Anisotropic distributions and applications

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H-measures[1,2]

Theorem 1. *If* $(u_n)_{n\in\mathbf{N}}$ *is a sequence in* $\mathrm{L}^2_{loc}(\Omega;\mathbf{R}^r)$, $\Omega\subset\mathbf{R}^{d+1}$, *such that* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{loc}(\Omega)$, *then there exists subsequence* $(u_{n'})_{n'}\subset(u_n)_n$ *and positive complex bounded measure* $\mu=\{\mu^{jk}\}_{j,k=1,\dots,r}$ *on* $\mathbf{R}^{d+1}\times\mathrm{S}^d$ *such that for all* $\varphi_1,\varphi_2\in\mathrm{C}_0(\Omega)$ *and* $\psi\in\mathrm{C}(\mathrm{S}^d)$,

$$\lim_{n'\to\infty}\int_\Omega(\varphi_1 u^j_{n'})(\boldsymbol{\xi})\overline{\mathcal{A}_{\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}(\varphi_2 u^k_{n'})(\boldsymbol{\xi})}d\mathbf{x}=\langle\mu^{jk},\varphi_1\bar{\varphi}_2\psi\rangle$$
$$=\int_{\mathbf{R}^{d+1}\times\mathrm{S}^d}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}\psi(\boldsymbol{\xi})d\mu^{jk}(\mathbf{x},\boldsymbol{\xi})$$

where $\mathcal{A}_{\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)}$ *is the multiplier operator with the symbol* $\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$. ∎

[1] L. Tartar, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations*, Proc. Roy. Soc. Edinburgh **115A** (1990) 193–230.

[2] P. Gérard, *Microlocal defect measures*, Comm. Partial Diff. Eq. **16** (1991) 1761–1794.

H-distributions[5]

Theorem 2. *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p_{\rm loc}(\mathbf{R}^d)$ *and* $v_n \stackrel{*}{\longrightarrow} v$ *in* $\mathrm{L}^q_{\rm loc}(\mathbf{R}^d)$ *for some* $p \in \langle 1, \infty\rangle$ *and* $q \geq p'$, *then there exist subsequences* $(u_{n'})$, $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$, *such that, for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, *for* $\kappa = [d/2]+1$, *one has:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x} = \lim_{n'\to\infty} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})}d\mathbf{x}$$
$$= \langle \mu, \varphi_1 \overline{\varphi}_2 \boxtimes \psi\rangle,$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ *is the Fourier multiplier operator with symbol* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$. ∎

Existing applications are related to the velocity averaging[3] and $\mathrm{L}^p - \mathrm{L}^q$ compactness by compensation[4].

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[3] M. Lazar, D. Mitrović, *On an extension of a bilinear functional on* $\mathrm{L}^p(\mathbf{R}^d) \times \mathrm{E}$ *to Bochner spaces with an application to velocity averaging*, C. R. Math. Acad. Sci. paris **351** (2013) 261–264.

[4] M. Mišur, D. Mitrović, *On a generalization of compensated compactness in the* $\mathrm{L}^p - \mathrm{L}^q$ *setting*, Journal of Functional Analysis **268** (2015) 1904–1927.

[5] N. Antonić, D. Mitrović, H*-distributions: An Extension of H-Measures to an* $\mathrm{L}^p - \mathrm{L}^q$ *Setting*, Abs. Appl. Analysis Volume 2011, Article ID 901084, 12 pages.

Anisotropic distributions

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds of dimenions d and r) and $\Omega \subseteq X \times Y$ an open set. By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \leq l$ and $|\boldsymbol{\beta}| \leq m$, $\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega)$.

$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}| \leq l, |\boldsymbol{\beta}| \leq m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} \ ,$$

where $K_n \subseteq \Omega$ are compacts, such that $\Omega = \cup_{n \in \mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int} K_{n+1}$, Consider the space

$$\mathrm{C}_c^{l,m}(\Omega) := \bigcup_{n \in \mathbf{N}} \mathrm{C}_{K_n}^{l,m}(\Omega) \ ,$$

and equip it by the topology of *strict inductive limit*.

Anisotropic distributions II

Definition. A *distribution of order l in $\mathbf{x}$ and order m in $\mathbf{y}$* is any linear functional on $\mathrm{C}_c^{l,m}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Conjecture. *Let X, Y be C^∞ manifolds and let u be a linear functional on $\mathrm{C}_c^{l,m}(X \times Y)$. If $u \in \mathcal{D}'(X \times Y)$ and satisfies*
$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y))(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(X))(\forall \psi \in \mathrm{C}_L^\infty(Y))$

$$|\langle u, \varphi \boxtimes \psi \rangle| \leq C p_K^l(\varphi) p_L^m(\psi),$$

then u can be uniquely extended to a continuous functional on $\mathrm{C}_c^{l,m}(X \times Y)$ (i.e. it can be considered as an element of $\mathcal{D}'_{l,m}(X \times Y)$). ■

The Schwartz kernel theorem[6]

Let X and Y be two C^∞ manifolds. Then the following statements hold:

Theorem 3. a) *Let $K \in \mathcal{D}'(X \times Y)$. Then for every $\varphi \in \mathcal{D}(X)$, the linear form K_φ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathcal{D}(X)$ to $\mathcal{D}'(Y)$ is linear and continuous.*

b) *Let $A : \mathcal{D}(X) \to \mathcal{D}'(Y)$ be a continous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$*

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

∎

[6] J. Dieudonné, *Éléments d'Analyse, Tome VII, Éditions Jacques Gabay, 2007.*

The Schwartz kernel theorem for anisotropic distributions

Let X and Y be two C^∞ manifolds of dimensions d and r, respectively. Then the following statements hold:

Theorem 4. a) *Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for every $\varphi \in \mathrm{C}^l_c(X)$, the linear form K_φ defined as $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$ is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.*

b) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continous linear operator. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$*

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$. ∎

How to prove it?

Standard attempts:

- ◦ regularisation? (Schwartz)
- ◦ constructive proof? (Simanca, Gask, Ehrenpreis)
- ◦ nuclear spaces? (Treves)

Use the structure theorem of distributions (Dieudonne). There are two steps:

Step I: assume the range of A is $\mathrm{C}(Y)$
Step II: use structure theorem and go back to Step I

Consequence: H-distributions are of order 0 in $\mathbf{x}$ and of finite order not greater than $d(\kappa+2)$ with respect to $\boldsymbol{\xi}$.

A variant by Bogdanovicz's result[7]

We can reformulate the main result of Bogdanowicz's article to our setting:

Theorem 5. *For every bilinear functional B on the space $\mathrm{C}_c^\infty(X_1) \times \mathrm{C}_c^l(X_2)$ which is continuous with respect to each variable separately, there exists a unique anisotropic distribution $T \in \mathcal{D}'_{\infty,l}(X_1 \times X_2)$ such that*

$$B(\varphi, \phi) = \langle T, \varphi \otimes \phi \rangle , \qquad \varphi \in \mathrm{C}_c^\infty(X_1), \phi \in \mathrm{C}_c^l(X_2) .$$

∎

It is worth noting that Bogdanowicz's result also holds when X_2 is a smooth manifold and that only elementary properties of Frechet and (LF)-spaces were used to prove it.

The same result can be obtained using the adjoint of operator A.

[7] W. M. Bogdanowicz: *A proof of Schwartz's theorem on kernels*, Studia Math. **20** (1961) 77–85.

Additional results[8]

Lema 1. *If* $u \in \mathcal{D}'_{l,m}(X_1 \times X_2)$ *is of compact support such that* $\operatorname{supp} u \subset \{0\} \times X_2$*, then for any* $\Phi \in \mathrm{C}_c^\infty(X_1 \times X_2)$ *it holds:*

$$u = \sum_{\boldsymbol{\alpha} \in \mathbf{N}_0^d, |\boldsymbol{\alpha}| \le l} \langle u_{\boldsymbol{\alpha}}, \Phi_{\boldsymbol{\alpha}} \rangle \,,$$

where $u_{\boldsymbol{\alpha}} \in \mathcal{D}'_m(X_2)$ *and* $\Phi_{\boldsymbol{\alpha}}(\mathbf{y}) = D_{\mathbf{x}}^{\boldsymbol{\alpha}} \Phi(0, \mathbf{y})$. ∎

Corollary 1. *If* $u \in \mathcal{D}'_{l,m}(X_1 \times X_2)$ *has compact support such that* $\operatorname{supp} u \subset \{\mathbf{x}_0\} \times X_2$*, for some* $\mathbf{x}_0 \in X_1$*, then*

$$u = \sum_{\boldsymbol{\alpha} \in \mathbf{N}_0^d, |\boldsymbol{\alpha}| \le l} D_{\mathbf{x}}^{\boldsymbol{\alpha}} \delta_{\mathbf{x}_0} \otimes u_{\boldsymbol{\alpha}} \,,$$

where $u_{\boldsymbol{\alpha}} \in \mathcal{D}'_m(X_2)$. ∎

Theorem 6. *Let* $A : \mathrm{C}_c^\infty(X) \to \mathcal{D}'_m(X)$ *be a continuous map. Its kernel is supported by the diagonal* $\{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in X\}$ *if and only if for any* $\varphi \in \mathrm{C}_c^\infty(X)$*:*

$$A\varphi = \sum_{\boldsymbol{\alpha} \in \mathbf{N}_0^d} a_{\boldsymbol{\alpha}} \otimes D^{\boldsymbol{\alpha}} \varphi,$$

where $a_{\boldsymbol{\alpha}} \in \mathcal{D}'_m(X)$ *and the above sum is locally finite. Moreover, this representation is unique.* ∎

[8] L. Hörmander: *The Analysis of Linear Partial Differential Operators I*, Springer, 1990.

Peetre's results[9]

Theorem 7. *Let* $A : \mathrm{C}_c^\infty(\Omega) \to \mathrm{C}_c^\infty(\Omega)$ *be a linear mapping such that the following holds:*

$$\operatorname{supp}(Au) \subset \operatorname{supp}(u)\,, \qquad u \in \mathrm{C}_c^\infty(\Omega).$$

Then A *is a differential operator on* Ω *with* C^∞ *coefficients.* ∎

Theorem 8. *Let* $A : \mathrm{C}_c^\infty(\Omega) \to \mathcal{D}'(\Omega)$ *be linear operator such that* $\operatorname{supp}(Af) \subset \operatorname{supp} f$ *for* $f \in \mathrm{C}_c^\infty(\Omega)$. *Then there exists locally finite family of distributions* $(a_{\boldsymbol{\alpha}}) \in \mathcal{D}'(\Omega)$, *unique on* $\Omega \setminus \Lambda$, *such that it holds:*

$$\operatorname{supp}\Big(Af - \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} D^{\boldsymbol{\alpha}} f\Big) \subset \Lambda\,, \qquad f \in \mathrm{C}_c^\infty(\Omega)\,.$$

∎

[9] J. Peetre: *Une caracterisation abstraite des operateurs differentiels*, Math. Scand. **7** (1959) 211–218; *Rectification*, ibid **8** (1960) 116–120.

Theorem 9. *Let $A : \mathrm{C}_c^\infty(\Omega) \to \mathcal{D}'_m(\Omega)$ be linear operator such that*

$$\operatorname{supp}(Af) \subset \operatorname{supp} f\,, \qquad f \in \mathrm{C}_c^\infty(\Omega)\,. \tag{1}$$

Then there exists locally finite family of distributions $(a_{\boldsymbol{\alpha}}) \in \mathcal{D}'_m(\Omega)$, unique on $\Omega \setminus \Lambda$, such that it holds:

$$\operatorname{supp}\Big(Af - \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} D^{\boldsymbol{\alpha}} f\Big) \subset \Lambda\,, \qquad f \in \mathrm{C}_c^\infty(\Omega)\,.$$

∎

Idea of the proof

Proof. Let $U \subset \Omega$ be an open and relatively compact set. Then there exists $j = j(U) \in \mathbf{N}$ such that for any $\mathbf{x}_0 \in U \setminus \Lambda$, there is a neighbourhood V of $\mathbf{x}_0$ such that

$$|\langle Af, g\rangle| \leq C\|f\|_j \|g\|_m \,, \qquad f \in \mathrm{C}_c^\infty(V), g \in \mathrm{C}_c^m(V)\,.$$

Schwartz kernel theorem gives existence of $K \in \mathcal{D}'_{\infty,m}(V \times V)$ such that $\langle Af, g\rangle = \langle K, f \otimes g\rangle$.

The locality assumption implies that the distribution K is supported on a diagonal of a set V:

$$\operatorname{supp} K \subset \{(\mathbf{x},\mathbf{x}) : \, \mathbf{x} \in V\}\,.$$

The theorem on the diagonal support gives:

$$A\varphi = \sum_{\boldsymbol{\alpha} \in \mathbf{N}_0^d} a_{\boldsymbol{\alpha}} \otimes D^{\boldsymbol{\alpha}}\varphi\,, \qquad \varphi \in \mathrm{C}_c^\infty(V)\,,$$

where family $(a_{\boldsymbol{\alpha}}) \subset \mathcal{D}'_m(V)$ is locally finite.

Idea of the proof - continued

Taking $\varphi \in \mathrm{C}_c^\infty(\Omega)$ equal to one on V, we obtain $A\varphi = a_0$ in V.

By using monomials $\mathbf{x}^{\boldsymbol{\alpha}}$, we can obtain the same conclusion for other $a_{\boldsymbol{\alpha}}$.

Now, we conclude:

$$\operatorname{supp}\Big(Af - \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} D^{\boldsymbol{\alpha}} f\Big) \subset \Lambda\,, \qquad f \in \mathrm{C}_c^\infty(U)\,.$$

Since $U \subset \Omega$ was arbitrary, the claim of the theorem follows.

Counterexample

As already noticed by Peetre in the standard case, the result in the statement of the preceeding theorem is the best possible.
Namely, it can happen $A - \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} D^{\boldsymbol{\alpha}} \neq 0$, as we can easily see from the following example:

for $\mathbf{x}_0 \in \Omega$, take a linear form F defined for sequence $(c_{\boldsymbol{\alpha}})$ such that it can not be written in the form $F = \sum_{\boldsymbol{\alpha}} b^{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}}$, for any finite collection of $b^{\boldsymbol{\alpha}}$. Then

$$(Af)(\mathbf{x}) = F(D^{\boldsymbol{\alpha}} f(\mathbf{x}_0))\delta_0(\mathbf{x} - \mathbf{x}_0)$$

has desired properties without being continuous: we have $\operatorname{supp}(Af) \subset \{\mathbf{x}_0\}$ and A is continuous everywhere except at the point $\mathbf{x}_0$.

Reference

- M. Mišur, *On Peetre's theorem*, in preparation, 13 pages

- N. Antonić, M. Erceg, M. Mišur, *Distributions of anisotropic order and applications*, preprint éternelle, 24 pages

- M. Mišur, Lj. Palle, *Paley-Wiener-Schwartz type results for anisotropic distributions*, in progress