Exact solutions in optimal design problems for stationary diffusion equation

Krešimir Burazin

University J. J. Strossmayer of Osijek
Department of Mathematics
Trg Ljudevita Gaja 6
31000 Osijek, Croatia

http://www.mathos.unios.hr/~kburazin
kburazin@mathos.hr

WeConMApp

Joint work with Marko Vrdoljak, University of Zagreb

Multiple state optimal design problem

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f_1, \ldots, f_m \in \mathrm{L}^2(\Omega)$ given; stationary diffusion equations with homogenous Dirichlet b. c.:

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{array} \right. , \quad i = 1, \ldots, m \tag{1}$$

Multiple state optimal design problem

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f_1, \ldots, f_m \in \mathrm{L}^2(\Omega)$ given; stationary diffusion equations with homogenous Dirichlet b. c.:

$$\left\{ \begin{array}{l} -\mathrm{div}\left(\mathbf{A}\nabla u_i\right) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. , \quad i = 1, \ldots, m \tag{1}$$

where **A** is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$:

Multiple state optimal design problem

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f_1, \ldots, f_m \in \mathrm{L}^2(\Omega)$ given; stationary diffusion equations with homogenous Dirichlet b. c.:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases}, \quad i = 1, \ldots, m \tag{1}$$

where $\mathbf{A}$ is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$: $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$, $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.

Multiple state optimal design problem

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f_1, \ldots, f_m \in \mathrm{L}^2(\Omega)$ given; stationary diffusion equations with homogenous Dirichlet b. c.:

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. , \quad i = 1, \ldots, m \tag{1}$$

where $\mathbf{A}$ is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$: $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$, $\int_\Omega \chi \, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.

For given Ω, α, β, q_α, f_i, and some given weights $\mu_i > 0$, we want to find such material $\mathbf{A}$ which minimizes the weighted sum of compliances (total amounts of heat/electrical energy dissipated in Ω):

Multiple state optimal design problem

$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f_1, \dots, f_m \in \mathrm{L}^2(\Omega)$ given; stationary diffusion equations with homogenous Dirichlet b. c.:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases}, \quad i = 1, \dots, m \tag{1}$$

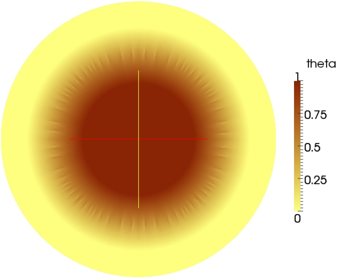
where $\mathbf{A}$ is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$: $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$, where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$, $\int_\Omega \chi\, d\mathbf{x} = q_\alpha$, for given $0 < q_\alpha < |\Omega|$.

For given Ω, α, β, q_α, f_i, and some given weights $\mu_i > 0$, we want to find such material $\mathbf{A}$ which minimizes the weighted sum of compliances (total amounts of heat/electrical energy dissipated in Ω):

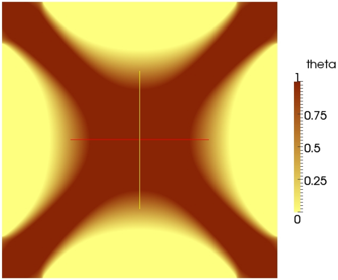
$$I(\chi) := \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min, \quad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$$

single state, $f \equiv 1$, Ω circle / square

Murat & Tartar

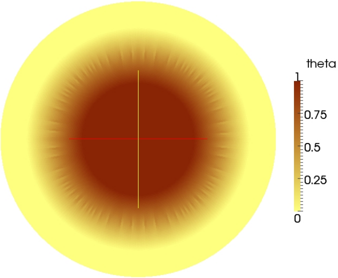


Lurie & Cherkaev

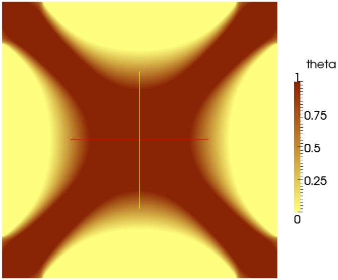


single state, $f \equiv 1$, Ω circle / square

Murat & Tartar



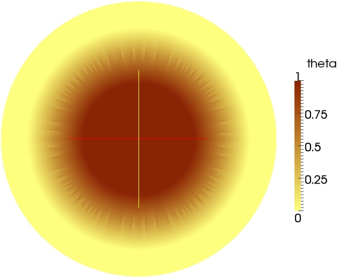
Lurie & Cherkaev



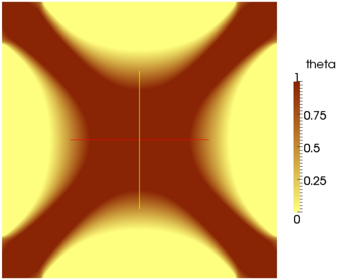
$$\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\})$$
$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$$
classical material

single state, $f \equiv 1$, Ω circle / square

Murat & Tartar



Lurie & Cherkaev



$$\chi \in \mathrm{L}^\infty(\Omega; \{0, 1\}) \quad \cdots \quad \theta \in \mathrm{L}^\infty(\Omega; [0, 1])$$

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1 - \chi)\beta\mathbf{I} \qquad \mathbf{A} \in \mathcal{K}(\theta) \quad \text{a.e. on } \Omega$$

classical material — composite mateiral - relaxation

Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0, 1\})$ *and conductivities*

$$\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1 - \chi_\varepsilon(x))\beta\mathbf{I}$$

satisfy $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε)*.*

Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *and conductivities*

$$\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$$

satisfy $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε)*.*

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = diag(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+),$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}.$$

Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0,1\})$ *and conductivities*

$$\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$$

satisfy $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε).

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = diag(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

Set of all composites:

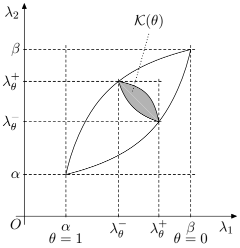
$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.}\}$$

Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

Effective conductivities – set $\mathcal{K}(\theta)$

2D:



G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

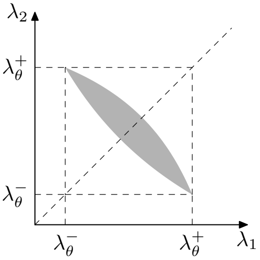
$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

Effective conductivities – set $\mathcal{K}(\theta)$

2D:



G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

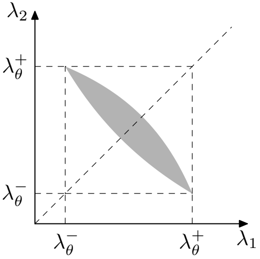
$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

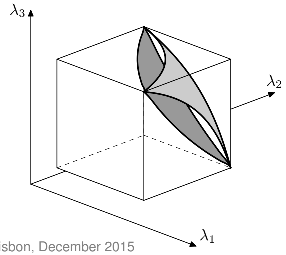
$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

2D:



3D:



Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

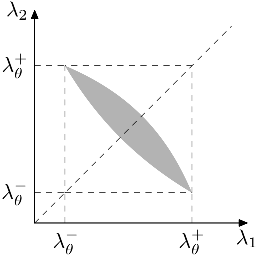
$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+ \quad j = 1, \ldots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

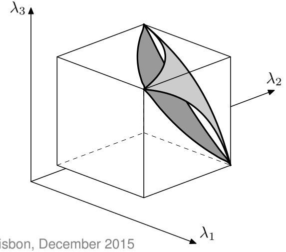
$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{\mathrm{L}^\infty(\Omega;\{0,1\})} I$

2D:



3D:



How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]

This problem can be rewritten as a
simpler convex minimization problem.

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]

This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega fu \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]

This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha \right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \qquad \Longleftrightarrow \qquad \min_{\mathcal{T}} I$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega fu \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{ \theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha \right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \qquad \Longleftrightarrow \qquad \min_{\mathcal{T}} I$$

B. Multiple state equations: Simpler relaxation fails, but in spherically symmetric case it can be done!

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar]
This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega fu\, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega;[0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$\min_{\mathcal{A}} J \quad \Longleftrightarrow \quad \min_{\mathcal{T}} I$

B. Multiple state equations: Simpler relaxation fails, but in spherically symmetric case it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i\, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega;[0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \ldots, m,$$

How do we find a solution?

Goal: find explicit solution for some simple domains (circle)

Motivation: test examples for robust numerical algorithms

A. Single state equation: [Murat & Tartar] This problem can be rewritten as a simpler convex minimization problem.

$$I(\theta) = \int_\Omega f u \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

$\theta \in \mathcal{T}$, and u determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$$\min_{\mathcal{A}} J \quad \Longleftrightarrow \quad \min_{\mathcal{T}} I$$

B. Multiple state equations: Simpler relaxation fails, but in spherically symmetric case it can be done!

$$I(\theta) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \longrightarrow \min$$

$$\mathcal{T} = \left\{\theta \in \mathrm{L}^\infty(\Omega; [0,1]) : \int_\Omega \theta = q_\alpha\right\}$$

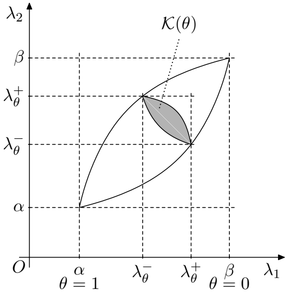
$\theta \in \mathcal{T}$, and u_i determined uniquely by

$$\begin{cases} -\mathrm{div}\,(\lambda_\theta^+ \nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{cases} \quad i = 1, \ldots, m,$$

$$\min_{\mathcal{A}} J \Longleftarrow \min_{\mathcal{B}} J \Longleftrightarrow \min_{\mathcal{T}} I$$

Minimization problem $\min_{\mathcal{B}} J$

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \; \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$



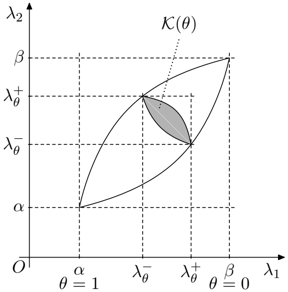
Minimization problem $\min_{\mathcal{B}} J$

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

Further relaxation:

$$\mathcal{B} \qquad \ldots \quad \int_\Omega \theta \, d\mathbf{x} = q_\alpha$$

$$\lambda_\theta^- \le \lambda_i(\mathbf{A}) \le \lambda_\theta^+$$



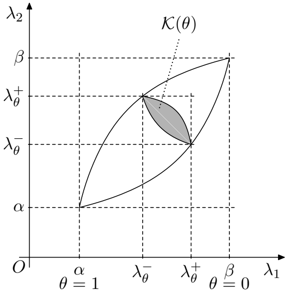
Minimization problem $\min_{\mathcal{B}} J$

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times \mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta \, d\mathbf{x} = q_\alpha \,, \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.} \}$$

Further relaxation:

$$\mathcal{B} \qquad \ldots \quad \int_\Omega \theta \, d\mathbf{x} = q_\alpha$$
$$\lambda_\theta^- \le \lambda_i(\mathbf{A}) \le \lambda_\theta^+$$

$\mathcal{B}$ is convex and compact and J is continuous on $\mathcal{B}$, so there is a solution of $\min_{\mathcal{B}} J$.



$\min_{\mathcal{B}} J \Longleftrightarrow \min_{\mathcal{T}} I$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$.

$\min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$.
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$.

$\min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$*.*
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$*.*
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$*, then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda^+_{\theta^*} \nabla u_i^*$ *almost everywhere on* Ω *(e.g.* $\mathbf{A}^* = \lambda^+_{\theta^*} \mathbf{I}$*) is an optimal design for the problem* $\min_{\mathcal{B}} J$*.*

$\min_{\mathcal{B}} J \iff \min_{\mathcal{T}} I$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$*.*
- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$*.*
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$*, then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda^+_{\theta^*} \nabla u_i^*$ *almost everywhere on* Ω *(e.g.* $\mathbf{A}^* = \lambda^+_{\theta^*}\mathbf{I}$*) is an optimal design for the problem* $\min_{\mathcal{B}} J$*.*
- *If* $m < d$*, then there exists minimizer* $(\theta^*, \mathbf{A}^*)$ *for* J *on* $\mathcal{B}$*, such that* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$*, and thus it is also minimizer for* J *on* $\mathcal{A}$*.*

Spherical symmetry: $\min_{\mathcal{A}} J \Longleftarrow \min_{\mathcal{B}} J \Longleftrightarrow \min_{\mathcal{T}} I$

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric, and let the right-hand sides $f_i = f_i(r)$, $r \in \omega$, $i = 1, \ldots, m$ be radial functions. Then there exists a minimizer $(\theta^*, \mathbf{A}^*)$ of the optimal design problem $\min_{\mathcal{A}} J$ which is a radial function.

Spherical symmetry: $\min_{\mathcal{A}} J \Longleftarrow \min_{\mathcal{B}} J \Longleftrightarrow \min_{\mathcal{T}} I$

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric, and let the right-hand sides $f_i = f_i(r)$, $r \in \omega$, $i = 1, \ldots, m$ be radial functions. Then there exists a minimizer $(\theta^*, \mathbf{A}^*)$ of the optimal design problem $\min_{\mathcal{A}} J$ which is a radial function. More precisely

a) *For any minimizer θ of functional I over $\mathcal{T}$, let us define a radial function $\theta^* : \Omega \longrightarrow \mathbf{R}$ as the average value over spheres of θ: for $r \in \omega$ we take*

$$\theta^*(r) := \fint_{\partial B(\mathbf{0},r)} \theta \, dS \,,$$

where S denotes the surface measure on a sphere. Then θ^* is also minimizer for I over $\mathcal{T}$.

Spherical symmetry. . . cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^*$ as a simple laminate layered with respect to a radial direction $\mathbf{e}_r$, as below, and local proportion of the first material θ^*. To be specific, we can define $\mathbf{A}^* : \Omega \longrightarrow \mathrm{M}_d(\mathbf{R})$ in the following way:*

Spherical symmetry... cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^*$ as a simple laminate layered with respect to a radial direction $\mathbf{e}_r$, as below, and local proportion of the first material θ^*. To be specific, we can define $\mathbf{A}^* : \Omega \longrightarrow \mathrm{M}_d(\mathbf{R})$ in the following way:*

- *If $\mathbf{x} = re_1 = (r, 0, 0, \ldots, 0)$, then*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(r), \lambda^-_{\theta^*}(r), \lambda^+_{\theta^*}(r), \ldots, \lambda^+_{\theta^*}(r)\right) \; .$$

Spherical symmetry...cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define* $\mathbf{A}^*$ *as a simple laminate layered with respect to a radial direction* $\mathbf{e}_r$*, as below, and local proportion of the first material* θ^**. To be specific, we can define* $\mathbf{A}^* : \Omega \longrightarrow \mathrm{M}_d(\mathbf{R})$ *in the following way:*

- *If* $\mathbf{x} = r\mathrm{e}_1 = (r, 0, 0, \ldots, 0)$, *then*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(r), \lambda^-_{\theta^*}(r), \lambda^+_{\theta^*}(r), \ldots, \lambda^+_{\theta^*}(r)\right) .$$

- *For all other* $\mathbf{x} \in \Omega$, *we take the unique rotation* $\mathbf{R}(\mathbf{x}) \in SO(d)$ *such that* $\mathbf{x} = |\mathbf{x}|\mathbf{R}(\mathbf{x})\mathrm{e}_1$, *and define*

$$\mathbf{A}^*(\mathbf{x}) := \mathbf{R}(\mathbf{x})\mathbf{A}^*(\mathbf{R}^\tau(\mathbf{x})\mathbf{x})\mathbf{R}^\tau(\mathbf{x}) .$$

Spherical symmetry. . . cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define* $\mathbf{A}^*$ *as a simple laminate layered with respect to a radial direction* $\mathbf{e}_r$*, as below, and local proportion of the first material* θ^**. To be specific, we can define* $\mathbf{A}^* : \Omega \longrightarrow \mathrm{M}_d(\mathbf{R})$ *in the following way:*

- *If* $\mathbf{x} = r\mathrm{e}_1 = (r, 0, 0, \ldots, 0)$*, then*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(r), \lambda^-_{\theta^*}(r), \lambda^+_{\theta^*}(r), \ldots, \lambda^+_{\theta^*}(r)\right) .$$

- *For all other* $\mathbf{x} \in \Omega$*, we take the unique rotation* $\mathbf{R}(\mathbf{x}) \in SO(d)$ *such that* $\mathbf{x} = |\mathbf{x}|\mathbf{R}(\mathbf{x})\mathrm{e}_1$*, and define*

$$\mathbf{A}^*(\mathbf{x}) := \mathbf{R}(\mathbf{x})\mathbf{A}^*(\mathbf{R}^\tau(\mathbf{x})\mathbf{x})\mathbf{R}^\tau(\mathbf{x}) .$$

Then $(\theta^*, \mathbf{A}^*)$ *is a radial optimal design for* $\min_{\mathcal{B}} J$*.*

Spherical symmetry... cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^*$ as a simple laminate layered with respect to a radial direction $\mathbf{e}_r$, as below, and local proportion of the first material θ^*. To be specific, we can define $\mathbf{A}^* : \Omega \longrightarrow \mathrm{M}_d(\mathbf{R})$ in the following way:*

- *If $\mathbf{x} = r\mathrm{e}_1 = (r, 0, 0, \ldots, 0)$, then*

$$\mathbf{A}^*(\mathbf{x}) := diag\left(\lambda^+_{\theta^*}(r), \lambda^-_{\theta^*}(r), \lambda^+_{\theta^*}(r), \ldots, \lambda^+_{\theta^*}(r)\right) .$$

- *For all other $\mathbf{x} \in \Omega$, we take the unique rotation $\mathbf{R}(\mathbf{x}) \in SO(d)$ such that $\mathbf{x} = |\mathbf{x}|\mathbf{R}(\mathbf{x})\mathrm{e}_1$, and define*

$$\mathbf{A}^*(\mathbf{x}) := \mathbf{R}(\mathbf{x})\mathbf{A}^*(\mathbf{R}^\tau(\mathbf{x})\mathbf{x})\mathbf{R}^\tau(\mathbf{x}) .$$

Then $(\theta^*, \mathbf{A}^*)$ is a radial optimal design for $\min_{\mathcal{B}} J$.
Moreover, $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$, and thus it is also a solution for $\min_{\mathcal{A}} J$.

Optimality conditions for $\min_{\mathcal{T}} I$

Lemma

$\theta^* \in \mathcal{T}$ is a solution $\min_{\mathcal{T}} I$ if and only if there exists a Lagrange multiplier $c \geq 0$ such that

$$\begin{aligned} \theta^* \in \langle 0, 1\rangle \quad &\Rightarrow \quad \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 = c\,, \\ \theta^* = 0 \quad &\Rightarrow \quad \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 \geq c\,, \\ \theta^* = 1 \quad &\Rightarrow \quad \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 \leq c\,, \end{aligned}$$

or equivalently

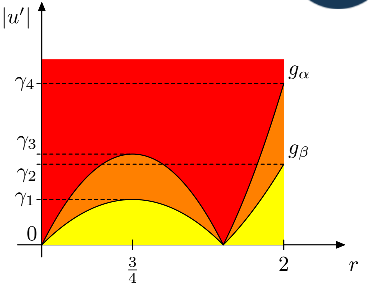
$$\begin{aligned} \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 > c \quad &\Rightarrow \quad \theta^* = 0\,, \\ \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 < c \quad &\Rightarrow \quad \theta^* = 1\,. \end{aligned}$$

Ball $\Omega = B(\mathbf{0}, 2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

Ball $\Omega = B(0,2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

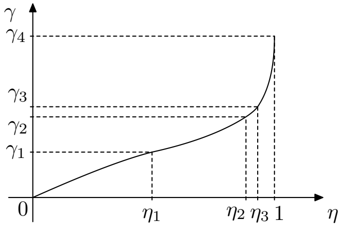


optimality conditions: $\gamma := \sqrt{c} > 0$

γ is uniquely determined by

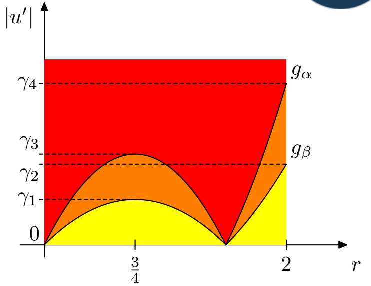
$$\int_\Omega \theta^* \, d\mathbf{x} = \eta := \frac{q_\alpha}{|\Omega|} \in [0,1]\,,$$

which is an algebraic equation for γ.



Ball $\Omega = B(0,2) \subseteq \mathbf{R}^2$ with nonconstant right-hand side

In all examples $\alpha = 1$, $\beta = 2$, one state equation $f(r) = 1 - r$

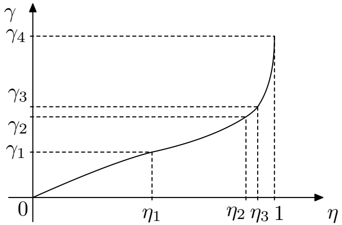


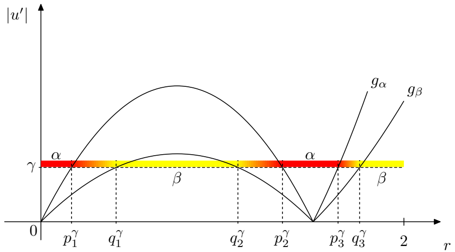
optimality conditions: $\gamma := \sqrt{c} > 0$

γ is uniquely determined by

$$\fint_\Omega \theta^* \, d\mathbf{x} = \eta := \frac{q_\alpha}{|\Omega|} \in [0,1],$$

which is an algebraic equation for γ.





Two state equations on a ball $\Omega = B(\mathbf{0}, 2)$

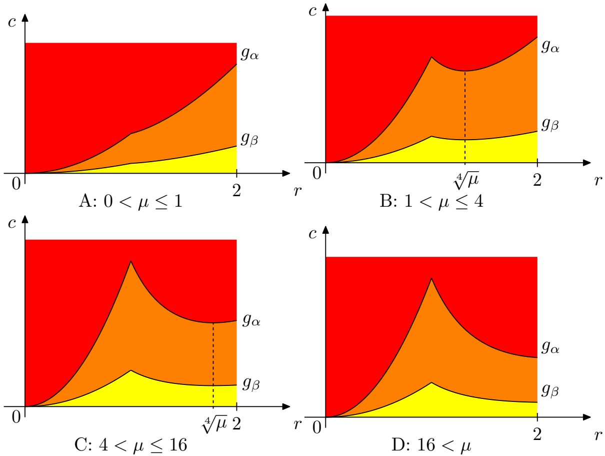
- $f_1 = \chi_{B(\mathbf{0},1)}\,,\ f_2 \equiv 1\,,$

Two state equations on a ball $\Omega = B(\mathbf{0}, 2)$

- $f_1 = \chi_{B(\mathbf{0},1)}\ ,\ f_2 \equiv 1\ ,$
- $\mu \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \min$

Two state equations on a ball $\Omega = B(\mathbf{0}, 2)$

- $f_1 = \chi_{B(\mathbf{0},1)}$, $f_2 \equiv 1$,
- $\mu \int_\Omega f_1 u_1 \, d\mathbf{x} + \int_\Omega f_2 u_2 \, d\mathbf{x} \to \min$

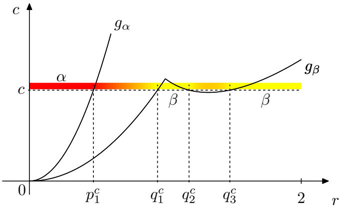


Optimal θ^* for case B

As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $\fint_\Omega \theta^* \, d\mathbf{x} = \eta$.

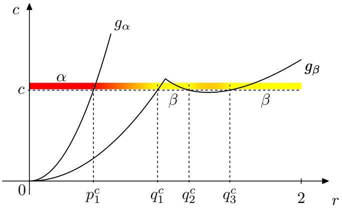
Optimal θ^* for case B

As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $⨍_\Omega \theta^* \, d\mathbf{x} = \eta$.



Optimal θ^* for case B

As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $⨍_\Omega \theta^* \, d\mathbf{x} = \eta$.



Thank you for your attention!

WeConMApp