Homogenization of Kirchhoff plate equation

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The physical idea of homogenization is to average a heterogeneous media in order to derive effective properties.

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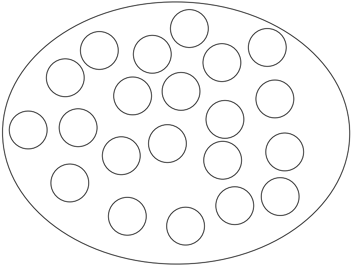

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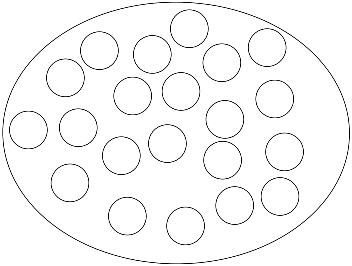

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Kirchhoff plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

- $\Omega \subseteq \mathbb{R}^2$ bounded domain
- $f \in H^{-2}(\Omega)$ external load
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{N} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\, \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \text{ and } \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S} \text{ a.e.}\mathbf{x}\}$ describes properties of material of the given plate
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Antonić, Balenović, 1999.

Definition

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

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coverges weakly to a limit u in $H^2_0(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.

Theorem

Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

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Theorem (Locality of the H-convergence)

Let $(\mathbf{M}^n)$ and $(\mathbf{O}^n)$ be two sequences of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, which H-converge to $\mathbf{M}$ and $\mathbf{O}$, respectively. Let ω be an open subset compactly embedded in Ω. If $\mathbf{M}^n(\mathbf{x}) = \mathbf{O}^n(\mathbf{x})$ in ω, then $\mathbf{M}(\mathbf{x}) = \mathbf{O}(\mathbf{x})$ in ω.

Theorem (Irrelevance of boundary conditions)

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $\mathbf{M}$. For any sequence (z_n) such that

$$\begin{aligned} z_n &\longrightarrow z \quad \text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \\ \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla z_n) = f_n &\longrightarrow f \quad \text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega), \end{aligned}$$

the weak convergence $\mathbf{M}^n \nabla\nabla z_n \rightharpoonup \mathbf{M}\nabla\nabla z$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$ holds.

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Theorem (Energy convergence)

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to $\mathbf{M}$. For any $f \in H^{-2}(\Omega)$, the sequence (u_n) of solutions of

$$\left\{ \begin{array}{l} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \,. \end{array} \right.$$

satisfies $\mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup \mathbf{M}\nabla\nabla u : \nabla\nabla u$ weakly-$*$ in the space of Radon measures and $\int_\Omega \mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} \to \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla u \, d\mathbf{x}$, where u is the solution of the homogenized equation

$$\left\{ \begin{array}{l} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega) \,. \end{array} \right.$$

Theorem (Ordering property)

Let $(\mathbf{M}^n)$ and $(\mathbf{O}^n)$ be two sequences of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converge to the homogenized tensors $\mathbf{M}$ and $\mathbf{O}$, respectively. Assume that, for any n,

$$\mathbf{M}^n \xi : \xi \leq \mathbf{O}^n \xi : \xi, \quad \forall \xi \in \mathrm{Sym}.$$

Then the homogenized limits are also ordered:

$$\mathbf{M} \xi : \xi \leq \mathbf{O} \xi : \xi, \quad \forall \xi \in \mathrm{Sym}.$$

Theorem

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that either converges strongly to a limit tensor $\mathbf{M}$ in $\mathrm{L}^1(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, or converges to $\mathbf{M}$ almost everywhere in Ω. Then, $\mathbf{M}^n$ also H-converges to $\mathbf{M}$.

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Theorem

Let $F = \{f_n : n \in \mathbf{N}\}$ be a dense countable family in $H^{-2}(\Omega)$, **M** *and* **O** *tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, and (u_n), (v_n) sequences of solutions to*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u_n) = f_n \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

and

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{O}\nabla\nabla v_n) = f_n \\ v_n \in \mathrm{H}^2_0(\Omega) \end{cases},$$

respectively. Then,

$$d(\mathbf{M}, \mathbf{O}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{M}\nabla\nabla u_n - \mathbf{O}\nabla\nabla v_n\|_{H^{-1}(\Omega;\mathrm{Sym})}}{\|f_n\|_{H^{-2}(\Omega)}}$$

is a metric function on $\mathfrak{M}_2(\alpha, \beta; \Omega)$ and H-convergence is equivalent to the convergence with respect to d.

Theorem (Compactness by compensation result)

Let the following convergences be valid:

$$w^n \rightharpoonup w^\infty \quad \text{in} \quad \mathrm{H}^2_{\mathrm{loc}}(\Omega)\,,$$
$$\mathbf{D}^n \rightharpoonup \mathbf{D}^\infty \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,,$$

with an additional assumption that the sequence $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ *is contained in a precompact (for the strong topology) set of the space* $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$. *Then we have*

$$\mathbf{E}^n : \mathbf{D}^n \xrightharpoonup{*} \mathbf{E}^\infty : \mathbf{D}^\infty$$

in the space of Radon measures, where we denote $\mathbf{E}^n := \nabla\nabla w^n$, *for* $n \in \mathbf{N} \cup \{\infty\}$.

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Definition

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$. Let $(w_n^{ij})_{1 \leq i,j \leq N}$ be a family of test functions satisfying

$$w_n^{ij} \rightharpoonup \frac{1}{2} x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)$$

$$\operatorname{div} \operatorname{div} (\mathbf{M}^n \nabla\nabla w_n^{ij}) \to \cdot \quad \text{in}\, \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$$

$$\mathbf{M}^n \nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym}).$$

The tensor $\mathbf{W}^n$ defined as $[a_{ijkm}]_{ij} = [\nabla\nabla w_n^{km}]_{ij}$ is called a corrector tensor.

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to a tensor* $\mathbf{M}$*. A sequence of correctors* $(\mathbf{W}^n)$ *is unique in the sense that, if there exist two sequences of correctors* $(\mathbf{W}^n)$ *and* $(\tilde{\mathbf{W}}^n)$*, their difference* $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ *converges strongly to zero in* $L^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Theorem (Corrector result)

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ which H-converges to $\mathbf{M}$. For $f \in H^{-2}(\Omega)$, let (u_n) be the solution of

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,. \end{cases}$$

Let u be the weak limit of (u_n) in $H_0^2(\Omega)$, i.e., the solution of the homogenized equation

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Then, $r_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u \to 0$ strongly in $L^1_{\mathrm{loc}}(\Omega; Sym)$.

Small-amplitude homogenization

$$\mathbf{A}_p^n(\mathbf{y}) := \mathbf{A}_0 + p\mathbf{B}^n(\mathbf{y}),\ p \in \mathbf{R}$$

$$\mathbf{A}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2),\ p \in \mathbf{R}$$

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If $p \mapsto \mathbf{A}_n^p$ is a C^k mapping (for any $n \in \mathbf{N}$) from some subset of $\mathbf{R}$ to $L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, what can we say about $p \mapsto \mathbf{A}_p$?

Theorem

Let $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*, where* $P \subseteq \mathbf{R}$ *is an open set. Assume that (for some* $k \in \mathbf{N}_0$*) a mapping* $p \mapsto \mathbf{M}^n(\cdot, p)$ *is of class* C^k *from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives which are equicontinuous on every compact set* $K \subseteq P$ *up to order k:*

$$(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})$$
$$(\forall i \in \{0, \dots, k\})$$
$$|p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))} < \varepsilon.$$

Then there is a subsequence $(\mathbf{M}^{n_k})$ such that for every $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightharpoonup{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

and $p \mapsto \mathbf{M}(\cdot, p)$ is a C^k mapping from P to $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Periodic case

- $Y = [0,1]^d$
- $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), x \in \Omega$
- $H^2_{\#}(Y) := \{f \in H^2_{\text{loc}}(\mathbf{R}^d) \, \text{such that f is Y} - \text{periodic}\}$ with the norm $\|\cdot\|_{H^2(Y)}$
- $H^2_{\#}(Y)/\mathbf{R}$ equipped with the norm $\|\nabla\nabla\cdot\|_{L^2(Y)}$
- $E_{ij}, 1 \leq i,j \leq d$ are $M_{d\times d}$ matrices defined as

$$[E_{ij}]_{kl} = \begin{cases} 1, & \text{if } i=j=k=l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors defined by* $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), x \in \Omega$. *Then* $(\mathbf{M}^n)$ *H-converges to a constant tensor* $\mathbf{M}^* \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *defined as*

$$m^*_{klij} = \int_Y \mathbf{M}(\mathbf{y})(E_{ij} + \nabla\nabla w_{ij}(\mathbf{y})) : (E_{kl} + \nabla\nabla w_{kl}(\mathbf{y}))\, d\mathbf{y},$$

where $(w_{ij})_{1\le i,j\le d}$ *is the family of unique solutions in* $H^2_\#(Y)/\mathbf{R}$ *of boundary value problems*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}(\mathbf{y})(E_{ij} + \nabla\nabla w_{ij}(\mathbf{y}))) = 0 \text{ in } \mathrm{Y}, \mathrm{i}, \mathrm{j} = 1, \dots, \mathrm{d} \\ \mathbf{y} \to w_{ij}(\mathbf{y}). \end{cases}$$

Theorem

Let $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym};\mathrm{Sym})$ *be a constant coercive tensor,* $\mathbf{B}^n(y) := \mathbf{B}(ny)$, $y \in \Omega$, *where* $\Omega \subseteq \mathbf{R}^d$ *is a bounded, open set.* $\mathbf{B}$ *is a Y-periodic,* L^∞ *tensor function, for which we assume that* $\int_Y \mathbf{B}(y)\,dy = 0$, $p \in P$ *where* $P \subseteq \mathbf{R}$ *is an open set, and*

$$\mathbf{A}_p^n(\mathbf{y}) = \mathbf{A}_0 + p\mathbf{B}^n(\mathbf{y}).$$

Then

$$\mathbf{A}_p^n(\mathbf{y}) := \mathbf{A}_0 + p\mathbf{B}^n(y)$$

H-converges to a tensor

$$\mathbf{A}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2)$$

with coefficients $\mathbf{B}_0 = 0$ and

$$\begin{aligned}\mathbf{C}_0 E_{mn} : E_{rs} =& (2\pi i)^2 \int_Y \sum_{k\in J} a_{-k}^{mn}\mathbf{B}_k(kk^T) : E_{rs}\, dy \\ &+ (2\pi i)^4 \int_Y \sum_{k\in J} a_k^{mn}\mathbf{A}_0(kk^T) : a_{-k}^{rs} kk^T\, dy \\ &+ (2\pi i)^2 \int_Y \sum_{k\in J} \mathbf{B}_k E_{mn} : a_{-k}^{rs} kk^T\, dy,\end{aligned}$$

where $m, n, r, s \in \{1, 2, \cdots, d\}$, $J := \mathbf{Z}^d/\{0\}$,

$$a_k^{mn} = -\frac{\mathbf{B}_k E_{mn} k \cdot k}{(2\pi i)^2\,(\mathbf{A}_0(k\cdot k^T)k)\cdot k}, \quad k \in J,$$

and B_k, $k \in J$, are Fourier coefficients of functions w_1^{mn} and $\mathbf{B}$, respectively.

Thank you for your attention!