Homogenization of elastic plate equation

Jelena Jankov

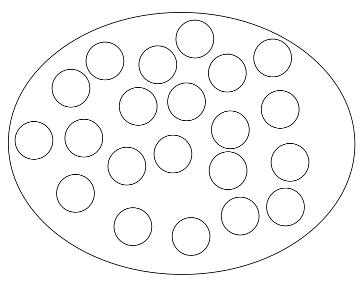
J. J. Strossmayer University of Osijek/Department of Mathematics/Joint work with K. Burazin and M. Vrdoljak

WeConMApp

Homogenization theory

The physical idea of homogenization is to average a heterogeneous media in order to derive effective properties.

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} \end{cases}$$



The mathematical theory of homogenization:
we consider a sequence of problems

$$\begin{cases} A_n u_n = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition}\,. \end{cases}$$

If $u_n \to u$, $A_n \to A$ the limit (effective) problem is

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition}\ \ldots \end{cases}$$

The mathematical problem is to determine an adequate topologies for these convergences.

Elastic plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \mathrm{divdiv}(\mathbf{M}\nabla\nabla \mathrm{u}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega) \end{cases}.$$

- $\Omega \subseteq \mathbb{R}^2$ bounded domain
- $f \in H^{-2}(\Omega)$ external load
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{M} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\ \mathbf{M}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S}$ and $\mathbf{M}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S}$ a. e. $\mathbf{x}\}$ describes elastic properties of the given plate
- u transversal displacement of the plate

Antonić, Balenović, 1999:

Definition 1 *A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems*

$$\begin{cases} \mathrm{divdiv}\,(\mathbf{M}^n\nabla\nabla \mathrm{u_n}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

coverges weakly to a limit u in $H_0^2(\Omega)$, while the sequence $(\mathbf{M}^n\nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.

Theorem 1 *Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.*

Properties of H-convergence and corrector result

Theorem 2 (Irrelevance of boundary conditions) *Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $\mathbf{M}$. For any sequence (z_n) such that*

$$\begin{aligned} z_n &\rightharpoonup z \quad \text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \\ \mathsf{div\,div}\,(\mathbf{M}^n\nabla\nabla z_n) = f_n &\to f \quad \text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega), \end{aligned}$$

the weak convergence $\mathbf{M}^n\nabla\nabla z_n \rightharpoonup \mathbf{M}\nabla\nabla z$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$ holds.

Theorem 3 *Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that either converges strongly to a limit tensor $\mathbf{M}$ in $L^1(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, or converges to $\mathbf{M}$ almost everywhere in Ω. Then $\mathbf{M}^n$ H-converges to $\mathbf{M}$.*

Theorem 4 *Let $F = \{f_n : n \in \mathbf{N}\}$ be a dense countable family in $H^{-2}(\Omega)$, $\mathbf{M}$ and $\mathbf{O}$ tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, and (u_n), (v_n) sequences of solutions to*

$$\begin{cases} \mathrm{divdiv}(\mathbf{M}\nabla\nabla \mathrm{u_n}) = \mathrm{f_n} \\ u_n \in H_0^2(\Omega) \end{cases}$$

and

$$\begin{cases} \mathrm{divdiv}(\mathbf{O}\nabla\nabla \mathrm{v_n}) = \mathrm{f_n} \\ v_n \in H_0^2(\Omega) \end{cases},$$

respectively. Then,

$$d(\mathbf{M}, \mathbf{O}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{L^2(\Omega)} + \|\mathbf{M}\nabla\nabla u_n - \mathbf{O}\nabla\nabla v_n\|_{H^{-1}(\Omega;\mathrm{Sym})}}{\|f_n\|_{H^{-2}(\Omega)}}$$

is a metric function on $\mathfrak{M}_2(\alpha, \beta; \Omega)$ and H-convergence is equivalent to the convergence with respect to d.

Definition 2 *Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$. Let $(w_n^{ij})_{1\leq i,j\leq N}$ be a family of test functions satisfying*

$$w_n^{ij} \rightharpoonup \frac{1}{2}x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)$$

$$\mathrm{divdiv}(\mathbf{M}^{\mathrm{n}}\nabla\nabla \mathrm{w_n^{ij}}) \to \cdot \quad \text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$$

$$\mathbf{M}^n\nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym}).$$

The tensor $\mathbf{W}^n$ defined as $[a_{ijkm}]_{ij} = [\nabla\nabla w_n^{km}]_{ij}$ is called a corrector tensor.

Theorem 5 (Corrector result) *Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ which H-converges to $\mathbf{M}$. For $f \in H^{-2}(\Omega)$, let (u_n) be the solution of*

$$\begin{cases} \mathrm{divdiv}(\mathbf{M}^{\mathrm{n}}\nabla\nabla \mathrm{u_n}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,. \end{cases}$$

Let u be the weak limit of (u_n) in $H_0^2(\Omega)$, i. e., the solution of the homogenized equation

$$\begin{cases} \mathrm{divdiv}(\mathbf{M}\nabla\nabla \mathrm{u}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Then, $r_n := \nabla\nabla u_n - \mathbf{W}^n\nabla\nabla u \to 0$ strongly in $L^1_{\mathrm{loc}}(\Omega; Sym)$.

Small-amplitude homogenization

Theorem 6 *Let $(\mathbf{M}^n)$ be a sequence of tensors defined by $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), x \in \Omega$, $Y = [0,1]^d$, $H^2_\#(Y) := \{f \in H^2_{\mathrm{loc}}(\mathbf{R}^d)\,\text{such that f is Y} - \text{periodic}\}$ with the norm $\|\cdot\|_{H^2(Y)}$ and $\mathbf{E}_{ij}, 1 \leq i,j \leq d$ are $M_{d\times d}$ matrices defined as*

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \textit{if } i = j = k = l \\ \frac{1}{2}, & \textit{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \textit{otherwise.} \end{cases}$$

Then $(\mathbf{M}^n)$ H-converges to a constant tensor $\mathbf{M}^* \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ defined as

$$m^*_{klij} = \int_Y \mathbf{M}(\mathbf{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{y})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\mathbf{y}))\,d\mathbf{y},$$

where $(w_{ij})_{1\leq i,j\leq d}$ is the family of unique solutions in $H^2_\#(Y)/\mathbf{R}$ of boundary value problems

$$\begin{cases} \mathsf{div\,div}\,(\mathbf{M}(\mathbf{y})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{y}))) = 0 \text{ in Y}, \mathrm{i,j} = 1, \ldots, \mathrm{d} \\ \mathbf{y} \to w_{ij}(\mathbf{y}). \end{cases}$$

Theorem 7 *Let $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym}; \mathrm{Sym})$ be a constant coercive tensor, $Y = [0,1]^d$, $\mathbf{B}^n(y) := \mathbf{B}(ny)$, $y \in \Omega$, where $\Omega \subseteq \mathbf{R}^d$ is a bounded, open set. Additionally, let $\mathbf{B}$ be a Y-periodic, L^∞ tensor function, for which we assume that $\int_Y \mathbf{B}(y)\,dy = 0$, $p \in P$ where $P \subseteq \mathbf{R}$ is an open set, and*

$$\mathbf{A}_p^n(\mathbf{y}) = \mathbf{A}_0 + p\mathbf{B}^n(\mathbf{y}).$$

Then

$$\mathbf{A}_p^n(\mathbf{y}) := \mathbf{A}_0 + p\mathbf{B}^n(y)$$

H-converges to a tensor

$$\mathbf{A}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2)$$

with coefficients $\mathbf{B}_0 = 0$ and

$$\begin{aligned} \mathbf{C}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} = (2\pi i)^2 &\sum_{\mathbf{k}\in J} a^{mn}_{-\mathbf{k}}\mathbf{B}_{\mathbf{k}}(\mathbf{k}\otimes\mathbf{k}) : \mathbf{E}_{rs} + \\ &+ (2\pi i)^4 \sum_{\mathbf{k}\in J} a^{mn}_{\mathbf{k}}\mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : a^{rs}_{-\mathbf{k}}\mathbf{k}\otimes\mathbf{k} + \\ &+ (2\pi i)^2 \sum_{\mathbf{k}\in J} \mathbf{B}_{\mathbf{k}}\mathbf{E}_{mn} : a^{rs}_{-\mathbf{k}}\mathbf{k}\otimes\mathbf{k}\,, \end{aligned} \tag{1}$$

where $m, n, r, s \in \{1, 2, \cdots, d\}$,

$$a^{mn}_{\mathbf{k}} = -\frac{\mathbf{B}_{\mathbf{k}}\mathbf{E}_{mn}\mathbf{k}\cdot\mathbf{k}}{(2\pi i)^2\mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k} \in J, \quad m, n \in \{1, 2, \cdots, d\}$$

and $\mathbf{B}_k$, $k \in J$, are Fourier coefficients of functions w_1^{mn} and $\mathbf{B}$, respectively.