Fractional H-measures and transport property

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Torino, February 1, 2017.

Conference in memory of Todor V. Gramchev

WeConMApp

Classical H-measures

H-measures were introduced independently by Luc Tartar and Patrick Gérard in the late 1980s and their existence is established by the following theorem.

Theorem 1. *If* (u_n) *is a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$ *such that* $\mathrm{u}_n \rightharpoonup 0$*, then there exist a subsequence* $(\mathrm{u}_{n'})$ *and an* $r \times r$ *Hermitian complex matrix Radon measure* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *one has:*

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$$\begin{aligned}\lim_{n'}\int_{\mathbf{R}^d}(\varphi_1\mathsf{u}_{n'})\otimes\mathcal{A}_\psi(\varphi_2\mathsf{u}_{n'})\,d\mathbf{x} &= \langle\boldsymbol{\mu},(\varphi_1\overline{\varphi_2})\boxtimes\overline{\psi}\rangle\\ &= \int_{\mathbf{R}^d\times\mathrm{S}^{d-1}}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})}\,d\boldsymbol{\mu}(\mathbf{x},\boldsymbol{\xi})\,,\end{aligned}$$

where $\mathcal{F}(\mathcal{A}_\psi v)(\boldsymbol{\xi})=\psi(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})\mathcal{F}v(\boldsymbol{\xi})$. ∎

First commutation lemma

The crucial step in Tartar's construction of H-measures is the result called *the First commutation lemma* . More precisely, for $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ we define *the Fourier multiplier operator* by

$$P_\psi : \mathrm{L}^2(\mathbf{R}^d) \longrightarrow \mathrm{L}^2(\mathbf{R}^d)\,, \quad P_\psi u := (\psi \hat{u})^\vee\,,$$

and the operator of multiplication by $\phi \in \mathrm{L}^\infty(\mathbf{R}^d)$ by

$$M_\phi : \mathrm{L}^2(\mathbf{R}^d) \longrightarrow \mathrm{L}^2(\mathbf{R}^d)\,, \quad M_\phi u := \phi u\,.$$

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is also bounded on $\mathrm{L}^2(\mathbf{R}^d)$. Furthermore, Tartar proved that if we take ψ to be homogeneous of order zero and continuous (except at the origin), while $\phi \in \mathrm{C}_0(\mathbf{R}^d)$, then K is compact on $\mathrm{L}^2(\mathbf{R}^d)$.

Theorem 2. *Let Q be an ellipsoid*

$$\frac{\xi_1^2}{\alpha_1} + \frac{\xi_2^2}{\alpha_2} + \cdots + \frac{\xi_d^2}{\alpha_d} = \frac{1}{\alpha_{\min}},$$

and for each $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d) \in Q$ we define

$$\varphi_{\boldsymbol{\eta}}(s) = \mathrm{diag}\,\{s^{\frac{1}{\alpha_1}}, \ldots, s^{\frac{1}{\alpha_d}}\}\boldsymbol{\eta},$$

where $\alpha_k \in \langle 0, 1]$. Also, π_Q is a projection on Q along $\varphi_{\boldsymbol{\eta}}$.

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where $\alpha_k \in \langle 0, 1]$. Also, π_Q is a projection on Q along $\varphi_{\boldsymbol{\eta}}$.
If $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$, then there exist a subsequence $(\mathsf{u}_{n'})$ and a Hermitian matrix Radon measure $\boldsymbol{\mu} = \{\mu^{ij}\}_{i,j=1,\ldots,r}$ on $\mathbf{R}^d \times Q$ so that for $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$, $\psi \in \mathrm{C}(Q)$, and $i, j = 1, \ldots, r$:

$$\lim_{n' \to \infty} \int_{\mathbf{R}^d} (\varphi_1 u^i_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\psi \circ \pi_Q}(\varphi_2 u^j_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \mu^{ij}, \varphi_1 \overline{\varphi_2 \psi} \rangle$$

$$= \int_{\mathbf{R}^d \times Q} \varphi_1(\mathbf{x}) \overline{\varphi_2(\mathbf{x}) \psi(\boldsymbol{\xi})}\, d\mu^{ij}(\mathbf{x}, \boldsymbol{\xi}).$$

∎

Properties of projections

The projection is given by the formula

$$\pi_Q(\boldsymbol{\xi}) = \left(\frac{\xi_1}{s(\boldsymbol{\xi})^{\frac{1}{\alpha_1}}}, \dots, \frac{\xi_d}{s(\boldsymbol{\xi})^{\frac{1}{\alpha_d}}} \right),$$

where $s(\boldsymbol{\xi})$ is the positive solution of the equation

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$i)$ $s \in \mathrm{C}^\infty(\mathbf{R}^d \setminus \{0\}; \mathbf{R}^+)$ and $s \in \mathrm{C}(\mathbf{R}^d)$ with $s(0) = 0$,

$ii)$ $s\left(\lambda^{\frac{1}{\alpha_1}} \xi_1, \dots, \lambda^{\frac{1}{\alpha_d}} \xi_d\right) = \lambda s(\boldsymbol{\xi}), \quad \lambda \in \mathbf{R}^+$,

$iii)$ $|\eta_k| \geqslant |\xi_k|,\ k = 1, \dots, d \quad \Longrightarrow \quad s(\boldsymbol{\eta}) \geqslant s(\boldsymbol{\xi})$,

$iv)$ $(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d) \quad C_1 \sum_{k=1}^d |\xi_k|^{\alpha_k} \leqslant s(\boldsymbol{\xi}) \leqslant C_2 \sum_{k=1}^d |\xi_k|^{\alpha_k}$,

$v)$ $d_s(\boldsymbol{\xi}, \boldsymbol{\eta}) := s(\boldsymbol{\xi} - \boldsymbol{\eta})$ defines a metric on $\mathbf{R}^d$.

Anisotropic Tartar spaces

For $m \in \mathbf{N}$ and $\boldsymbol{\alpha} \in \langle 0,1]^d$ we define

$$\mathrm{X}^{m\boldsymbol{\alpha}}(\mathbf{R}^d) := \{u \in \mathcal{S}' : k_{\boldsymbol{\alpha}}^m \hat{u} \in \mathrm{L}^1(\mathbf{R}^d)\},$$

where

$$k_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) := \Big(1 + \sum_{k=1}^d |\xi_k|^{\alpha_k}\Big).$$

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$\mathrm{X}^{m\boldsymbol{\alpha}}(\mathbf{R}^d)$ is a Banach space with the norm

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Assumption: $\alpha_1, \alpha_2, \dots, \alpha_m < 1$, and $\alpha_{m+1} = \cdots = \alpha_d = 1$

Notation: $\mathbf{x} = (\bar{\mathbf{x}}, \mathbf{x}')$, $\bar{\mathbf{x}} = (x_1, \dots, x_m)$, $\mathbf{x}' = (x_{m+1}, \dots, x_d)$, $0 \leqslant m \leqslant d$

Crutial properties

Lemma 1. *Let* $m \in \mathbf{N}$ *and* $\boldsymbol{\alpha} \in \langle 0,1]^d$. *For* $\phi \in \mathrm{X}^{m\boldsymbol{\alpha}}(\mathbf{R}^d)$ *we have*

$$(\forall\, \boldsymbol{\beta} \in [0,\infty\rangle^d) \quad \frac{\beta_1}{\alpha_1} + \ldots + \frac{\beta_d}{\alpha_d} \leqslant m \quad \Longrightarrow \quad \partial_{\boldsymbol{\beta}}\phi \in \mathrm{C}_0(\mathbf{R}^d)\,.$$

Moreover, the following estimate holds

$$\|\partial_{\boldsymbol{\beta}}\phi\|_{\mathrm{L}^\infty(\mathbf{R}^d)} \leqslant \|\widehat{\partial_{\boldsymbol{\beta}}\phi}\|_{\mathrm{L}^1(\mathbf{R}^d)} \leqslant (2\pi)^m \|\phi\|_{\mathrm{X}^{m\boldsymbol{\alpha}}(\mathbf{R}^d)}\,.$$

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Lemma 2. *Let* $m \in \mathbf{N}$ *and* $\boldsymbol{\alpha} \in \langle 0, 1]^d$*. If* $s > m + \frac{1}{2\alpha_1} + \ldots + \frac{1}{2\alpha_d}$*, then we have a continuous embedding*

$$\mathrm{H}^{s\boldsymbol{\alpha}}(\mathbf{R}^d) \hookrightarrow \mathrm{X}^{m\boldsymbol{\alpha}}(\mathbf{R}^d).$$

∎

Second commutation lemma

Theorem 3. *Let* P_ψ *and* M_ϕ *be a Fourier and pointwise multiplier operators on* $\mathrm{L}^2(\mathbf{R}^d)$ *defined by* $\mathcal{F}(P_\psi u) = \psi \mathcal{F} u, \quad M_\phi u = \phi u$*, with associated symbols* $\psi \in \mathrm{C}^1(P^d)$ *and* $\phi \in \mathrm{X}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$ *respectively.*

Second commutation lemma

Theorem 3. *Let P_ψ and M_ϕ be a Fourier and pointwise multiplier operators on $\mathrm{L}^2(\mathbf{R}^d)$ defined by $\mathcal{F}(P_\psi u) = \psi \mathcal{F} u, \quad M_\phi u = \phi u$, with associated symbols $\psi \in \mathrm{C}^1(P^d)$ and $\phi \in \mathrm{X}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$ respectively. Then for a commutator $K := [P_\psi, M_\phi] = P_\psi M_\phi - M_\phi P_\psi$ we have (up to a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$):*

$$\partial_j^{\alpha_j} K = P_{\frac{(2\pi i \xi_j)^{\alpha_j}}{2\pi i} \nabla^{\boldsymbol{\xi}'} \psi^Q} M_{\nabla^{\mathbf{x}'} \phi},$$

where $\psi^Q = \psi \circ \pi_Q$. ∎

The first step in the proof

I. *It is sufficient to consider* $\phi \in \mathcal{S}(\mathbf{R}^d)$ *such that* $\hat{\phi}$ *has compact support*

This is based on the fact that such functions are dense in $\mathrm{X}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$ and on the following continuity result.

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This is based on the fact that such functions are dense in $\mathrm{X}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$ and on the following continuity result.

Theorem 4. *For $\psi \in \mathrm{C}^1(Q)$ and $\phi \in \mathrm{X}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$ a commutator $K := [P_{\psi_Q}, M_\phi]$ is continuous from $\mathrm{L}^2(\mathbf{R}^d)$ to $\mathrm{H}^{\boldsymbol{\alpha}}(\mathbf{R}^d)$. Moreover, there exists $C > 0$ (depending on ψ) such that*

$$\|K\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{H}^{\boldsymbol{\alpha}}(\mathbf{R}^d))} \leqslant C\|\phi\|_{\mathrm{X}^{\boldsymbol{\alpha}}(\mathbf{R}^d)} .$$

∎

The second step in the proof

II. *Cutoff around the origin in the Fourier space*

By a simple application of the theory of Hilbert-Schmidt operators we can replace ψ^Q with $\tilde{\psi} = (1-\theta)\psi^Q$, where $\theta \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ is equal to 1 on a neighbourhood of the origin, and so it remains to prove that

$$\tilde{D} := \partial_j^{\alpha_j}[P_{\tilde{\psi}}, M_\phi] - P_{\frac{(2\pi i \xi_j)^{\alpha_j}}{2\pi i} \nabla_{\boldsymbol{\xi}'} \tilde{\psi}} M_{\nabla_{\mathbf{x}'}\phi}$$

is compact.

The third step in the proof

III. *The decomposition:* $\mathcal{F}\tilde{D} = A_m + B_m$, $m \in \mathbf{N}$

We decompose $\mathcal{F}\tilde{D} = A_m + B_m$, where

$$(A_m u)(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} \chi_{\mathcal{K}_m}(\boldsymbol{\xi}) \Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) \cdot \widehat{\nabla_{\mathbf{x}}\phi}(\boldsymbol{\xi} - \boldsymbol{\eta}) \hat{u}(\boldsymbol{\eta}) \, d\boldsymbol{\eta} \,,$$

$$(B_m u)(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} (1 - \chi_{\mathcal{K}_m}(\boldsymbol{\xi})) \Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) \cdot \widehat{\nabla_{\mathbf{x}}\phi}(\boldsymbol{\xi} - \boldsymbol{\eta}) \hat{u}(\boldsymbol{\eta}) \, d\boldsymbol{\eta} \,,$$

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$$\mathcal{K}_m := \{\boldsymbol{\xi} \in \mathbf{R}^d : s(\boldsymbol{\xi}) \leqslant m\} \,,$$

$$\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}) := \frac{(2\pi i \xi_j)^{\alpha_j}}{2\pi i} \begin{bmatrix} \nabla^{\bar{\boldsymbol{\xi}}} \tilde{\psi}(\boldsymbol{\zeta}) \\ \nabla^{\boldsymbol{\xi}'} \tilde{\psi}(\boldsymbol{\zeta}) - \nabla^{\boldsymbol{\xi}'} \tilde{\psi}(\boldsymbol{\xi}) \end{bmatrix} \,,$$

and then prove that $(\forall\, m \in \mathbf{N})$ A_m is compact, while $B_m \longrightarrow 0$.

An example application

We study sequence of equations

$$iu^n_t + (a(t,x)u^n_{xx})_{xx} = f^n,$$

where $a \in \mathrm{X}^{(\frac{1}{4},1)}(\mathbf{R}^2)$, $f \in \mathrm{L}^2(\mathbf{R}^2)$ and a is real.

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Using second commutation lemma and assumptions

$$u_n \longrightarrow 0 \text{ in } \mathrm{L}^2, \quad u_x^n \longrightarrow 0 \text{ in } \mathrm{L}^2$$

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$$f_n \rightharpoonup 0 \text{ in } \mathrm{L}^2, \quad u^n_{xx} \rightharpoonup 0 \text{ in } \mathrm{L}^2$$

we obtain

$$4\langle \mu, a\phi_x \boxtimes \psi\rangle - \langle \mu, a_x\phi \boxtimes (\psi + \xi(\psi^Q)_\xi)\rangle = 0,$$

where μ is a fractional $(\alpha_1 = \frac{1}{4}, \alpha_2 = 1)$ H-measure associated with the sequence (u^n_{xx}) and $\psi \in \mathrm{C}^1(Q)$, $\phi \in \mathrm{C}^1_c(\mathbf{R}^2)$.

Sketch of the proof

For $\psi \in \mathrm{C}^1(Q)$ and $\phi \in \mathrm{C}^1_c(\mathbf{R}^2)$ we apply operators P_ψ and M_ϕ on our equation, and then form a scalar product in $\mathrm{L}^2(\mathbf{R}^2)$ with u^n_x obtaining

$$\langle\, i\phi P_\psi u^n_t \mid u^n_x \,\rangle + \langle\, \phi P_\psi (a u^n_{xx})_{xx} \mid u^n_x \,\rangle = \langle\, \phi P_\psi f^n \mid u^n_x \,\rangle \,.$$

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The idea is to get $(a u^n_{xx})_{xx}$ in the second argument, by using integration by parts.

The second commutation lemma comes in the following calculation

$$\begin{aligned} \lim_n \langle\, \phi([P_\psi, M_a)] u^n_{xx})_x \mid u^n_{xx} \,\rangle &= \lim_n \langle\, \phi P_{\xi(\psi^Q)^\xi} (a_x u^n_{xx}) \mid u^n_{xx} \,\rangle \\ &= \langle \mu, a_x \phi \boxtimes \xi(\psi^Q)_\xi \rangle \,. \end{aligned}$$

Concluding remarks

After some manipulations we also obtain

$$\langle \mu, \{\Psi, W\} \rangle + \left\langle \mu, \Psi \frac{3\kappa^2(5-\kappa^2)}{16(\kappa^2-1)} \xi W_x \right\rangle = 0\,,$$

where $\Psi = \phi \boxtimes \psi^Q$, $W = 2\pi\tau - 16\pi^4\xi^4 a$ and $\kappa = (\tau_0^2 + \frac{\xi_0^2}{16})^{-\frac{1}{2}}$.

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Also, under assumption that μ is absolutely continuous with respect to the Lebesgue measure, we get

$$\partial_x \mu \left(\partial^\xi W - \left(\frac{\kappa^2}{16} + \frac{\kappa^2}{4} + \frac{3\kappa^2(5-\kappa^2)}{16(\kappa^2-1)} \right) \xi W \right) - \nabla^{\tau,\xi} \mu \cdot \left(\begin{bmatrix} 0 \\ \partial_x W \end{bmatrix} - \left(\begin{bmatrix} 0 \\ \partial_x W \end{bmatrix} \cdot \mathsf{n} \right) \mathsf{n} \right) = 0$$

which we call the propagation principle for μ associated to our equation.