Homogenisation of elastic plate equation

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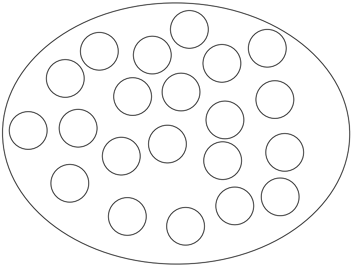
The physical idea of homogenisation is to average a heterogeneous media in order to derive effective properties.

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} \end{cases}$$



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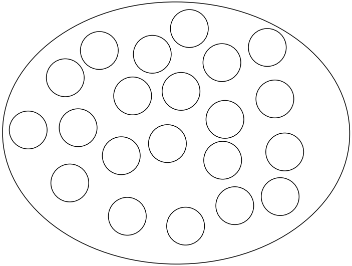

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Elastic plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}\operatorname{div}(M^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega) \end{cases}$$

- $\Omega \subseteq \mathbb{R}^2$ bounded domain
- $f \in H^{-2}(\Omega)$ external load
- $M \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{M \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall S \in \mathrm{Sym})\, M(x)S : S \geq \alpha S : S \text{ and } M^{-1}S : S \geq \frac{1}{\beta} S : S\, a.e.x\}$ describes properties of material of the given plate
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Antonić, Balenović, 1999.

Definition

A sequence of tensor functions (M^n) in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $M \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

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coverges weakly to a limit u in $H_0^2(\Omega)$, while the sequence $(M^n \nabla\nabla u_n)$ converges to $M \nabla\nabla u$ weakly in the space $L^2(\Omega, \mathrm{Sym})$.

Theorem

Let (M^n) be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence (M^{n_k}) and a tensor function $M \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that (M^{n_k}) H-converges to M.

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Theorem (Locality of the H-convergence)

Let (M^n) *and* (O^n) *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, which H-converge to* M *and* O*, respectively. Let* ω *be an open subset compactly embedded in* Ω*. If* $M^n(x) = O^n(x)$ *in* ω*, then* $M(x) = O(x)$ *in* ω*.*

Theorem (Irrelevance of the boundary condition)

Let (M^n) *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to M. For any sequence* (z_n) *such that*

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M^n *satisfies* $M^n\nabla\nabla z_n \rightharpoonup M\nabla\nabla z$ *in* $L^2_{\mathrm{loc}}(\Omega; Sym)$*.*

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Theorem (Energy convergence)

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to M. For any $f \in H^{-2}(\Omega)$, the sequence (u_n) of solutions of

$$\begin{cases} \mathrm{divdiv}(\mathrm{M^n \nabla\nabla u_n}) = \mathrm{f} \quad \mathrm{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,. \end{cases}$$

satisfies $M^n \nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup M\nabla\nabla u : \nabla\nabla u$ in $M_b(\Omega)$ and $\int_\Omega M^n \nabla\nabla u_n : \nabla\nabla u_n \, dx \to \int_\Omega M\nabla\nabla u : \nabla\nabla u \, dx$, where u is the solution of the homogenized equation

$$\begin{cases} \mathrm{divdiv}(\mathrm{M \nabla\nabla u}) = \mathrm{f} \quad \mathrm{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Theorem (Ordering property)

Let (M^n) *and* (O^n) *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converge to the homogenized tensors* M *and* O*, respectively. Assume that, for any* n*,*

$$M^n\xi : \xi \leq O^n\xi : \xi, \quad \forall \xi \in \mathrm{Sym}.$$

Then the homogenized limits are also ordered:

$$M\xi : \xi \leq O\xi : \xi, \quad \forall \xi \in \mathrm{Sym}.$$

Theorem

Let (M^n) *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that either converges strongly to a limit tensor M in* $L^1(\Omega; L(\mathrm{Sym}, \mathrm{Sym}))$*, or converges to M almost everywhere in* Ω*. Then,* M^n *also H-converges to M.*

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Let the following convergences be valid: $w_n \rightharpoonup w$ *in* $H^2_{\mathrm{loc}}(\Omega)$ *and* $D^n \rightharpoonup D$ *in* $L^2_{\mathrm{loc}}(\Omega; M_{2\times 2})$ *with an additional assumption that the sequence* $(\mathrm{divdivD^n})$ *is contained in a precompact (for the strong topology) set of the space* $H^{-2}_{\mathrm{loc}}(\Omega)$. *Then we have that* $E^n : D^n \rightharpoonup E : D$ *weakly-*$*$ *in the space of Radon measures, where we denote* $E^n := \nabla\nabla w^n$, *for* $n \in \mathbb{N} \cup \{\infty\}$.

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Definition

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit M. Let $(w_n^{ij})_{1 \le i,j \le N}$ be a family of test functions satisfying

$$w_n^{ij} \rightharpoonup \frac{1}{2} x_i x_j \quad \text{in} \quad H^2(\Omega)$$

$$\mathrm{divdiv}(M^n \nabla\nabla w_n^{ij}) \to \cdot \quad \text{in } H^{-2}_{\mathrm{loc}}(\Omega)$$

$$M^n \nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad L^2_{\mathrm{loc}}(\Omega; \mathrm{Sym}).$$

The tensor W^n defined as $[a_{ijkm}]_{ij} = [\nabla\nabla w_n^{km}]_{ij}$ is called a corrector tensor.

Theorem

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor M. A sequence of correctors (W^n) is unique in the sense that, if there exist two sequences of correctors (W^n) and $(\tilde{W}^n)$, their difference $(W^n - \tilde{W}^n)$ converges strongly to zero in $L^2_{loc}(\Omega; \mathcal{L}(Sym, Sym))$.

Theorem (Corrector result)

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ which H-converges to M. For $f \in H^{-2}(\Omega)$, let (u_n) be the solution of

$$\begin{cases} \mathrm{divdiv}(\mathrm{M^n \nabla\nabla u_n}) = \mathrm{f} \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,. \end{cases}$$

Let u be the weak limit of (u_n) in $H_0^2(\Omega)$, i.e., the solution of the homogenized equation

$$\begin{cases} divdiv(M\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Then, $r_n := \nabla\nabla u_n - W^n \nabla\nabla u \to 0$ strongly in $L^1_{loc}(\Omega; Sym)$.

Thank you for your attention!