Abstract Friedrichs systems and universal operator extension

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Joint work with Marko Erceg and Alessandro Michelangeli
Friedrichs systems
- Classical theory of Friedrichs systems
- Boundary conditions for Friedrichs systems
- Abstract formulation
- Interdependence of different representations of boundary conditions

Hilbert space framework
- Bijective realisations with signed boundary map
- Universal classification
- One dimensional example
Symmetric positive systems


Unified treatment of linear hyperbolic systems like Maxwell’s, Dirac’s, or higher order equations (e.g. the wave equation).

A generalisation:


Goals:

– treating the equations of mixed type, such as the Tricomi equation:

\[ y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \]

– unified treatment of equations and systems of different type;

– more recently: better numerical properties.

All of Gårding’s theory of general elliptic equations, or Lerray’s of general hyperbolic equations, is not covered.
Friedrichs’ system (KOF1958)

Assumptions:
\( d, r \in \mathbb{N}, \ \Omega \subseteq \mathbb{R}^d \) open and bounded with Lipschitz boundary \( \Gamma \);
\( A_k \in W^{1,\infty}(\Omega; M_r(\mathbb{C})), \ k \in 1..d, \) and \( B \in L^\infty(\Omega; M_r(\mathbb{C})) \) satisfying
\[
(F1) \quad \text{matrix functions } A_k \text{ are hermitian: } A_k = A_k^* ;
\]
\[
(F2) \quad (\exists \mu_0 > 0) \quad B + B^* + \sum_{k=1}^d \partial_k A_k \geq 2\mu_0 \mathbf{I} \quad \text{ (ae on } \Omega). \]

The operator \( \mathcal{L} : L^2(\Omega; \mathbb{C}^r) \rightarrow D'(\Omega; \mathbb{C}^r) \)
\[
\mathcal{L}u := \sum_{k=1}^d \partial_k (A_k u) + Bu
\]
is called \textit{the symmetric positive operator} or \textit{the Friedrichs operator}, and
\[
\mathcal{L}u = f
\]

\textit{the symmetric positive system} or \textit{the Friedrichs system}. 
Symmetric hyperbolic systems (KOF1954)

\[ \sum_{k=1}^{d} A^k \partial_k u + Du = f \]

In divergence form:

\[ \sum_{k=1}^{d} \partial_k (A^k u) + (D - \partial_k A^k)u = f \]

It is symmetric if all matrices \( A^k \) are real and symmetric; and uniformly hyperbolic if there is a \( \xi \in \mathbb{R}^d \) such that for any \( x \in \text{Cl} \Omega \) the matrix \( \xi_k A^k(x) \) is positive definite.

Such systems can easily be transformed into Friedrichs' systems.

It is known that the wave equation, the Maxwell and the Dirac system can be written as an equivalent symmetric hyperbolic system.
An example – scalar elliptic equation

\( \Omega \subseteq \mathbb{R}^2, \mu > 0 \) and \( f \in L^2(\Omega) \) given.

\[-\Delta u + \mu u = f\]

can be written as a first-order system

\[
\begin{cases}
\mathbf{p} + \nabla u = 0 \\
\mu u + \text{div}\mathbf{p} = f
\end{cases},
\]

which is a Friedrichs system with the choice of

\[
\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.
\]
Example – heat equation

... with zero initial and Dirichlet boundary condition:

\[
\begin{aligned}
\partial_t u - \text{div}_x (A \nabla_x u) + b \cdot \nabla_x u + cu &= f \text{ in } \Omega_T \\
    u &= 0 \text{ on } (0, T) \times \Gamma \\
    u(0, \cdot) &= 0 \text{ on } \Omega
\end{aligned}
\]

...as a Friedrichs system:

\[
\begin{aligned}
\nabla_x u_{d+1} + A^{-1} u_d &= 0 \\
\partial_t u_{d+1} + \text{div}_x u_d + cu_{d+1} - A^{-1} b \cdot u_d &= f
\end{aligned}
\]

(note that we use \( u = (u_d, u_{d+1})^\top \), where \( u_d = -A \nabla u \), and \( u_{d+1} = u \)). Indeed

\[
\begin{bmatrix}
0 & 0 \\
0^\top & 1
\end{bmatrix}
\partial_t \begin{bmatrix} u_d \\ u \end{bmatrix} + \sum_{i=1}^d \begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\partial_{x_i} \begin{bmatrix} u_d \\ u \end{bmatrix} + \begin{bmatrix}
-A^{-1}b & 0 \\
-(A^{-1}b)^\top & c
\end{bmatrix}
\begin{bmatrix} u_d \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.
\]

The condition (F1) holds. The positivity condition \(B + B^\top \geq 2\mu_0 I\) is fulfilled if and only if \(c - \frac{1}{4} A^{-1} b \cdot b\) is uniformly positive.
Boundary conditions are enforced via a matrix valued boundary field:

\[ A_\nu := \sum_{k=1}^{d} \nu_k A_k \in L^\infty(\Gamma; M_r(C)), \]

where \( \nu = (\nu_1, \nu_2, \cdots, \nu_d) \) is the outward unit normal on \( \Gamma \), and \( M \in L^\infty(\Gamma; M_r(C)) \).

Boundary condition

\[ (A_\nu - M)u|_\Gamma = 0 \]

is sufficient for treatment of different types of usual boundary conditions.
Assumptions on boundary matrix $M$

We assume (for ae $x \in \Gamma$) \[ (\forall \xi \in C^r) \ (M(x) + M(x)^*)\xi \cdot \xi \geq 0, \]

\[ (FM2) \quad C^r = \ker\left(A_\nu(x) - M(x)\right) + \ker\left(A_\nu(x) + M(x)\right). \]

Such $M$ is called the admissible boundary condition.

The boundary problem: for given $f \in L^2(\Omega; C^r)$ find $u$ such that

$$\begin{cases} 
\mathcal{L}u = f \\
(A_\nu - M)u\big|_\Gamma = 0 \end{cases}.$$
Elliptic equation – different boundary conditions

\[
\begin{align*}
M &= \begin{bmatrix} 0 & 0 & -\nu_1 \\ 0 & 0 & -\nu_2 \\ \nu_1 & \nu_2 & 0 \end{bmatrix}, & \quad A_\nu - M &= \begin{bmatrix} 0 & 0 & 2\nu_1 \\ 0 & 0 & 2\nu_2 \\ 0 & 0 & 0 \end{bmatrix}, & \quad (A_\nu - M) \begin{bmatrix} p \\ u \end{bmatrix} |_\Gamma &= 0 \\
\nu \cdot (\nabla u) |_\Gamma &= 0, & \quad \nu \cdot (\nabla u) |_\Gamma + \alpha u |_\Gamma &= 0 \\
\end{align*}
\]

All above matrices \( M \) satisfy (FM).

\[
A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix}.
\]
Instead of
\[(A - M)u = 0 \text{ on } \Gamma,\]
Lax proposed boundary conditions with
\[u(x) \in N(x), \quad x \in \Gamma,\]
where \(N = \{N(x) : x \in \Gamma\}\) is a family of subspaces of \(C^r\).

Boundary problem:
\[
\begin{cases}
\mathcal{L}u = f \\
u(x) \in N(x), \quad x \in \Gamma
\end{cases}
\]
Assumptions on $N$

**maximal boundary conditions:** (for ae $x \in \Gamma$)

1. $(FX1)$ \[ N(x) \text{ is non-negative with respect to } A_\nu(x): \]
   \[ (\forall \xi \in N(x)) \quad A_\nu(x)\xi \cdot \xi \geq 0; \]

2. $(FX2)$ there is no non-negative subspace with respect to $A_\nu(x)$, which (properly) contains $N(x)$;

or

Let $N(x)$ and $\tilde{N}(x) := (A_\nu(x)N(x))^\perp$ satisfy (for ae $x \in \Gamma$)

1. $(FV1)$ \[ (\forall \xi \in N(x)) \quad A_\nu(x)\xi \cdot \xi \geq 0 \]

2. $(FV2)$ \[ \tilde{N}(x) = (A_\nu(x)N(x))^\perp \quad \text{and} \quad N(x) = (A_\nu(x)\tilde{N}(x))^\perp. \]
Equivalence of different descriptions of boundary conditions

**Theorem.** It holds

\[(FM1)-(FM2) \iff (FX1)-(FX2) \iff (FV1)-(FV2),\]

with

\[N(x) := \ker\left( A_\nu(x) - M(x) \right).\]

In fact, for a weak existence result some additional assumptions are needed [JR1994], [MJ2004].
Classical results on well-posedness

Friedrichs:
– uniqueness of the classical solution
– existence of a weak solution (under some additional assumptions)

Contributions (and particular cases):
C. Morawetz, P. Lax, L. Sarason, R. S. Phillips, J. Rauch, ...
– the meaning of traces for functions in the graph space,
– weak well-posedness results under additional assumptions (on $A_{\nu}$),
– regularity of solution,
– numerical treatment.

Shortcomings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

However, since the beginning of 21st century the numerical advantages of FS have overshadowed that.
New approach...


– abstract setting (operators on Hilbert spaces),
– intrinsic criterion for the bijectivity of *Friedrichs’* operator,
– avoiding the question of traces for functions in the graph space,
– investigation of different formulations of boundary conditions,

... and new open questions.

They considered only the real case.
Assumptions

Let $L$ be real (complex) Hilbert space ($L'$ is (anti)dual of $L$), $\mathcal{D} \subseteq L$ a dense subspace, and $T, \tilde{T} : \mathcal{D} \rightarrow L$ linear unbounded operators satisfying

\begin{align*}
&T1 \quad (\forall \varphi, \psi \in \mathcal{D}) \quad \langle T\varphi \mid \psi \rangle_L = \langle \varphi \mid \tilde{T}\psi \rangle_L, \\
&T2 \quad (\exists c > 0)(\forall \varphi \in \mathcal{D}) \quad \|(T + \tilde{T})\varphi\|_L \leq c\|\varphi\|_L, \\
&T3 \quad (\exists \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \quad \langle (T + \tilde{T})\varphi \mid \varphi \rangle_L \geq 2\mu_0\|\varphi\|^2_L.
\end{align*}

$(T, \tilde{T})$ is referred to as a joint pair of abstract Friedrichs operators.

Recall the Friedrichs operator: $\mathcal{D} := C_c^{\infty}(\Omega; C^r)$, $L = L^2(\Omega; C^r)$ and $T, \tilde{T} : \mathcal{D} \rightarrow L$ be defined by

\begin{align*}
T\mathbf{u} &:= \sum_{k=1}^{d} \partial_k(A_k\mathbf{u}) + \mathbf{B}\mathbf{u}, \\
\tilde{T}\mathbf{u} &:= -\sum_{k=1}^{d} \partial_k(A_k\mathbf{u}) + (\mathbf{B}^* + \sum_{k=1}^{d} \partial_k A_k)\mathbf{u},
\end{align*}

where $A_k$ and $\mathbf{B}$ are as above (they satisfy (F1)–(F2)).

Then $T$ and $\tilde{T}$ satisfy (T1)–(T3) ... fits in this framework.
Extension of operators, starting from \((T, \tilde{T}) = (T_1, \tilde{T}_1)\)

\(\mathcal{D}\) is an inner product space when equipped with *graph norm* stemming from

\[
\langle \cdot | \cdot \rangle_T := \langle \cdot | \cdot \rangle_L + \langle T \cdot | T \cdot \rangle_L.
\]

By \(W_0\) denote the completion of \(\mathcal{D}\) in the graph norm, same for \(\tilde{T}\) by (T2). \(W_0 \leq L\) by (T1), and both \(T\) and \(\tilde{T}\) extend to bounded operators from \(W_0\) to \(L\), which we denote by \((T_2, \tilde{T}_2)\).

The following embedding are dense and continuous (we have a Gel’fand triplet):

\[
W_0 \hookrightarrow L \equiv L' \hookrightarrow W'_0.
\]

Let \(T_3 := \tilde{T}_2' \in \mathcal{L}(L; W'_0)\) be the Banach adjoint of \(\tilde{T}_2 : W_0 \rightarrow L\), and \(\tilde{T}_3 := T_2'\). Thus we have defined \((T_3, \tilde{T}_3)\).

Note that the *graph space*

\[
W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\} \leq L
\]

is a Hilbert space with respect to \(\langle \cdot | \cdot \rangle_T\).

\((T_4, \tilde{T}_4)\) are defined as restrictions of \(T_3\) and \(\tilde{T}_3\) to \(W\).
This produces the maximal pair of abstract Friedrichs operators \((T_4, \tilde{T}_4)\), mapping \(T_4, \tilde{T}_4 : W \rightarrow L\), which are associated to the initial pair \((T, \tilde{T})\).

Find sufficient conditions for a subspace \(W_0 \leq V \leq W\) such that \(T_4|_V : V \rightarrow L\) is an isomorphism.

As the continuity in the graph norm holds for any restriction to a closed subspace \(V\) of \(W\), the key question is bijectivity.
Boundary operator

Sufficient conditions were obtained by [EGC2007] and [AB2010] using

**Boundary operator** $D \in \mathcal{L}(W; W')$:

$$W' \langle Du, v \rangle_W := \langle Tu | v \rangle_L - \langle u | \tilde{T}v \rangle_L, \quad u,v \in W.$$  

$D$ is symmetric: $W' \langle Du, v \rangle_W = W' \langle Dv, u \rangle_W$ and satisfies  

$$\ker D = W_0$$  

$$\text{im } D = W_0^0 := \{ g \in W' : (\forall u \in W_0) \quad W' \langle g, u \rangle_W = 0 \}.$$  

For a given joint pair of abstract FO $(T, \tilde{T})$, a pair $(V, \tilde{V})$ of subspaces of $W$ is said to allow the $(V)$-boundary conditions relative to $(T, \tilde{T})$ when:

(V1) the boundary operator has opposite sign on $V$ and on $\tilde{V}$, in the sense that  

$$(\forall u \in V) \quad W' \langle Du, u \rangle_W \geq 0,$$  

$$(\forall v \in \tilde{V}) \quad W' \langle Dv, v \rangle_W \leq 0;$$  

(V2) the image via $D$ of either space has, as annihilator, the other space, namely  

$V = D(\tilde{V})^0$ and $\tilde{V} = D(V)^0$.  

For classical Friedrichs operator

If $T$ is the Friedrichs operator $\mathcal{L}$, then for $u, v \in C_c^\infty(\mathbb{R}^d; \mathbb{C}^n)$ we have

$$w'\langle Du, v \rangle_W = \int_\Gamma A_\nu(x)u|_\Gamma(x) \cdot v|_\Gamma(x) dS(x).$$

With the assumptions:

(FV1)

$$(\forall \xi \in N(x)) \quad A_\nu(x)\xi \cdot \xi \geq 0,$$

$$(\forall \xi \in \tilde{N}(x)) \quad A_\nu(x)\xi \cdot \xi \leq 0,$$

(FV2)

$$\tilde{N}(x) = (A_\nu(x)N(x))^\perp \quad \text{and} \quad N(x) = (A_\nu(x)\tilde{N}(x))^\perp,$$

we are lead to consider subspaces $V$ and $\tilde{V}$ in the functional framework:

(V1)

$$(\forall u \in V) \quad w'\langle Du, u \rangle_W \geq 0,$$

$$(\forall v \in \tilde{V}) \quad w'\langle Dv, v \rangle_W \leq 0,$$

(V2)

$$V = D(\tilde{V})^0, \quad \tilde{V} = D(V)^0.$$
Well-posedness theorem

\[ [u | v] := \langle Du, v \rangle_W = \langle Tu | v \rangle_L - \langle u | \tilde{T}v \rangle_L , \quad u, v \in W \]

is an indefinite inner product on \( W \), and we consider subspaces \( V \) and \( \tilde{V} \) satisfying:

(V1) \[
(\forall v \in V) \quad [v | v] \geq 0 , \\
(\forall v \in \tilde{V}) \quad [v | v] \leq 0 ;
\]

(V2) \[
V = \tilde{V}^{[\perp]} , \quad \tilde{V} = V^{[\perp]} .
\]

([\perp] \text{ stands for } [ \cdot | \cdot ]\text{-orthogonal complement})

Theorem. Under assumptions \((T1) - (T3)\) and \((V1) - (V2)\), the operators \( T|_V : V \to L \) and \( \tilde{T}|_{\tilde{V}} : \tilde{V} \to L \) are isomorphisms.

In the real case \([EGC2007]\).
maximal boundary conditions: (for ae $x \in \Gamma$)

(FX1) \[(\forall \xi \in N(x)) \quad A_\nu(x)\xi \cdot \xi \geq 0,\]

FX2) there is no non-negative subspace with respect to $A_\nu(x)$, which contains $N(x)$,

subspace $V$ is maximal non-negative in $(W, [\cdot | \cdot])$:

(X1) $V$ is non-negative in $(W, [\cdot | \cdot])$: $(\forall u \in V) \quad [u | v] \geq 0$,

(X2) there is no non-negative subspace in $(W, [\cdot | \cdot])$ containing $V$. 

Correspondence — maximal b.c.
Correspondence — admissible b.c.

**admissible boundary condition:** there exists a matrix function \( M : \Gamma \rightarrow M_r(C) \) such that (for all \( x \in \Gamma \))

\[
\text{(FM1)} \quad (\forall \xi \in C^r) \quad (M(x) + M(x)^*)\xi \cdot \xi \geq 0 ,
\]

\[
\text{(FM2)} \quad C^r = \ker\left(A_{\nu}(x) - M(x)\right) + \ker\left(A_{\nu}(x) + M(x)\right).
\]

**abstract admissible boundary condition:** there exists \( M \in \mathcal{L}(W; W') \) such that

\[
\text{(M1)} \quad (\forall u \in W) \quad W'\langle (M + M^*)u, u \rangle_W \geq 0 ,
\]

\[
\text{(M2)} \quad W = \ker(D - M) + \ker(D + M) .
\]
Equivalence of different descriptions of b.c.

**Theorem.** (classical) It holds

\[(FM1)–(FM2) \iff (FV1)–(FV2) \iff (FX1)–(FX2),\]

with

\[N(x) := \ker(A_\nu(x) - M(x)).\]

**Theorem.** [EGC2007, AB2010] It holds

\[(M1)–(M2) \iff (V1)–(V2) \iff (X1)–(X2),\]

with

\[V := \ker(D - M).\]
Friedrichs systems
- Classical theory of Friedrichs systems
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Hilbert space framework
- Bijective realisations with signed boundary map
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- One dimensional example
Theorem. [Ern, Guermond, Caplain, 2007] Let \((T, \tilde{T})\) be a joint pair of Friedrichs systems and let \((V, \tilde{V})\) satisfy (V1)–(V2). Then \(T_4|_V : V \to L\) and \(\tilde{T}_4|_{\tilde{V}} : \tilde{V} \to L\) are closed bijective realisations of \(T\) and \(\tilde{T}\), respectively.

Can we say something more about extensions \(T_4, \tilde{T}_4\), and conditions (V)?

Theorem. \((T, \tilde{T})\) is a joint pair of abstract Friedrichs operators iff

(i) \(T \subseteq \tilde{T}^*\) and \(\tilde{T} \subseteq T^*\);

(ii) \(T + \tilde{T}\) is a bounded self-adjoint operator in \(L\) with strictly positive bottom;

(iii) \(\text{dom } T = \text{dom } \tilde{T} = W_0\) and \(\text{dom } T^* = \text{dom } \tilde{T}^* = W\).

In fact: \(T_4 = \tilde{T}^*\) and \(\tilde{T}_4 = T^*\).
Theorem. Let \((T, \tilde{T})\) be a pair of operators on the Hilbert space \(L\) satisfying conditions (T1)–(T2), and let \((V, \tilde{V})\) be a pair of subspaces of \(L\). Then (V2) is equivalent to

\begin{enumerate}
\item \(W_0 \subseteq V \subseteq W, \ W_0 \subseteq \tilde{V} \subseteq W\),
\item \(V\) and \(\tilde{V}\) are closed in \(W\), and
\item \((\tilde{T}^*|_V)^* = T^*|_\tilde{V}, \ (T^*|_\tilde{V})^* = \tilde{T}^*|_V\).
\end{enumerate}

We are seeking bijective closed operators \(S \equiv \tilde{T}^*|_V\) such that

\[
\overline{T} \subseteq S \subseteq \tilde{T}^*,
\]

and thus also \(S^*\) is bijective and \(\overline{T} \subseteq S^* \subseteq T^*\).

In the following we work with closed \(T\) and \(\tilde{T}\).

Let \((T, \tilde{T})\) be a joint pair of closed abstract Friedrichs operators on the Hilbert space \(L\). For a closed \(T \subseteq S \subseteq \tilde{T}^*\) such that \((\text{dom } S, \text{dom } S^*)\) satisfies (V1) we call \((S, S^*)\) an adjoint pair of bijective realisations with signed boundary map relative to \((T, \tilde{T})\).
Questions:

1) **Sufficient** conditions on $V$? ✓

2) **Existence** of $V \subseteq W$ such that $(\tilde{T}^*|_V, (\tilde{T}^*|_V)^*)$ is an adjoint pair of bijective realisations with signed boundary map relative to $(T, \tilde{T})$?

3) Existence of **infinitely many** such $V$?

4) **Classification** of all such $V$?
Existence of infinitely many $V$'s

**Theorem.** Let $(T, \tilde{T})$ be a joint pair of closed abstract Friedrichs operators on the Hilbert space $L$.

(i) There is an adjoint pair of bijective realisations with signed boundary map. Furthermore, there is an adjoint pair $(T_r, T_r^*)$ of bijective realisations with signed boundary map relative to $(T, \tilde{T})$ such that

$$W_0 + \ker T^* \subseteq \text{dom } T_r \quad \text{and} \quad W_0 + \ker \tilde{T}^* \subseteq \text{dom } T_r^*.$$ 

(ii) If both $\ker \tilde{T}^* \neq \{0\}$, $\ker T^* \neq \{0\}$, then $(T, \tilde{T})$ admits uncountably many adjoint pairs of bijective realisations with signed boundary map. Else, if either $\ker \tilde{T}^* = \{0\}$ or $\ker T^* = \{0\}$, then there is exactly one adjoint pair of bijective realisations with signed boundary map relative to $(T, \tilde{T})$. Such a pair is precisely $(\tilde{T}^*, \tilde{T})$ when $\ker \tilde{T}^* = \{0\}$, and $(T, T^*)$ when $\ker T^* = \{0\}$. 

\[\blacksquare\]
Start with a pair \((A_0, A'_0)\) of closed operators on \(L\) such that
\[
A_0 \subseteq (A'_0)^* =: A_1 \quad \text{and} \quad A'_0 \subseteq (A_0)^* =: A'_1,
\]
\(A_0 + A'_0\) is bounded on \(L\) and extends to an everywhere defined, bounded, self-adjoint operator in \(L\) with strictly positive bottom. We refer to any such \((A_0, A'_0)\) as a joint pair of closed abstract Friedrichs operators. This definition implies that
\[
\text{dom } A_0 = \text{dom } A'_0 =: W_0 \quad \text{and} \quad \text{dom } A_1 = \text{dom } A'_1 =: W.
\]

We are interested in restrictions \(A_1|_V\) and \(A'_1|_{\tilde{V}}\) onto suitable subspaces \(V\) and \(\tilde{V}\) of \(L\) which satisfy conditions (V1)–(V2). Equivalently, this is the class of restrictions such that
\[
W_0 \subseteq V \subseteq W \quad \text{and} \quad W_0 \subseteq \tilde{V} \subseteq W,
\]
which satisfy that \(A_1|_V\) and \(A'_1|_{\tilde{V}}\) are mutually adjoint (thus, in particular, \(A_1|_V\) and \(A'_1|_{\tilde{V}}\) are closed operators) and
\[
(\forall u \in V) \quad W'\langle Du, u \rangle_W = \langle A_1 u \mid u \rangle_L - \langle u \mid A'_1 u \rangle_L \geq 0,
\]
\[
(\forall v \in \tilde{V}) \quad W'\langle Dv, v \rangle_W = \langle A_1 v \mid v \rangle_L - \langle v \mid A'_1 v \rangle_L \leq 0.
\]
Grubb's universal classification (cont.)

We shall refer to any such pair \((A_1|_V, A'_1|_{\bar{V}})\) as an adjoint pair of bijective realisations with signed boundary map relative to the given joint pair of closed abstract Friedrichs operators \((A_0, A'_0) = (\overline{T}, \overline{T})\).

For their adjoints we have

\[
A_1 := (A'_0)^* = \overline{T}^* \quad \text{and} \quad A'_1 := (A_0)^* = T^*.
\]

It is immediate that there is a one-to-one correspondence between all pairs of isomorphisms induced by \((T, \overline{T})\), and all adjoint pairs of bijective realisations with signed boundary map relative to \((A_0, A'_0)\), i.e. \((\overline{T}, \overline{T})\).

Since \(A_R = A_1|_V\) is closed and bijective onto \(L\), then \((A_1|_V)^{-1}\) is necessarily everywhere defined and bounded, so we may also speak of \(A_1|_V\) as of an isomorphic realisation of \(A_0\) with signed boundary map.

It is worth remarking that the fact that a closed operator \(S\) satisfies \(A_0 \subseteq S \subseteq A_1\) is equivalent to \(A'_0 \subseteq S^* \subseteq A'_1\).

The interest towards such pairs \((A_1|_V, A'_1|_{\bar{V}})\) is two-fold: first, when \((V1)–(V2)\) hold, \(A_1|_V\) and \(A'_1|_{\bar{V}}\) are bijections onto \(L\), thus providing a sufficient criterion of well-posedness of the abstract Friedrichs system; moreover, \((V1)–(V2)\) encode the most relevant class of boundary conditions, as it may be seen from a large variety of concrete examples of boundary value problems on which such conditions are modelled.
Grubb’s universal classification (cont.)

Let $(A_0, A_0^*)$ and $(A_1, A_1^*)$ be two pairs of mutually adjoint, closed and densely defined operators in $L$, with properties as above, which admit a further pair $(A_r, A_r^*)$ of reference operators that are closed, satisfy $A_0 \subseteq A_r \subseteq A_1$, equivalently $A_0' \subseteq A_r^* \subseteq A_1'$, and are invertible with everywhere defined bounded inverses $A^{-1}_r$ and $(A_r^*)^{-1}$. Then there are decompositions

$$\text{dom } A_1 = \text{dom } A_r + \ker A_1 \quad \text{and} \quad \text{dom } A_1' = \text{dom } A_r^* + \ker A_1'$$

$$p_r = A_r^{-1} A_1, \quad p_r' = (A_r^*)^{-1} A_1', \quad p_k = I - p_r, \quad p_k' = I - p_r'.$$

There is a one-to-one correspondence between

$$(A, A^*) \begin{cases} \{ (B, B^*) \} \\
A_0 \subseteq A \subseteq A_1 \\
A_0' \subseteq A^* \subseteq A_1' \end{cases} \leftrightarrow \begin{cases} \{ (B, B^*) \} \\
\mathcal{V} \subseteq \ker A_1 \text{ closed} \\
\mathcal{W} \subseteq \ker A_1' \text{ closed} \\
B : \mathcal{V} \to \mathcal{W} \text{ densely defined} \end{cases}$$

$$B \mapsto A_B : \quad \text{dom } A_B = \left\{ u \in \text{dom } A_1 : p_k u \in \text{dom } B, \ P_W(A_1 u) = B(p_k u) \right\},$$

$$A \mapsto B_A : \quad \text{dom } B_A = p_k \text{dom } A, \quad \mathcal{V} = \overline{\text{dom } B_A}, \quad B_A(p_k u) = P_W(A_1 u),$$

where $P_W$ is the orthogonal projections from $L$ onto $\mathcal{W}$.

Important: $A$ is injective, resp. surjective, resp. bijective, if and only if so is $B$. 
Grubb’s universal classification (cont.)

When \( A_B \) corresponds to \( B \) as above, then

\[
\text{dom } A_B = \{ w_0 + (A_r)^{-1}(B\nu + \nu') + \nu \mid w_0 \in \text{dom } A_0 \\
&\& \nu \in \text{dom } B \& \nu' \in \ker A_1 \ominus \mathcal{W} \}
\]

\[
A_B(w_0 + (A_r)^{-1}(B\nu + \nu') + \nu) = A_0 w_0 + B\nu + \nu'
\]

We shall apply this theory on a joint pair of closed abstract Friedrichs systems.
Classification of bijective realisations with signed boundary map

For simplicity here we use the notation of Grubb's universal classification. 

\((A_0, A'_0)\) a joint pair of closed abstract Friedrichs operators, \(A_1 := (A'_0)^*\), \(A'_1 := A_0^*\), and let \((A_r, A^*_r)\) be an adjoint pair of bijective realisations with signed boundary map relative to \((A_0, A'_0)\).

\((A_B, A^*_B)\) a generic pair of closed extensions \(A_0 \subseteq A_B \subseteq A_1\).
Classification of bijective realisations with signed boundary map (cont.)

\[ \left( \forall \nu \in \text{dom } B \right) \quad \left( \forall \nu' \in \ker A_1' \ominus W \right) \]
\[ \left\{ \langle \nu | A_1' \nu \rangle_L - 2 \text{Re} \langle p_{k'} \nu | B \nu \rangle_L \leq 0 \right. \]
\[ \left. \langle p_{k'} \nu | \nu' \rangle_L = 0 \right\} \]

\[ \left( \forall \mu' \in \text{dom } B^* \right) \quad \left( \forall \mu \in \ker A_1 \ominus \mathcal{V} \right) \]
\[ \left\{ \langle A_1 \mu' | \mu' \rangle_L - 2 \text{Re} \langle B^* \mu' | p_{k} \mu' \rangle_L \leq 0 \right. \]
\[ \left. \langle \mu | p_{k} \mu' \rangle_L = 0 \right\} \]

**Theorem.** Any of the following three facts,

(a) conditions (1) and (2) hold true, or

(b) condition (1) holds true and \( B : \text{dom } B \to \mathcal{W} \) is a bijection, or

(c) condition (2) holds true and \( B^* : \text{dom } B^* \to \mathcal{V} \) is a bijection, is sufficient for \((A_B, A_B^*)\) to be another adjoint pair of bijective realisations with signed boundary map relative to \((A_0, A_0')\).

Assume further that \( \text{dom } A_r = \text{dom } A_r^* \). Then the following properties are equivalent:

(i) \((A_B, A_B^*)\) is another adjoint pair of bijective realisations with signed boundary map relative to \((A_0, A_0')\);

(ii) the mirror conditions (1) and (2) are satisfied.
Example: First order ode

Take \( L := L^2(0, 1) \), \( \mathcal{D} := C_c^\infty(0, 1) \) and define \( T, \tilde{T} : \mathcal{D} \to L \) by

\[
T \phi := \frac{d}{dx} \phi + \phi \quad \text{and} \quad \tilde{T} \phi := -\frac{d}{dx} \phi + \phi.
\]

We have

\[
\text{dom } \overline{T} = \text{dom } \overline{\tilde{T}} = H^1_0(0, 1) =: W_0
\]

\[
\text{dom } T^* = \text{dom } \tilde{T}^* = H^1(0, 1) =: W,
\]

Further define

\[
A_0 := \overline{T}, \quad A'_0 := \overline{\tilde{T}}, \quad A_1 := \tilde{T}^*, \quad A'_1 := T^*.
\]

As \( W \langle Du, v \rangle_W = u(1)v(1) - u(0)v(0) \), for

\[
V := \tilde{V} := \{ u \in H^1(0, 1) : u(0) = u(1) \},
\]

we have that \( A_r := A_1|_V, A^*_r = A'_1|_V \) for an adjoint pair of bijective realisations with signed boundary map.

As \( \ker A_1 = \text{span}\{e^{-x}\} \) and \( \ker A'_1 = \text{span}\{e^x\} \), so

\[
p_k u = -\frac{u(1) - u(0)}{1 - e^{-1}} e^{-x}, \quad p_k' u = \frac{u(1) - u(0)}{e - 1} e^x.
\]
Example (cont.)

The corresponding spaces are \( \mathcal{V} = \ker A_1 \), \( \mathcal{W} = \ker A'_1 \), while \( B_{\alpha,\beta} : \mathcal{V} \to \mathcal{W} \) is defined by

\[
B_{\alpha,\beta}e^{-x} = (\alpha + i\beta)e^x
\]

where \( (\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\} \).

(1) simplifies to check

\[
\langle e^{-x} \mid A'_1e^{-x} \rangle_L - 2\text{Re} \langle p_{k'}e^{-x} \mid B_{\alpha,\beta}e^{-x} \rangle_L \leq 0
\]

\( \iff \alpha \leq -e^{-1} \)

\[
\{(A_{\alpha,\beta}, A^*_{\alpha,\beta}) : \alpha \leq -e^{-1}, \beta \in \mathbb{R}\} \cup \{(A_r, A^*_r)\}
\]

\[
\text{dom } A^{(*)}_{\alpha,\beta} = \left\{ u \in H^1(0, 1) : \left(2e^{-1}-(+)\alpha(1+e) - i\beta(1+e)\right)u(1)
\right.
\]

\[
= \left(2 + \alpha(1+e) - (+)i\beta(1+e)\right)u(0) \right\}
\]
Thank you for your attention!