Complex Friedrichs systems and applications

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Abstract settings

- A. Ern, J. L. Guermond, G. Caplain (2007), N. Antonić, K. Burazin (2009, 2010, 2011), N.A., K. B., M. Vrdoljak (2013, 2014), K. B., M. Erceg (2016)...
- L - complex Hilbert space (L' antidual of L),
- $\mathcal{D} \subseteq L$ - dense subspace
- $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \to L$ linear unbounded operators satisfying

$$(\forall \varphi, \psi \in \mathcal{D}) \qquad \langle \mathcal{L}\varphi | \psi \rangle_L = \langle \varphi | \tilde{\mathcal{L}}\psi \rangle_L, \tag{T1}$$

$$(\exists c > 0)(\forall \varphi \in \mathcal{D}) \qquad \|(\mathcal{L} + \tilde{\mathcal{L}})\varphi\|_L \leq c\|\varphi\|_L, \tag{T2}$$

$$(\exists \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \quad \langle (\mathcal{L} + \tilde{\mathcal{L}})\varphi | \varphi \rangle_L \geq 2\mu_0 \|\varphi\|_L^2. \tag{T3}$$

Classical complex Freidrichs operator

Let $d, r \in \mathbf{N}$, $\Omega \subseteq \mathbf{R}^d$ open and bounded with Lipshitz boundary, $\mathcal{D} = \mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$ and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{C}))$ satisfying

$$(F1) \qquad \mathbf{A}_k = \mathbf{A}_k^*$$

$$(F2) \qquad (\exists \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geq 2\mu_0 I \qquad \text{(ae on } \Omega\text{)}.$$

Operators $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \to L$ defined as

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

$$\tilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^* \mathsf{u}) + (\mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k^*)\mathsf{u}$$

satisfy (T1)–(T3).

Operator $\mathcal{L}$ is called the symmetric positive operator or the Friedrichs operator and

$$\mathcal{L}\mathsf{u} = \mathsf{f}$$

the symmetric positive system or the Friedrichs system.

- introduced in K. O. Friedrichs: Symmetric positive linear differential equations, Communications on Pure and Applied Mathematics $\mathbf{11}$ (1958), 333-418
- goal: treatment of the equations of mixed type, such as the Tricommi equation

  $$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

  unified tretment of equations and systems of different type
- convenient for the numerical treatment

Formulation of the problem

- $(\mathcal{D}, \langle\,\cdot\mid\cdot\,\rangle_{\mathcal{L}})$ is an inner product space, where

$$\langle\,\cdot\mid\cdot\,\rangle_{\mathcal{L}} := \langle\,\cdot\mid\cdot\,\rangle_{L} + \langle\,\mathcal{L}\cdot\mid\mathcal{L}\cdot\,\rangle_{L}\,.$$

- $\|\cdot\|_{\mathcal{L}}$ is called the graph norm.
- W_0 - the completion of $\mathcal{D}$ in the graph norm $\ldots\ \mathcal{L}, \tilde{\mathcal{L}} \in \mathcal{L}(L; W_0')$
- $W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'$

Lemma

The graph space

$$W := \{u \in L : \mathcal{L}u \in L\} = \{u \in L : \tilde{\mathcal{L}}u \in L\}$$

is Hilbert space with respect to $\langle\,\cdot\mid\cdot\,\rangle_{\mathcal{L}}$.

Problem: for given $f \in L$ find $u \in W$ such that $\mathcal{L}u = f$.

Find sufficient conditions on $V \leq W$ such that $\mathcal{L}_{|V} : V \to L$ is an isomorphism.

Boundary operator

Boundary operator $D \in \mathcal{L}(W; W')$ is defined by

$$ {}_{W'}\langle Du, v \rangle_W := \langle\, \mathcal{L}u \,|\, v \,\rangle_L - \langle\, u \,|\, \tilde{\mathcal{L}}v \,\rangle_L \qquad u,\, v \in W. $$

Lemma

Under assumptions (T1)–(T2), operator D is selfadjoint

$$ {}_{W'}\langle Du, v \rangle_W = \overline{{}_{W'}\langle Dv, u \rangle_W} $$

and satisfies

$$ \begin{aligned} \ker D &= W_0 \\ \operatorname{im} D &= W_0^0 := \{g \in W' : (\forall u \in W_0) \quad {}_{W'}\langle g, u \rangle_W = 0\}. \end{aligned} $$

In particular, $\operatorname{im} D$ *is closed in* W'*.*

Different sets of boundary conditions

I. Let V and $\tilde{V}$ be subspaces of W that satisfy

$$\begin{aligned} (\forall u \in V) \quad {}_{W'}\langle Du, u\rangle_W \geq 0 \\ (\forall v \in \tilde{V}) \quad {}_{W'}\langle Dv, v\rangle_W \leq 0 \end{aligned} \tag{V1}$$

$$V = D(\tilde{V})^o, \qquad \tilde{V} = D(V)^o. \tag{V2}$$

II. A subspace V of W is maximal nonnegative if

$$(\forall u \in V) \qquad {}_{W'}\langle\, Du\,,\, u\,\rangle_W \geq 0, \tag{X1}$$

there is no subspace which is larger then V and satisfies (X1). (X2)

III. Let $M \in \mathcal{L}(W; W')$ be an operator satisfying

$$(\forall u \in W) \qquad {}_{W'}\langle\, (M + M^*)u\,,\, u\,\rangle_W \geq 0, \tag{M1}$$

$$W = \ker(D - M) + \ker(D + M). \tag{M2}$$

$$(V1) - (V2) \iff (X1) - (X2) \iff (M1) - (M2)$$

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there is no subspace which is larger then V and satisfies (X1). (X2)

III. Let $M \in \mathcal{L}(W; W')$ be an operator satisfying

$$(\forall u \in W) \qquad {}_{W'}\langle\, (M + M^*)u\,,\, u\,\rangle_W \geq 0, \tag{M1}$$

$$W = \ker(D - M) + \ker(D + M). \tag{M2}$$

$$(V1) - (V2) \quad \Longleftrightarrow \quad (X1) - (X2) \quad \Longleftrightarrow \quad (M1) - (M2)$$

Well-posedness theorem

Theorem

Let L be complex Hilbert space, $\mathcal{D}$ its dense subspace and define linear operators $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \to L$ satisfying

$$(\forall \varphi, \psi \in \mathcal{D}) \qquad \langle \mathcal{L}\varphi | \psi \rangle_L = \langle \varphi | \tilde{\mathcal{L}}\psi \rangle_L, \tag{T1}$$

$$(\exists c > 0)(\forall \varphi \in \mathcal{D}) \qquad \|(\mathcal{L} + \tilde{\mathcal{L}})\varphi\|_L \leq c\|\varphi\|_L, \tag{T2}$$

$$(\exists \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \quad \langle (\mathcal{L} + \tilde{\mathcal{L}})\varphi | \varphi \rangle_L \geq 2\mu_0 \|\varphi\|_L^2. \tag{T3}$$

Let $V, \tilde{V} \leq W$ satisfy

$$\begin{aligned} (\forall u \in V) \quad & {}_{W'}\langle Du, u \rangle_W \geq 0 \\ (\forall v \in \tilde{V}) \quad & {}_{W'}\langle Dv, v \rangle_W \leq 0 \end{aligned} \tag{V1}$$

$$V = D(\tilde{V})^o, \qquad \tilde{V} = D(V)^o. \tag{V2}$$

Operators $\mathcal{L}_{|V} : V \to L$ and $\tilde{\mathcal{L}}_{|\tilde{V}} : \tilde{V} \to L$ are isomorphisms.[1]

[1] **In real case: [AE&JLG&GC2007].**

Non-stationary complex Friedrichs systems

Consider abstract Cauchy problem

$$\begin{cases} \mathsf{u}'(t) + \mathcal{L}\mathsf{u}(t) = \mathsf{f} \\ \mathsf{u}(0) = \mathsf{u}_0, \end{cases}$$

where $\mathsf{u} : [0, T\rangle \to L$, $T > 0$ is the unknown function, $\mathsf{f} : \langle 0, T\rangle \to L$, $\mathsf{u}_0 \in L$ and $\mathcal{L}$ is abstract Friedrichs operator that satisfy (T1)–(T2) and

$$(\forall \varphi \in \mathcal{D}) \qquad \mathrm{Re} \left\langle (\mathcal{L} + \tilde{\mathcal{L}})\varphi \,|\, \varphi \right\rangle_L \geq 0. \tag{T3'}$$

Let $V \leq W$ satisfy (V1)–(V2). Then the following is valid

Theorem

$-\mathcal{L}_{|V}$ is the infinitesimal generator of a contraction C_0-semigroup $(T(t))_{t\geq 0}$ on L.[2]

[2] **In real case: [BE2016]**.

Friedrichs systems in H^s spaces

Let $s \in \mathbf{R}$, $L = \mathrm{H}^s(\mathbf{R}^d; \mathbf{C}^r)$, $\mathcal{D} = \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ and assume that constant matrices $\mathbf{C}$, $\mathbf{A}_k$, $k \in 1..d$, satisfy (F1) and (F2):

$$\mathbf{A}_k = \mathbf{A}_k^* \; ,$$

$$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* \geq 2\mu_0 \mathbf{I} \qquad (\text{ae on } \Omega) \; .$$

Operators

$$\mathcal{L}\mathsf{u} := \sum_{\mathsf{k}=1}^{\mathsf{d}} \partial_{\mathsf{k}}(\mathbf{A}_{\mathsf{k}}\mathsf{u}) + \mathbf{C}\mathsf{u}$$

and

$$\tilde{\mathcal{L}}\mathsf{u} := -\sum_{\mathsf{k}=1}^{\mathsf{d}} \partial_{\mathsf{k}}(\mathbf{A}_{\mathsf{k}}\mathsf{u}) + \mathbf{C}^*\mathsf{u}$$

satisfy (T1)–(T3), boundary operator D is trival and $V = \tilde{V} = W$.

Linear Dirac system

Consider system of equations

$$\gamma^0 \partial_t \psi + \gamma^1 \partial_1 \psi + \gamma^2 \partial_2 \psi + \gamma^3 \partial_3 \psi + \mathbf{B}\psi = \mathsf{f}, \tag{1}$$

where $\psi : [0, T\rangle \times \mathbf{R}^3 \to \mathbf{C}^4$ is unknown function, f : $\langle 0, \mathrm{T}\rangle \to \mathbf{C}^4$, $\mathbf{B} = \begin{bmatrix} b_1\mathbf{I} & 0 \\ 0 & b_2\mathbf{I} \end{bmatrix}$ for $b_1, b_2 : \mathbf{R}^3 \to \mathbf{C}$ and 2×2 unit matrix $\mathbf{I}$, while

$$\gamma^0 = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \quad k = 1, 2, 3.$$

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

System (1) can be written as evolution Friedrichs system

$$\partial_t \psi + \mathcal{L}\psi = \mathsf{F},$$

where $\mathsf{F} = \gamma^0 \mathsf{f}$ and $\mathcal{L}\psi = \sum_{k=1}^{3} \mathbf{A}_k \partial_k \psi + \mathbf{C}\psi$ for

$$\mathbf{A}_k = \begin{bmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{bmatrix}, \qquad \mathbf{C} = \gamma^0 \mathbf{B}.$$

Spaces involved:

$$\begin{aligned}
\mathcal{D} &= \mathrm{C}_c^\infty(\mathbf{R}^3; \mathbf{C}^4) \\
L &= \mathrm{L}^2(\mathbf{R}^3; \mathbf{C}^4), (\text{or } \mathrm{H}^s(\mathbf{R}^3; \mathbf{C}^4)) \\
W &= \{\mathsf{u} \in \mathrm{L}^2(\mathbf{R}^3; \mathbf{C}^4) : \sum_{k=1}^{3} \mathbf{A}_k \partial_k \mathsf{u} \in \mathrm{L}^2(\mathbf{R}^3; \mathbf{C}^4)\}
\end{aligned}$$

D is trivial, $V = \tilde{V} = W$.

Dirac-Klein-Gordon system

$$\begin{cases} -i(\gamma^0\partial_t + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3 + M)\psi = \phi\psi \\ \partial_t^2\phi - \Delta\phi + m^2\phi = \psi^*\gamma^0\psi \end{cases} \tag{2}$$

where unknown functions are $\psi = \psi(t,x) : \mathbf{R}^{1+3} \to \mathbf{C}^4$ and $\phi : \mathbf{R}^{1+3} \to \mathbf{R}$, while $M, m \geq 0$ and $\gamma^k,\ k = 1..3$ are same as in previous example.

Remark

For two Friedrichs systems

$$\begin{aligned} \partial_t \mathsf{u}_1 + \mathcal{L}_1\mathsf{u}_1 &= \mathsf{f}_1 \\ \partial_t \mathsf{u}_2 + \mathcal{L}_2\mathsf{u}_2 &= \mathsf{f}_2 \end{aligned}$$

system

$$\partial_t \mathsf{u} + \mathcal{L}\mathsf{u} = \mathsf{f}$$

is also a Friedrichs system with $\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{bmatrix}$, $\mathsf{u} = \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \end{bmatrix}$, $\mathsf{f} = \begin{bmatrix} \mathsf{f}_1 \\ \mathsf{f}_2 \end{bmatrix}$.

For the second system of equations in (2) we introduce

$$\mathsf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \mathsf{v}_3 \end{bmatrix} = \begin{bmatrix} \phi \\ \partial_t \phi \\ -\nabla \phi \end{bmatrix}$$

in order to get an evolution Friedrichs system

$$\partial_t \mathsf{v} + \mathcal{L}_2 \mathsf{v} = \mathsf{f}_2,$$

where $\mathcal{L}_2 \mathsf{v} = \sum_{\mathsf{k}=1}^{3} \partial_{\mathsf{k}} \mathbf{A}_{\mathsf{k}} \mathsf{v} + \mathbf{C}_2 \mathsf{v}, \;\; \mathsf{f}_2 = \begin{bmatrix} 0 \\ |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Dirac-Maxwell system

$$\begin{cases} -\frac{i}{2\pi}(\gamma^0\partial_t + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3)\psi + m\beta\psi = \sum_{k=0}^{3}\mathcal{A}_k\gamma^k\psi \\ \qquad (-\frac{\partial^2}{\partial t} + \Delta)\mathcal{A}_k = -\gamma^k\psi\cdot\psi, \quad k = 0..3, \end{cases} \tag{3}$$

where $\gamma^0 = \mathbf{I}$ and γ^k, $k = 1, 2, 3$ as before. Unknown functions are $\psi : \mathbf{R}^{1+3} \to \mathbf{C}^4$ and $\mathcal{A} = \begin{bmatrix}\mathcal{A}_0 & \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3\end{bmatrix}^\top$, while $m \geq 0$ and $\beta = \begin{bmatrix}\mathbf{I} & 0 \\ 0 & -\mathbf{I}\end{bmatrix}$. Analog procedure as in previous example gives us Friedrichs system

$$\partial_t \mathsf{u} + \mathcal{L}\mathsf{u} = \mathsf{F},$$

where $\mathsf{u} = \begin{bmatrix}\psi \\ \mathsf{v}_0 \\ \mathsf{v}_1 \\ \mathsf{v}_2 \\ \mathsf{v}_3\end{bmatrix}$, $\mathsf{v}_\mathsf{k} = \begin{bmatrix}\mathcal{A}_k \\ \partial_t\mathcal{A}_k \\ -\nabla\mathcal{A}_k\end{bmatrix}$, $\mathsf{f}_\mathsf{k} = \begin{bmatrix}0 \\ \gamma^k\psi\cdot\psi \\ 0 \\ 0 \\ 0\end{bmatrix}$, $\mathsf{k} = 0..3$

Thank you for your attention!

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