The mathematical theory of homogenization: The physical idea of homogenization is to average a heterogeneous media in order to derive effective properties.

\[ \text{u} \in \text{W} \]

Theorem 3

Let \( M \) be the weak limit of \( u \) and \( A \rightarrow 0 \) the limit (effective) problem is

\[ \begin{cases} A_n u = f \quad \text{in} \ \Omega \\ \text{initial/boundary condition} \end{cases} \]

The mathematical problem is to determine an adequate topologies for these convergences.

Homogenization theory

Properties of H-convergence and corrector result

Theorem 2 (Irrelevance of boundary conditions) Let \( \{M\} \) be a sequence of tensors in \( M_0(\alpha, \beta; \Omega) \) that H-converges to \( M \). For any sequence \( \{u_n\} \) such that

\[ \text{div} (M_n \nabla u_n) \rightarrow \text{div} (M \nabla u) \]

the weak convergence \( M_n \nabla u_n \rightarrow M \nabla u \) in \( L^2(\Omega; \text{Sym}) \) holds.

Theorem 4

Let \( F = \{ f_n; n \in N \} \) be a dense countable family in \( H^{-2}(\Omega) \) and \( (u_n, v_n) \) sequence of solutions to

\[ \begin{cases} \text{div} (M_n \nabla u_n) = f_n \quad \text{in} \ \Omega \\ u_n \in H_0^1(\Omega) \end{cases} \]

and

\[ \begin{cases} \text{div} (O(M_n \nabla u_n)) = f_n \quad \text{in} \ \Omega \\ u_n \in H_0^1(\Omega) \end{cases} \]

respectively. Then,

\[ a(M; \Omega) = \sum_{n=1}^{\infty} \left[ \frac{\| \nabla u_n - O(M_n \nabla u_n) \|_{L^2(\Omega; \text{Sym})}^2}{\| f_n \|^2_{H^{-2}(\Omega)}} \right] \]

is a metric function on \( M_0(\alpha, \beta; \Omega) \) and H-convergence is equivalent to the convergence with respect to \( d \).

Definition 2

Let \( \{M\} \) be a sequence of tensors in \( M_0(\alpha, \beta; \Omega) \) that H-converges to a limit \( M \). Let \( \{u_n\}_{n \in \mathbb{N}} \) be a family of test functions satisfying

\[ \begin{cases} \text{div} (M_n \nabla u_n) = f_n \quad \text{in} \ \Omega \\ u_n \in H_0^1(\Omega) \end{cases} \]

\[ \text{div} (M \nabla u) = f \quad \text{in} \ \Omega \\

The tensor \( W \) defined as \( [u_m, u_n] = [\nabla u_m, \nabla u_n]_{L^2(\Omega; \text{Sym})} \) is called a corrector tensor.

Theorem 5 (Corrector result) Let \( \{M\} \) be a sequence of tensors in \( M_0(\alpha, \beta; \Omega) \) which H-converges to \( M \). For \( f \in H^{-2}(\Omega) \), let \( (u_n, v_n) \) be the solutions of

Small-amplitude homogenization

Theorem 6

Let \( \{M\} \) be a sequence of tensors defined by \( M(x) = M(\alpha, \beta; x) \). Let \( \Omega = [0, 1]^d \) and \( H^2(\Omega) = \{ f \in H^2(\Omega) \text{ such that } f = Y \text{ periodic } \} \) with the norm \( \| \cdot \|_{H^2(\Omega)} \) and \( E_{ij}, 1 \leq i, j \leq d \) are Maxwell matrices defined as

\[ E_{ij} = \begin{cases} 1, & i = j = k \\ 0, & \text{otherwise} \end{cases} \]

Then \( \{M\} \) H-converges to a constant tensor \( M(\alpha, \beta; \Omega) \) defined as

\[ m_{ij} = \int_{\Omega} E_{ij}(y) d\Omega \]

where \( [m_{ij}]_{d \times d} \) is the family of unique solutions in \( H^2(\Omega); \mathbb{R} \) of boundary value problems

\[ \text{div} (M(y)E_{ij} + \nabla m_{ij}(y)) = 0 \text{ in } \Omega, i, j = 1, \ldots, d \]

\[ y = y_0(x) \]

Theorem 7

Let \( A_n \in C(\text{Sym}) \) be a constant coercive tensor, \( Y = [0, 1]^d \), \( B_n(y) : = B_n(y, \Omega) \in H^\infty(\Omega) \), \( y \in \Omega \), where \( \Omega \subseteq \mathbb{R} \) is a bounded, open set. Additionally, let \( B \) be a Y-periodic, L^2 tensor function, for which we assume that

\[ \int_{\Omega} B(y) d\Omega = 0, \ y \in P \text{ where } P \subseteq \mathbb{R} \text{ is an open set, and} \]

\[ A_n(y) = A + \varepsilon B_n(y) \]

Then

\[ A_n(y) = A + \varepsilon B_n(y) \]

H-converges to a tensor

\[ A_n = A + \varepsilon B_n + \varepsilon^2 C_n + o(\varepsilon^2) \]

with coefficients \( C_n = 0 \) and

\[ \begin{cases} C_{k,k} = 0 \\ C_{k,k} = (2\pi)^d \sum_{l \neq l} \varepsilon^2 [A_{kl} + \varepsilon B_{kl}] \cdot E_{kl} \end{cases} \]

where \( m, n, s, t \in \{1, 2, \ldots, d\} \)

\[ \text{d} \]