Abstract Friedrichs systems and universal operator extension

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WeConMApp

Symmetric positive systems

K. O. Friedrichs: *Symmetric hyperbolic linear differential equations, Commun. Pure Appl. Math.* **7** (1954) 345–392.

Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).

A generalisation:

K. O. Friedrichs: *Symmetric positive linear differential equations, Commun. Pure Appl. Math.* **11** (1958), 333–418.

Goals:

– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

– unified treatment of equations and systems of different type;

– more recently: better numerical properties.

All of Gårding's theory of general elliptic equations, or Lerray's of general hyperbolic equations, is not covered.

Friedrichs' system (KOF1958)

Assumptions:
$d, r \in \mathbf{N}$, $\Omega \subseteq \mathbf{R}^d$ open and bounded with Lipschitz boundary Γ;

$\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, and $\mathbf{B} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{C}))$ satisfying

(F1) matrix functions $\mathbf{A}_k$ are hermitian: $\mathbf{A}_k = \mathbf{A}_k^*$;

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad \text{(ae on } \Omega) \,.$$

The operator $\mathcal{L} : \mathrm{L}^2(\Omega; \mathbf{C}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{C}^r)$

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u}$$

is called *the symmetric positive operator* or *the Friedrichs operator*, and

$$\mathcal{L}\mathsf{u} = \mathsf{f}$$

the symmetric positive system or *the Friedrichs system*.

Symmetric hyperbolic systems (KOF1954)

$$\sum_{k=1}^{d} \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{D}\mathsf{u} = \mathsf{f}$$

In divergence form:

$$\sum_{k=1}^{d} \partial_k (\mathbf{A}^k \mathsf{u}) + (\mathbf{D} - \partial_k \mathbf{A}^k)\mathsf{u} = \mathsf{f}$$

It is symmetric if all matrices $\mathbf{A}^k$ are real and symmetric; and uniformly hyperbolic if there is a $\boldsymbol{\xi} \in \mathbf{R}^d$ such that for any $\mathbf{x} \in \mathrm{Cl}\,\Omega$ the matrix $\xi_k \mathbf{A}^k(\mathbf{x})$ is positive definite.

Such systems can easily be transformed into Friedrichs' systems.

It is known that the wave equation, the Maxwell and the Dirac system can be written as an equivalent symmetric hyperbolic system.

An example – scalar elliptic equation

$\Omega \subseteq \mathbf{R}^2$, $\mu > 0$ and $f \in \mathrm{L}^2(\Omega)$ given.

$$-\triangle u + \mu u = f$$

can be written as a first-order system

$$\begin{cases} \mathsf{p} + \nabla u = 0 \\ \mu u + \operatorname{div} \mathsf{p} = f \end{cases},$$

which is a Friedrichs system with the choice of

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Example – heat equation

... with zero initial and Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \operatorname{div}_{\mathbf{x}}(\mathbf{A}\nabla_{\mathbf{x}} u) + \mathbf{b}\cdot\nabla_{\mathbf{x}} u + cu = f \text{ in } \Omega_T \\ u = 0 \text{ on } \langle 0, T\rangle \times \Gamma \\ u(0,\cdot) = 0 \text{ on } \Omega \end{cases}$$

...as a Friedrichs system:

$$\begin{cases} \nabla_{\mathbf{x}} u_{d+1} + \mathbf{A}^{-1}\mathsf{u}_d = 0 \\ \partial_t u_{d+1} + \operatorname{div}_{\mathbf{x}}\mathsf{u}_d + c u_{d+1} - \mathbf{A}^{-1}\mathsf{b}\cdot\mathsf{u}_d = f \end{cases},$$

(note that we use $\mathsf{u} = (\mathsf{u}_d, u_{d+1})^\top$, where $\mathsf{u}_d = -\mathbf{A}\nabla u$, and $u_{d+1} = u$). Indeed

$$\begin{bmatrix} \mathbf{0} & 0 \\ 0^\top & 1 \end{bmatrix} \partial_t \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} + \sum_{i=1}^{d} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 1 \\ \vdots & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & 1 \end{bmatrix} \partial_{x^i} \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} + \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ -(\mathbf{A}^{-1}\mathsf{b})^\top & c \end{bmatrix} \begin{bmatrix} \mathsf{u}_d \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

The condition (F1) holds. The positivity condition $\mathbf{B} + \mathbf{B}^\top \geqslant 2\mu_0\mathbf{I}$ is fulfilled if and only if $c - \frac{1}{4}\mathbf{A}^{-1}\mathbf{b}\cdot\mathbf{b}$ is uniformly positive.

Boundary conditions

Boundary conditions are enforced via a matrix valued boundary field:

$$\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^{d} \nu_k \mathbf{A}_k \in \mathrm{L}^{\infty}(\Gamma; \mathrm{M}_r(\mathbf{C})),$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)$ is the outward unit normal on Γ, and

$$\mathbf{M} \in \mathrm{L}^{\infty}(\Gamma; \mathrm{M}_r(\mathbf{C})).$$

Boundary condition

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0$$

is sufficient for treatment of different types of usual boundary conditions.

Assumptions on boundary matrix $\mathbf{M}$

We assume (for ae $\mathbf{x} \in \Gamma$) [KOF1958]

(FM1) $$(\forall\, \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FM2) $$\mathbf{C}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,.$$

Such $\mathbf{M}$ is called *the admissible boundary condition*.

The boundary problem: for given $\mathsf{f} \in \mathrm{L}^2(\Omega; \mathbf{C}^r)$ find u such that

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases}.$$

Elliptic equation – different boundary conditions

$$\mathbf{M} \qquad \mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M} \qquad (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M}) \begin{bmatrix} \mathsf{p} \\ u \end{bmatrix}_{|\Gamma} = 0$$

$$\begin{bmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 2\nu_1 \\ 0 & 0 & 2\nu_2 \\ 0 & 0 & 0 \end{bmatrix} \qquad u_{|\Gamma} = 0$$

$$\begin{bmatrix} 0 & 0 & \nu_1 \\ 0 & 0 & \nu_2 \\ -\nu_1 & -\nu_2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\nu_1 & 2\nu_2 & 0 \end{bmatrix} \qquad \boldsymbol{\nu} \cdot (\nabla u)_{|\Gamma} = 0$$

$$\begin{bmatrix} 0 & 0 & \nu_1 \\ 0 & 0 & \nu_2 \\ -\nu_1 & -\nu_2 & 2\alpha \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\nu_1 & 2\nu_2 & 2\alpha \end{bmatrix} \qquad \boldsymbol{\nu} \cdot (\nabla u)_{|\Gamma} + \alpha u_{|\Gamma} = 0$$

All above matrices $\mathbf{M}$ satisfy (FM).

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Different ways to enforce boundary conditions

Instead of

$$(\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u} = 0 \quad \text{on } \Gamma\,,$$

Lax proposed boundary conditions with

$$\mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma\,,$$

where $N = \{N(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ is a family of subspaces of $\mathbf{C}^r$.

Boundary problem:

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ \mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma \end{cases}.$$

Assumptions on N

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$) [PDL]

(FX1) $N(\mathbf{x})$ is non-negative with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$:

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,;$$

(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which (properly) contains $N(\mathbf{x})$;

or [RSP&LS1966]

Let $N(\mathbf{x})$ and $\tilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ satisfy (for ae $\mathbf{x} \in \Gamma$)

(FV1)

$$(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0$$

$$(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0$$

(FV2) $\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ and $N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^{\perp}$.

Theorem. *It holds*

$$\textit{(FM1)–(FM2)} \quad \Longleftrightarrow \quad \textit{(FX1)–(FX2)} \quad \Longleftrightarrow \quad \textit{(FV1)–(FV2)},$$

with

$$N(\mathbf{x}) := \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big).$$

∎

In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].

Classical results on well-posedness

Friedrichs:
– uniqueness of the classical solution
– existence of a *weak* solution (under some additional assumptions)

Contributions (and particular cases):
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
– the meaning of traces for functions in the graph space,
– weak well-posedness results under additional assumptions (on $\mathbf{A}_{\boldsymbol{\nu}}$),
– regularity of solution,
– numerical treatment.

Shortcommings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

However, since the beginning of 21st century the numerical advantages of FS have overshadowed that.

New approach…

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Diff. Eq.* **32** (2007) 317–341.

– abstract setting (operators on Hilbert spaces),

– intrinsic criterion for the bijectivity of *Friedrichs'* operator,

– avoiding the question of traces for functions in the graph space,

– investigation of different formulations of boundary conditions,

. . . and new open questions.

They considered only the real case.

Assumptions

Let L be real (complex) Hilbert space (L' is (anti)dual of L), $\mathcal{D} \subseteq L$ a dense subspace, and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ linear unbounded operators satisfying

(T1) $$(\forall\, \varphi, \psi \in \mathcal{D}) \quad \langle T\varphi \mid \psi \rangle_L = \langle \varphi \mid \tilde{T}\psi \rangle_L \,,$$

(T2) $$(\exists\, c > 0)(\forall\, \varphi \in \mathcal{D}) \quad \|(T + \tilde{T})\varphi\|_L \leqslant c\|\varphi\|_L \,,$$

(T3) $$(\exists\, \mu_0 > 0)(\forall\, \varphi \in \mathcal{D}) \quad \langle (T + \tilde{T})\varphi \mid \varphi \rangle_L \geqslant 2\mu_0\|\varphi\|_L^2 \,.$$

$(T, \widetilde{T})$ is referred to as a joint pair of abstract Friedrichs operators.

Recall the Friedrichs operator: $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{C}^r)$ and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ be defined by

$$T\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{B}\mathsf{u} \,,$$

$$\tilde{T}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + (\mathbf{B}^* + \sum_{k=1}^{d} \partial_k\mathbf{A}_k)\mathsf{u} \,,$$

where $\mathbf{A}_k$ and $\mathbf{B}$ are as above (they satisfy (F1)–(F2)).

Then T and $\tilde{T}$ satisfy (T1)–(T3) ... fits in this framework.

Extension of operators, starting from $(T, \tilde{T}) = (T_1, \tilde{T}_1)$

$\mathcal{D}$ is an inner product space when equipped with *graph norm* stemming from

$$\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle_L + \langle T \cdot \mid T \cdot \rangle_L \,.$$

By W_0 denote the completion of $\mathcal{D}$ in the graph norm, same for $\tilde{T}$ by (T2). $W_0 \leqslant L$ by (T1), and both T and $\tilde{T}$ extend to bounded operators from W_0 to L, which we denote by $(T_2, \tilde{T}_2)$.

The following embedding are dense and continuous (we have a Gel'fand triplet):

$$W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0' \,.$$

Let $T_3 := \tilde{T}_2' \in \mathcal{L}(L; W_0')$ be the Banach adjoint of $\tilde{T}_2' : W_0 \longrightarrow L$, and $\tilde{T}_3 := T_2'$. Thus we have defined $(T_3, \tilde{T}_3)$.

Note that the *graph space*

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\} \leqslant L$$

is a Hilbert space with respect to $\langle \cdot \mid \cdot \rangle_T$.
$(T_4, \tilde{T}_4)$ are defined as restrictions of T_3 and $\tilde{T}_3$ to W.

Well-posedness for abstract Friedrichs operator

This produces the maximal pair of abstract Friedrichs operators $(T_4, \widetilde{T}_4)$, mapping $T_4, \widetilde{T}_4 : W \to L$, which are associated to the initial pair $(T, \widetilde{T})$.

Find sufficient conditions for a subspace $W_0 \leqslant V \leqslant W$ such that $T_{4|_V} : V \longrightarrow L$ is an isomorphism.

As the continuity in the graph norm holds for any restriction to a closed subspace V of W, the key question is bijectivity.

Boundary operator

Sufficient coditions were obtained by [EGC2007] and [AB2010] using *Boundary operator* $D \in \mathcal{L}(W; W')$:

$$ {}_{W'}\langle\, Du, v\,\rangle_W := \langle\, Tu \mid v\,\rangle_L - \langle\, u \mid \tilde{T}v\,\rangle_L \,, \qquad u, v \in W \,. $$

D is symmetric: ${}_{W'}\langle\, Du, v\,\rangle_W = \overline{{}_{W'}\langle\, Dv, u\,\rangle_W}$ and satisfies

$$ \begin{aligned} &\ker D = W_0 \\ &\operatorname{im} D = W_0^0 := \{g \in W' : (\forall\, u \in W_0) \quad {}_{W'}\langle\, g, u\,\rangle_W = 0\} \,. \end{aligned} $$

For a given joint pair of abstract FO $(T, \widetilde{T})$, a pair $(V, \widetilde{V})$ of subspaces of W is said to allow the (V)-boundary conditions relative to $(T, \widetilde{T})$ when:

(V1) the boundary operator has opposite sign on V and on $\widetilde{V}$, in the sense that
$(\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u\,\rangle_W \geqslant 0 \,,$
$(\forall\, v \in \widetilde{V}) \qquad {}_{W'}\langle\, Dv, v\,\rangle_W \leqslant 0 \,;$

(V2) the image via D of either space has, as annihilator, the other space, namely
$V = D(\widetilde{V})^0 \qquad$ and $\qquad \widetilde{V} = D(V)^0 \,.$

For classical Friedrichs operator

If T is the Friedrichs operator $\mathcal{L}$, then for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ we have

$$
{}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x})\,.
$$

With the assumptions:

(FV1)
$$
\begin{aligned}
(\forall\, \boldsymbol{\xi} \in N(\mathbf{x})) \quad & \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,, \\
(\forall\, \boldsymbol{\xi} \in \tilde{N}(\mathbf{x})) \quad & \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0\,,
\end{aligned}
$$

(FV2)
$$
\tilde{N}(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^\perp \quad \text{and} \quad N(\mathbf{x}) = (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\tilde{N}(\mathbf{x}))^\perp\,,
$$

we are lead to consider subspaces V and $\tilde{V}$ in the functional framework:

(V1)
$$
\begin{aligned}
(\forall\, u \in V) \qquad & {}_{W'}\langle\, Du, u\,\rangle_W \geqslant 0\,, \\
(\forall\, v \in \tilde{V}) \qquad & {}_{W'}\langle\, Dv, v\,\rangle_W \leqslant 0\,,
\end{aligned}
$$

(V2)
$$
V = D(\tilde{V})^0\,, \qquad \tilde{V} = D(V)^0\,.
$$

Well-posedness theorem

$$[\,u \mid v\,] := {}_{W'}\langle\, Du, v\,\rangle_W = \langle\, Tu \mid v\,\rangle_L - \langle\, u \mid \tilde{T}v\,\rangle_L\,, \qquad u, v \in W$$

is an indefinite inner product on W, and we consider subspaces V and $\tilde{V}$ satisfying:

(V1)
$$(\forall\, v \in V) \quad [\,v \mid v\,] \geqslant 0\,,$$
$$(\forall\, v \in \tilde{V}) \quad [\,v \mid v\,] \leqslant 0\,;$$

(V2)
$$V = \tilde{V}^{[\perp]}\,, \qquad \tilde{V} = V^{[\perp]}\,.$$

($^{[\perp]}$ stands for $[\,\cdot \mid \cdot\,]$-orthogonal complement)

Theorem. *Under assumptions* $(T1)-(T3)$ *and* $(V1)-(V2)$, *the operators* $T_{|_V} : V \longrightarrow L$ *and* $\tilde{T}_{|_{\tilde{V}}} : \tilde{V} \longrightarrow L$ *are isomorphisms.* ∎

In the real case [EGC2007].

Correspondence — maximal b.c.

maximal boundary conditions: (for ae $\mathbf{x} \in \Gamma$)

(FX1) $$(\forall \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,,$$

(FX2) there is no non-negative subspace with respect to $\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})$, which contains $N(\mathbf{x})\,,$

subspace V is maximal non-negative in $(W, [\,\cdot \mid \cdot\,])$:

(X1) V is non-negative in $(W, [\,\cdot \mid \cdot\,])$: $\quad (\forall\, v \in V) \quad [\,v \mid v\,] \geqslant 0\,,$

(X2) there is no non-negative subspace in $(W, [\,\cdot \mid \cdot\,])$ containing V .

Correspondence — admissible b.c.

admissible boundary condition: there exists a matrix function $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{C})$ such that (for ae $\mathbf{x} \in \Gamma$)

(FM1) $$(\forall \boldsymbol{\xi} \in \mathbf{C}^r) \quad (\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^*)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0 \, ,$$

(FM2) $$\mathbf{C}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big) \, .$$

abstract admissible boundary condition: there exists $M \in \mathcal{L}(W; W')$ such that

(M1) $$(\forall \, u \in W) \quad {}_{W'}\langle \, (M + M^*)u, u \, \rangle_W \geqslant 0 \, ,$$

(M2) $$W = \ker(D - M) + \ker(D + M) \, .$$

Equivalence of different descriptions of b.c.

Theorem. *(**classical**)* *It holds*

(FM1)–(FM2) $\iff$ *(FV1)–(FV2)* $\iff$ *(FX1)–(FX2)*,

with

$$N(\mathbf{x}) := \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big)\,.$$

∎

Theorem. *[EGC2007, AB2010]* *It holds*

(M1)–(M2) $\iff$ *(V1)–(V2)* $\iff$ *(X1)–(X2)*,

with

$$V := \ker(D - M)\,.$$

∎

Friedrichs systems

- Classical theory of Friedrichs systems
- Boundary conditions for Friedrichs systems
- Abstract formulation
- Interdependence of different representations of boundary conditions

Hilbert space framework

- Bijective realisations with signed boundary map
- Universal classification
- One dimensional example

Theorem. *[Ern, Guermond, Caplain, 2007] Let $(T, \widetilde{T})$ be a joint pair of Friedrichs systems and let $(V, \widetilde{V})$ satisfy (V1)–(V2). Then $T_4|_V : V \to L$ and $\widetilde{T}_4|_{\widetilde{V}} : \widetilde{V} \to L$ are closed bijective realisations of T and $\widetilde{T}$, respectively.* ∎

Can we say something more about extensions T_4, $\widetilde{T}_4$, and conditions (V)?

Theorem. *$(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators iff*

(i) $T \subseteq \widetilde{T}^*$ *and* $\widetilde{T} \subseteq T^*$;

(ii) $\overline{T + \widetilde{T}}$ *is a bounded self-adjoint operator in L with strictly positive bottom;*

(iii) dom $\overline{T}$ = dom $\overline{\widetilde{T}} = W_0$ *and* dom $T^* =$ dom $\widetilde{T}^* = W$. ∎

In fact: $T_4 = \widetilde{T}^*$ and $\widetilde{T}_4 = T^*$.

Bijective realisations with signed boundary map

Theorem. *Let* $(T, \widetilde{T})$ *be a pair of operators on the Hilbert space* L *satisfying conditions* $(\mathrm{T1})$–$(\mathrm{T2})$*, and let* $(V, \widetilde{V})$ *be a pair of subspaces of* L*. Then (V2) is equivalent to*

i) $W_0 \subseteq V \subseteq W, \;\; W_0 \subseteq \widetilde{V} \subseteq W$,

ii) V *and* $\widetilde{V}$ *are closed in* W*, and*

iii) $(\widetilde{T}^*|_V)^* \;=\; T^*|_{\widetilde{V}}, \;\; (T^*|_{\widetilde{V}})^* \;=\; \widetilde{T}^*|_V$. ∎

We are seeking bijective closed operators $S \equiv \widetilde{T}^*|_V$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^* ,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$.

In the following we work with closed T and $\widetilde{T}$.

Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space L. For a closed $T \subseteq S \subseteq \widetilde{T}^*$ such that $(\mathrm{dom}\, S, \mathrm{dom}\, S^*)$ satisfies $(V1)$ we call (S, S^*) an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.

Questions:

1) Sufficient conditions on V? ✓
2) Existence of $V \subseteq W$ such that $(\widetilde{T}^*|_V, (\widetilde{T}^*|_V)^*)$ is an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$?
3) Existence of infinitely many such V?
4) Classification of all such V?

Existence of infinitely many V's

Theorem. *Let $(T, \widetilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space L.*

(i) There is an adjoint pair of bijective realisations with signed boundary map. Furthermore, there is an adjoint pair (T_r, T_r^*) of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$ such that

$$W_0 + \ker T^* \subseteq \mathsf{dom}\, T_r \qquad \textit{and} \qquad W_0 + \ker \widetilde{T}^* \subseteq \mathsf{dom}\, T_r^* \,.$$

(ii) If both $\ker \widetilde{T}^* \neq \{0\}, \ker T^* \neq \{0\}$, then $(T, \widetilde{T})$ admits uncountably many adjoint pairs of bijective realisations with signed boundary map. Else, if either $\ker \widetilde{T}^* = \{0\}$ or $\ker T^* = \{0\}$, then there is exactly one adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$. Such a pair is precisely $(\widetilde{T}^*, \widetilde{T})$ when $\ker \widetilde{T}^* = \{0\}$, and (T, T^*) when $\ker T^* = \{0\}$.

∎

Grubb's universal classification [G1968, ...]

Start with a pair (A_0, A_0') of closed operators on L such that

$$A_0 \subseteq (A_0')^* =: A_1 \qquad \text{and} \qquad A_0' \subseteq (A_0)^* =: A_1' \,,$$

$A_0 + A_0'$ is bounded on L and extends to an everywhere defined, bounded, self-adjoint operator in L with strictly positive bottom.
We refer to any such (A_0, A_0') as a *joint pair of closed abstract Friedrichs operators*. This definition implies that

$$\text{dom}\, A_0 = \text{dom}\, A_0' =: W_0 \qquad \text{and} \qquad \text{dom}\, A_1 = \text{dom}\, A_1' =: W \,.$$

We are interested in restrictions $A_1|_V$ and $A_1'|_{\widetilde{V}}$ onto suitable subspaces V and $\widetilde{V}$ of L which satisfy conditions (V1)–(V2). Equivalently, this is the class of restrictions such that

$$W_0 \subseteq V \subseteq W \qquad \text{and} \qquad W_0 \subseteq \widetilde{V} \subseteq W \,,$$

which satisfy that $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are mutually adjoint (thus, in particular, $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are closed operators) and

$$(\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u \,\rangle_W = \langle\, A_1 u \mid u \,\rangle_L - \langle\, u \mid A_1' u \,\rangle_L \geqslant 0 \,,$$
$$(\forall\, v \in \widetilde{V}) \qquad {}_{W'}\langle\, Dv, v \,\rangle_W = \langle\, A_1 v \mid v \,\rangle_L - \langle\, v \mid A_1' v \,\rangle_L \leqslant 0 \,.$$

Grubb's universal classification (cont.)

We shall refer to any such pair $(A_1|_V, A_1'|_{\widetilde{V}})$ as an *adjoint pair of bijective realisations with signed boundary map* relative to the given joint pair of closed abstract Friedrichs operators $(A_0, A_0') = (\overline{T}, \overline{\widetilde{T}})$.

For their adjoints we have

$$A_1 \ := \ (A_0')^* \ = \ \widetilde{T}^* \qquad \text{and} \qquad A_1' \ := \ (A_0)^* \ = \ T^* \,.$$

It is immediate that there is a one-to-one correspondence between all pairs of isomorphisms induced by $(T, \widetilde{T})$, and all adjoint pairs of bijective realisations with signed boundary map relative to (A_0, A_0'), i.e. $\big(\overline{T}, \overline{\widetilde{T}}\big)$.

Since $A_R = A_1|_V$ is closed and bijective onto L, then $(A_1|_V)^{-1}$ is necessarily everywhere defined and bounded, so we may also speak of $A_1|_V$ as of an *isomorphic realisation of A_0 with signed boundary map*.

It is worth remarking that the fact that a closed operator S satisfies $A_0 \subseteq S \subseteq A_1$ is *equivalent* to $A_0' \subseteq S^* \subseteq A_1'$.

The interest towards such pairs $(A_1|_V, A_1'|_{\widetilde{V}})$ is two-fold: first, when (V1)–(V2) hold, $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are bijections onto L, thus providing a sufficient criterion of well-posedness of the abstract Friedrichs system; moreover, (V1)–(V2) encode the most relevant class of boundary conditions, as it may be seen from a large variety of concrete examples of boundary value problems on which such conditions are modelled.

Grubb's universal classification (cont.)

Let (A_0, A_0^*) and (A_1, A_1^*) be two pairs of mutually adjoint, closed and densely defined operators in L, with properties as above, which admit a further pair (A_r, A_r^*) of reference operators that are closed, satisfy $A_0 \subseteq A_r \subseteq A_1$, equivalently $A_0' \subseteq A_r^* \subseteq A_1'$, and are invertible with everywhere defined bounded inverses A_r^{-1} and $(A_r^*)^{-1}$. Then there are decompositions

$$\text{dom}\, A_1 \;=\; \text{dom}\, A_r \dot{+} \ker A_1 \quad \text{and} \quad \text{dom}\, A_1' \;=\; \text{dom}\, A_r^* \dot{+} \ker A_1'$$

$$\begin{aligned} p_r &= A_r^{-1} A_1\,, \qquad & p_{r'} &= (A_r^*)^{-1} A_1'\,, \\ p_k &= I - p_r\,, \qquad & p_{k'} &= I - p_{r'}\,. \end{aligned}$$

There is a one-to-one correspondence between

$$\left.\begin{array}{r} (A, A^*) \\ A_0 \subseteq A \subseteq A_1 \\ A_0' \subseteq A^* \subseteq A_1' \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} (B, B^*) \\ \mathcal{V} \subseteq \ker A_1 \text{ closed} \\ \mathcal{W} \subseteq \ker A_1' \text{ closed} \\ B : \mathcal{V} \to \mathcal{W} \text{ densely defined} \end{array}\right.$$

$$B \mapsto A_B : \quad \text{dom}\, A_B \;=\; \Big\{ u \in \text{dom}\, A_1 \;:\; p_k u \in \text{dom}\, B\,,\; P_{\mathcal{W}}(A_1 u) = B(p_k u) \Big\}\,,$$

$$A \mapsto B_A : \quad \text{dom}\, B_A \;=\; p_k \text{dom}\, A\,, \quad \mathcal{V} \;=\; \overline{\text{dom}\, B_A}\,, \quad B_A(p_k u) = P_{\mathcal{W}}(A_1 u)\,,$$

where $P_{\mathcal{W}}$ is the *orthogonal* projections from L onto $\mathcal{W}$.

Important: A is injective, resp. surjective, resp. bijective, if and only if so is B.

Grubb's universal classification (cont.)

When A_B corresponds to B as above, then

$$\begin{aligned}\mathsf{dom}\, A_B = \{w_0 + (A_r)^{-1}(B\nu + \nu') + \nu \mid w_0 \in \mathsf{dom}\, A_0 \\ \& \ \nu \in \mathsf{dom}\, B \ \& \ \nu' \in \ker A_1' \ominus \mathcal{W}\}\end{aligned}$$

$$A_B(w_0 + (A_r)^{-1}(B\nu + \nu') + \nu) \ = \ A_0 w_0 + B\nu + \nu'$$

We shall apply this theory on a joint pair of closed abstract Friedrichs systems.

Classification of bijective realisations with signed boundary map

For simplicity here we use the notation of Grubb's universal classification.

(A_0, A_0') a joint pair of closed abstract Friedrichs operators, $A_1 := (A_0')^*$, $A_1' := A_0^*$, and let (A_r, A_r^*) be an adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0').
(A_B, A_B^*) a generic pair of closed extensions $A_0 \subseteq A_B \subseteq A_1$.

Classification of bijective realisations with signed boundary map (cont.)

$$(1) \qquad \begin{array}{l}(\forall\, \nu \in \operatorname{dom} B)\\ (\forall\, \nu' \in \ker A_1' \ominus \mathcal{W})\end{array} \qquad \begin{cases} \langle\, \nu \mid A_1'\nu \,\rangle_L - 2\,\mathrm{Re}\,\langle\, p_{\mathrm{k}'}\nu \mid B\nu \,\rangle_L \leqslant 0 \\ \langle\, p_{\mathrm{k}'}\nu \mid \nu' \,\rangle_L = 0 \end{cases}$$

$$(2) \qquad \begin{array}{l}(\forall\, \mu' \in \operatorname{dom} B^*)\\ (\forall\, \mu \in \ker A_1 \ominus \mathcal{V})\end{array} \qquad \begin{cases} \langle\, A_1\mu' \mid \mu' \,\rangle_L - 2\,\mathrm{Re}\,\langle\, B^*\mu' \mid p_k\mu' \,\rangle_L \leqslant 0 \\ \langle\, \mu \mid p_k\mu' \,\rangle_L = 0\,, \end{cases}$$

Theorem. *Any of the following three facts,*

(a) conditions (1) and (2) hold true, or

(b) condition (1) holds true and $B : \operatorname{dom} B \to \mathcal{W}$ is a bijection, or

(c) condition (2) holds true and $B^* : \operatorname{dom} B^* \to \mathcal{V}$ is a bijection,

is sufficient for (A_B, A_B^*) to be another adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0').

Assume further that $\operatorname{dom} A_r = \operatorname{dom} A_r^*$. Then the following properties are equivalent:

(i) (A_B, A_B^*) is another adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0');

(ii) the mirror conditions (1) and (2) are satisfied. ∎

Example: First order ode

Take $L := \mathrm{L}^2(0,1)$, $\mathcal{D} := \mathrm{C}_c^\infty(0,1)$ and define $T, \widetilde{T} : \mathcal{D} \to L$ by

$$T\phi := \frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi \qquad \text{and} \qquad \widetilde{T}\phi := -\frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi\,.$$

We have

$$\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} = \mathrm{H}_0^1(0,1) =: W_0$$
$$\operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* = \mathrm{H}^1(0,1) =: W\,,$$

Further define

$$A_0 := \overline{T}\,, \qquad , A_0' := \overline{\widetilde{T}}\,, \quad A_1 := \widetilde{T}^*\,, A_1' := T^*\,.$$

As ${}_{W'}\langle\, Du, v\,\rangle_W = u(1)\overline{v(1)} - u(0)\overline{v(0)}$, for

$$V := \widetilde{V} := \{u \in \mathrm{H}^1(0,1) : u(0) = u(1)\}\,,$$

we have that $A_r := A_1|_V$, $A_r^* = A_1'|_V$ for an adjoint pair of bijective realisations with signed boundary map.

As $\ker A_1 = \operatorname{span}\{e^{-x}\}$ and $\ker A_1' = \operatorname{span}\{e^{x}\}$, so

$$p_k u = -\frac{u(1) - u(0)}{1 - e^{-1}} e^{-x}\,, \quad p_{k'} u = \frac{u(1) - u(0)}{e - 1} e^{x}\,.$$

Example (cont.)

The corresponding spaces are $\mathcal{V} = \ker A_1$, $\mathcal{W} = \ker A_1'$, while $B_{\alpha,\beta} : \mathcal{V} \to \mathcal{W}$ is defined by

$$B_{\alpha,\beta} e^{-x} = (\alpha + \mathrm{i}\beta) e^{x}$$

where $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(1) simplifies to check

$$\langle\, e^{-x} \mid A_1' e^{-x} \,\rangle_L - 2\mathrm{Re}\, \langle\, p_{k' e^{-x}} \mid B_{\alpha,\beta} e^{-x} \,\rangle_L \leqslant 0$$

$$\iff \alpha \leqslant -e^{-1}$$

$$\{(A_{\alpha,\beta}, A_{\alpha,\beta}^*) : \alpha \leqslant -e^{-1}\,,\ \beta \in \mathbb{R}\} \cup \{(A_r, A_r^*)\}$$

$$\text{dom}\, A_{\alpha,\beta}^{(*)} = \Big\{ u \in \mathrm{H}^1(0,1) : \Big(2e^{-1} - (+)\alpha(1+e) - \mathrm{i}\beta(1+e)\Big) u(1) = \Big(2 + \alpha(1+e) - (+)\mathrm{i}\beta(1+e)\Big) u(0) \Big\}$$

Thank you for your attention!