{x})|\} = 0 \,,$$

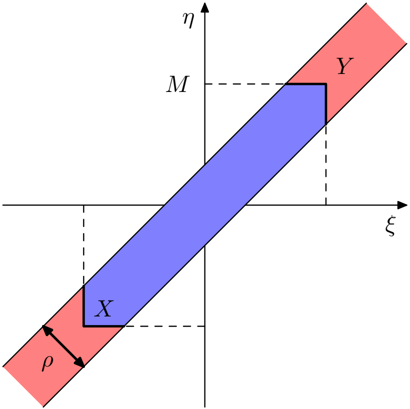
then the commutator $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$.

[1] H. O. Cordes, *On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators*, J. Funct. Anal. **18** (1975) 115–131.

Compactness on L^2 - Tartar's version

For given $M, \varrho \in \mathbf{R}^+$ we denote the set

$$Y(M,\varrho) = \{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geq M \& |\boldsymbol{\xi}-\boldsymbol{\eta}| \leq \varrho\} \ .$$



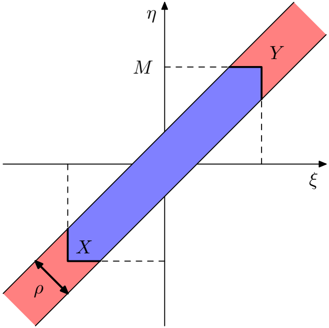
Compactness on L^2 - Tartar's version[2]

Lemma (general form of the First commutation lemma)

If $b \in \mathrm{C}_0(\mathbf{R}^d)$, while $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ satisfies the condition

$$(\forall\, \varrho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |\psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\text{s.s. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \varrho)) \,, \quad (1)$$

then $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$.



[2] L. Tartar, *The general theory of homogenization: A personalized introduction*, Springer, 2009.

Lemma

Let $\pi : \mathbf{R}^d{}_* \to \Sigma$ *be a smooth projection to a smooth compact hypersurface* Σ*, such that* $\|\nabla\pi(\boldsymbol{\xi})\| \to 0$ *for* $|\boldsymbol{\xi}| \to \infty$*, and let* $\psi \in \mathrm{C}(\Sigma)$*. Then* $\psi \circ \pi$ *(*ψ *extended by homogeneity of order 0) satisfies* (1)*.*

In the special case of the sphere, one has $\|\pi(\boldsymbol{\xi})\| \leq 1/|\boldsymbol{\xi}|$.

Where is it used?

- L. Tartar, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations,* Proc. Roy. Soc. Edinburgh **115A** (1990) 193–230.[3]
- E. Ju. Panov, *Ultra-parabolic H-measures and compensated compactness*, Ann. Inst. H. Poincaré Anal. Non Linéaire C **28** (2011) 47–62.
- N. Antonić, M. Lazar, *Parabolic H-measures*, J. Funct. Anal. **265** (2013) 1190–1239.
- Z. Lin, *Instability of nonlinear dispersive solitary waves*, J. Funct. Anal. **255** (2008) 1191–1224.
- Z. Lin, *On Linear Instability of 2D Solitary Water Waves*, International Mathematics Research Notices **2009** (2009) 1247–1303.
- S. Richard, R. T. de Aldecoa, *New Formulae for the Wave Operators for a Rank One Interaction*, Integr. Equ. Oper. Theory **66** (2010) 283–292.

[3] P. Gérard, *Microlocal defect measures,* Comm. Partial Diff. Eq. **16** (1991) 1761–1794.

Boundedness on L^p - the Hörmander-Mihlin theorem

Theorem

Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ *have partial derivatives of order less than or equal to* $\kappa = [d/2] + 1$*. If for some* $k > 0$

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \quad |\boldsymbol{\alpha}| \le \kappa \Longrightarrow \int_{r/2 \le |\boldsymbol{\xi}| \le r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \le k^2 r^{d-2|\boldsymbol{\alpha}|},$$

then for any $p \in \langle 1, \infty \rangle$ *and the associated multiplier operator* $\mathcal{A}_\psi$ *there exists a constant* C_d *such that*

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \le C_d \max\{p, 1/(p-1)\}(k + \|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}).$$

What about the L^p variant of the First commutation lemma?

One variant can be found in the article by Cordes - complicated proof and higher regularity assumptions. Namely, the symbol is requird to satisfy:

- $\psi \in \mathrm{C}^{2\kappa}(\mathbf{R}^d)$,
- for every $\boldsymbol{\alpha} \in \mathbf{N}_0^d, |\boldsymbol{\alpha}| \leq 2\kappa$:

$$(1+|\boldsymbol{\xi}|)^{|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi \qquad \text{is bounded.}$$

A different variant was given by Antonić and Mitrović[4]:

Lemma

Assume $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ *and* $b \in \mathrm{C}_0(\mathbf{R}^d)$. *Let* (v_n) *be a bounded sequence, both in* $\mathrm{L}^2(\mathbf{R}^d)$ *and in* $\mathrm{L}^r(\mathbf{R}^d)$, *for some* $r \in \langle 2, \infty]$, *and such that* $v_n \rightharpoonup 0$ *in the sense of distributions.*

Then $[\mathcal{A}_\psi, M_b]v_n \longrightarrow 0$ *strongly in* $\mathrm{L}^q(\mathbf{R}^d)$, *for any* $q \in [2, r\rangle$.

The proof was based on a simple interpolation inequality of L^p spaces: $\|f\|_{\mathrm{L}^q} \leqslant \|f\|_{\mathrm{L}^2}^\theta \|f\|_{\mathrm{L}^r}^{1-\theta}$, where $1/q = \theta/2 + (1-\theta)/r$.

[4] N. Antonić, D. Mitrović, *H-distributions: an extension of H-measures to an* $\mathrm{L}^p - \mathrm{L}^q$ *setting*, Abs. Appl. Analysis **2011** Article ID 901084 (2011) 12 pp.

A variant of Krasnoselskij's type of result[5]

Lemma

Assume that linear operator A *is compact on* $\mathrm{L}^2(\mathbf{R}^d)$ *and bounded on* $\mathrm{L}^r(\mathbf{R}^d)$*, for some* $r \in \langle 1, \infty\rangle \setminus \{2\}$*. Then* A *is also compact on* $\mathrm{L}^p(\mathbf{R}^d)$*, for any* p *between 2 and* r *(i.e. such that* $1/p = \theta/2 + (1-\theta)/r$*, for some* $\theta \in \langle 0, 1\rangle$*).*

Corollary

If $b \in \mathrm{C}_0(\mathbf{R}^d)$*, while* $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d)$ *satisfies the conditions of the Hörmander-Mihlin theorem and condition from the general form of the First commutation lemma, then the commutator* $[\mathcal{A}_\psi, M_b]$ *is a compact operator on* $\mathrm{L}^p(\mathbf{R}^d)$*, for any* $p \in \langle 1, \infty\rangle$*.*

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[5] M. A. Krasnoselskij, *On a theorem of M. Riesz*, Dokl. Akad. Nauk SSSR **131** (1960) 246–248 (in russian); translated as Soviet Math. Dokl. **1** (1960) 229–231.

Theorem

Let $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ be bounded and satisfy Hörmander's condition, while $b \in \mathrm{C}_c(\mathbf{R}^d)$. Then for any $u_n \xrightarrow{*} 0$ in $\mathrm{L}^\infty(\mathbf{R}^d)$ and $p \in \langle 1, \infty\rangle$ one has:

$$(\forall \varphi, \phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)) \qquad \phi C(\varphi u_n) \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^p(\mathbf{R}^d)\,.$$

Corollary

Let (u_n) be a bounded, uniformly compactly supported sequence in $\mathrm{L}^\infty(\mathbf{R}^d)$, converging to 0 in the sense of distributions. Assume that $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ satisfies Hörmander's condition and condition from the general form of the First commutation lemma.
Then for any $b \in \mathrm{L}^s(\mathbf{R}^d)$, $s > 1$ arbitrary, it holds

$$\lim_{n\to\infty} \|b\mathcal{A}_\psi(u_n) - \mathcal{A}_\psi(bu_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} = 0, \quad r \in \langle 1, s\rangle.$$

BMO and VMO spaces

A locally integrable function f is said to belong to $\mathrm{BMO}(\mathbf{R}^d)$ if there exists a constant $A > 0$ such that the following inequality holds for all balls $B \subseteq \mathbf{R}^d$:

$$\frac{1}{|B|} \int_B |f - f_B| d\mathbf{x} \leqslant A \, ,$$

where f_B is the mean value of f over the ball B.

$\mathrm{VMO}(\mathbf{R}^d)$ is the closure of $\mathrm{C}_c(\mathbf{R}^d)$ functions in the $\mathrm{BMO}(\mathbf{R}^d)$ norm.

Uchiyama's result[6]

Denote: $R_j := \mathcal{A}_{i\xi_j/|\boldsymbol{\xi}|}$, for $j \in \{1, \cdots, d\}$

Theorem

Let $b \in \cup_{q>1} \mathrm{L}^q_{\mathrm{loc}}(\mathbf{R}^d)$. Then the commutator $[M_b, R_j]$ is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty \rangle$, if and only if $b \in \mathrm{VMO}(\mathbf{R}^d)$.

Lemma

Let a be a function which is a polynomial in $\boldsymbol{\xi}/|\boldsymbol{\xi}|$ and $b \in \mathrm{VMO}(\mathbf{R}^d)$. Then the commutator $[M_b, \mathcal{A}_a]$ is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty \rangle$.

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[6] A. Uchiyama, *On the compactness of operators of Hankel type,* Tohoku Math. Journ. **30** (1978) 163–171.

Corollary

Let $b \in \mathrm{L}^\infty(\mathbf{R}^d) \cap \mathrm{VMO}(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$*. Then the commutator* $[M_b, \mathcal{A}_\psi]$ *is compact on* $\mathrm{L}^2(\mathbf{R}^d)$*.*[7]

Corollary

Let $b \in \mathrm{L}^\infty(\mathbf{R}^d) \cap \mathrm{VMO}(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*. Then the commutator* $[M_b, \mathcal{A}_\psi]$ *is compact on* $\mathrm{L}^p(\mathbf{R}^d)$*,* $p \in \langle 1, \infty \rangle$*.*

[7] L. Tartar, *The general theory of homogenization: A personalized introduction*, Springer, 2009.

Further comments...

- results of I.-L. Hwang and A. Stefanov...

N. Antonić, M. Mišur, D. Mitrović, *On the First commutation lemma*, 18pp, submitted