# Homogenisation of elastic plate equation


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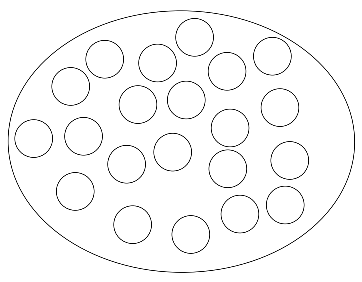
## Homogenisation theory

The physical idea of homogenisation is to average a heterogeneous media in order to derive effective properties.

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} \end{cases}$$



The mathematical theory of homogenisation:
we consider a sequence of problems

$$\begin{cases} A_n u_n = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} \,. \end{cases}$$

If $u_n \to u$, $A_n \to A$ the limit (effective) problem is

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} \ldots \end{cases}$$

The mathematical problem is to determine an adequate topologies for these convergences.

## Elastic plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \mathrm{divdiv}(\mathrm{M}\nabla\nabla \mathrm{u}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega) \end{cases}$$

- $\Omega \subseteq \mathbb{R}^2$ bounded domain
- $f \in H^{-2}(\Omega)$ external load
- $M \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{M \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathrm{S} \in \mathrm{Sym})\, \mathrm{M(x)S : S} \geq \alpha \mathrm{S : S}$ and $\mathrm{M^{-1}(x)S : S} \geq \frac{1}{\beta}\mathrm{S : S\, a.e.\, x}\}$ describes elastic properties of the given plate
- $u$ transversal displacement of the plate

Antonić, Balenović, 1999:

**Definition 1** *A sequence of tensor functions $(M^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $M \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions $(u_n)$ of problems*

$$\begin{cases} \mathrm{divdiv}\,(\mathrm{M^n}\nabla\nabla \mathrm{u_n}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

*coverges weakly to a limit $u$ in $H_0^2(\Omega)$, while the sequence $(M^n\nabla\nabla u_n)$ converges to $M\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.*

**Theorem 1** *Let $(M^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(M^{n_k})$ and a tensor function $M \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(M^{n_k})$ H-converges to $M$.*

## Properties of H-convergence

**Theorem 2 (Locality of the H-convergence)** *Let $(M^n)$ and $(O^n)$ be two sequences of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, which H-converge to $M$ and $O$, respectively. Let $\omega$ be an open subset compactly embedded in $\Omega$. If $M^n(x) = O^n(x)$ in $\omega$, then $M(x) = O(x)$ in $\omega$.*

**Theorem 3 (Irrelevance of the boundary condition)** *Let $(M^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $M$. For any sequence $(z_n)$ such that*

$$\begin{cases} \mathrm{divdiv}(\mathrm{M^n}\nabla\nabla \mathrm{z_n}) = \mathrm{f} \quad \text{in} \quad \Omega \\ z_n \rightharpoonup z \text{ in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \end{cases}$$

*$M^n$ satisfies $M^n\nabla\nabla z_n \rightharpoonup M\nabla\nabla z$ in $L^2_{\mathrm{loc}}(\Omega; Sym)$.*

**Theorem 4 (Energy convergence)** *Let $(M^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $M$. For any $f \in H^{-2}(\Omega)$, the sequence $(u_n)$ of solutions of*

$$\begin{cases} \mathrm{divdiv}(\mathrm{M^n}\nabla\nabla \mathrm{u_n}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

*satisfies*

$$M^n\nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup M\nabla\nabla u : \nabla\nabla u$$

*in $M_b(\Omega)$ and*

$$\int_\Omega M^n\nabla\nabla u_n : \nabla\nabla u_n \, dx \to \int_\Omega M\nabla\nabla u : \nabla\nabla u \, dx,$$

*where $u$ is the solution of the homogenised equation*

$$\begin{cases} \mathrm{divdiv}(\mathrm{M}\nabla\nabla \mathrm{u}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega) \,. \end{cases}$$

**Theorem 5 (Ordering property)** *Let $(M^n)$ and $(O^n)$ be two sequences of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converge to homogenised tensors $M$ and $O$, respectively. Assume that, for any $n$,*
*$M^n\xi : \xi \leq O^n\xi : \xi, \quad \forall \xi \in \mathrm{Sym}$.*
*Then the homogenised limits are also ordered:*
*$M\xi : \xi \leq O\xi : \xi, \quad \forall \xi \in \mathrm{Sym}$.*

**Theorem 6** *Let $(M^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that either converges strongly to a limit tensor $M$ in $L^1(\Omega; L(\mathrm{Sym}, \mathrm{Sym}))$, or converges to $M$ almost everywhere in $\Omega$. Then $M^n$ H-converges to $M$.*

## Corrector results

**Definition 2** *Let $(M^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $M$. Let $(w_n^{ij})_{1\leq i,j\leq N}$ be a family of test functions satisfying*

$$w_n^{ij} \rightharpoonup \frac{1}{2}x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)$$

$$\mathrm{divdiv}(\mathrm{M^n}\nabla\nabla \mathrm{w_n^{ij}}) \to \cdot \quad \text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$$

$$M^n\nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym}).$$

*The tensor $W^n$ defined as $[a_{ijkm}]_{ij} = [\nabla\nabla w_n^{km}]_{ij}$ is called a corrector tensor.*

**Theorem 7** *Let $(M^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $M$. A sequence of correctors $(W^n)$ is unique in the sense that, for any two sequences of correctors $(W^n)$ and $(\tilde{W}^n)$, their difference $(W^n - \tilde{W}^n)$ converges strongly to zero in $L^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.*

**Theorem 8 (Corrector result)** *Let $(M^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ which H-converges to $M$. For $f \in H^{-2}(\Omega)$, let $(u_n)$ be the solution of*

$$\begin{cases} \mathrm{divdiv}(\mathrm{M^n}\nabla\nabla \mathrm{u_n}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \,. \end{cases}$$

*Let $u$ be the weak limit of $(u_n)$ in $H_0^2(\Omega)$, i. e., the solution of the homogenised equation*

$$\begin{cases} \mathrm{divdiv}(\mathrm{M}\nabla\nabla \mathrm{u}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega) \,. \end{cases}$$

*Then, $r_n := \nabla\nabla u_n - W^n\nabla\nabla u \to 0$ strongly in $L^1_{\mathrm{loc}}(\Omega; Sym)$.*

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