Variant of Optimality Criteria Method for Multiple State Optimal Design Problems

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Stationary diffusion equation

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded, $\mathbf{A} = \mathbf{A}^\top \in \mathrm{L}^\infty(\Omega; \mathcal{M}_{\alpha,\beta})$,

$$\mathcal{M}_{\alpha,\beta} := \left\{ \mathbf{A} \in \mathrm{M}_d(\mathbf{R}) : (\forall \xi \in \mathbf{R}^d) \quad \mathbf{A}\xi \cdot \xi \geq \alpha|\xi|^2, \mathbf{A}^{-1}\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \right\}$$

and $f \in \mathrm{H}^{-1}(\Omega)$. Stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases}$$

Ω - mixture of two isotropic materials with conductivities $0 < \alpha < \beta$:

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I},$$

where $\chi \in \mathrm{L}^\infty(\Omega; \{0,1\})$.

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and $f \in \mathrm{H}^{-1}(\Omega)$. Stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

Ω - mixture of two isotropic materials with conductivities $0 < \alpha < \beta$:

$$\mathbf{A} = \chi \alpha \mathbf{I} + (1 - \chi)\beta \mathbf{I},$$

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Optimal design problem (single state)

$$\left\{\begin{array}{l} J(\chi) = \int_\Omega [\chi(\mathbf{x}) g_\alpha(\mathbf{x}, u(\mathbf{x})) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x}, u(\mathbf{x}))] \, d\mathbf{x} \longrightarrow \min\,, \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})\,, \; \int_\Omega \chi \, d\mathbf{x} = q_\alpha\,, \end{array}\right.$$

where $\Omega \subseteq \mathbf{R}^d$ open and bounded, $f \in \mathrm{H}^{-1}(\Omega)$, $0 < \alpha < \beta$, $0 < q_\alpha < |\Omega|$, and g_α, g_β Caratheodory functions which satisfies growth condition

$$g_j(x, u) \leq a|u|^s + b(x), \qquad j = \alpha, \beta,$$

for some $a > 0$, $b \in \mathrm{L}^1(\Omega)$ and $1 \leq s < \frac{2d}{d-2}$, and u is the solution of the state equation

$$\left\{\begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{array}\right.$$

Definition (Composite material)

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0,1\})$ and conductivities $\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1-\chi_\varepsilon(x))\beta\mathbf{I}$ satisfy

$$\begin{array}{ccc} \chi_\varepsilon & \stackrel{*}{\rightharpoonup} & \theta \\ \mathbf{A}^\varepsilon & \stackrel{H}{\rightharpoonup} & \mathbf{A}, \end{array}$$

then it is said that $\mathbf{A}$ is homogenised tensor of two-phase composite material with proportions θ of first material and microstructure defined by the sequence (χ_ε).

Definition (H-convergence)

A sequence of matrix functions $\mathbf{A}^\varepsilon$ is said to H-converge to $\mathbf{A}$ if for every f the sequence of solutions of

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}^\varepsilon \nabla u_\varepsilon) = f \\ u_\varepsilon \in \mathrm{H}_0^1(\Omega) \end{array} \right.$$

satisfies $u_\varepsilon \rightharpoonup u$ in $\mathrm{H}_0^1(\Omega)$, $\mathbf{A}^\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbf{A}\nabla u$ in $\mathrm{L}^2(\Omega; \mathbf{R}^d)$, where u is the solution of the homogenised equation

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A} \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega)\,. \end{array} \right.$$

Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A} = \mathrm{diag}(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \ldots, \lambda_\theta^+)\,,$$

where

$$\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\,.$$

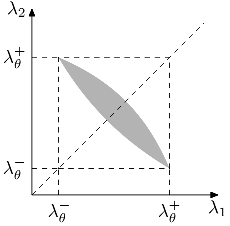
Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathcal{M}_{\alpha,\beta}) : \ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e. } \}$$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}$

$\mathcal{K}(\theta)$ is given in terms of eigenvalues (Murat & Tartar; Lurie & Cherkaev):

2D:

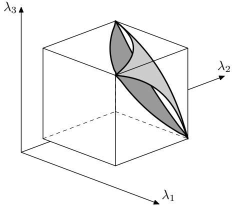


$$\lambda_\theta^- \leq \quad \lambda_j \quad \leq \lambda_\theta^+ \quad j = 1, \dots, d$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \quad \leq \quad \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \quad \leq \quad \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+},$$

3D:



Original problem:

$$\left\{\begin{array}{c} J(\chi) = \int_\Omega \chi(\mathbf{x}) g_\alpha(\mathbf{x}, u) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x}, u)\, d\mathbf{x} \longrightarrow \min\,, \\ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})\,,\ \int_\Omega \chi\, d\mathbf{x} = q_\alpha\,. \end{array}\right.$$

Generalized objective function is

$$J(\theta, \mathbf{A}) = \int_\Omega [\theta(\mathbf{x}) g_\alpha(\mathbf{x}, u(\mathbf{x})) + (1-\theta)(\mathbf{x})) g_\beta(\mathbf{x}, u(\mathbf{x}))]\, d\mathbf{x}$$

where u is the solution of the state equation

$$\left\{\begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,, \end{array}\right.$$

and relaxed problem is

$$\left\{\begin{array}{c} J(\theta, \mathbf{A}) \longrightarrow \min\,, \\ (\theta, \mathbf{A}) \in \mathcal{A}\,,\ \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,. \end{array}\right.$$

Multiple state optimal design problem

State equations

$$\left\{ \begin{array}{l} -\mathrm{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}_0^1(\Omega) \end{array} \right. \qquad i = 1, \ldots, m$$

State function u $= (u_1, \ldots, u_m)$

$$\left\{ \begin{array}{l} J(\chi) = \int_\Omega \chi(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u})\, d\mathbf{x} \longrightarrow \min\,, \\ \qquad \mathsf{u} = (u_1, \ldots, u_m) \text{ state function for } \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} \\ \qquad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\})\,, \ \displaystyle\int_\Omega \chi\, d\mathbf{x} = q_\alpha\,. \end{array} \right.$$

Relaxed problem:

$$\left\{ \begin{array}{l} J(\theta, \mathbf{A}) = \displaystyle\int_\Omega (\theta(\mathbf{x}) g_\alpha(\mathbf{x}, \mathsf{u}) + (1-\theta(\mathbf{x})) g_\beta(\mathbf{x}, \mathsf{u}))\, d\mathbf{x} \longrightarrow \min \\ (\theta, \mathbf{A}) \in \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1] \times \mathcal{M}_{\alpha,\beta}) : \mathbf{A} \in \mathcal{K}(\theta) a.e.\}\,, \ \displaystyle\int_\Omega \theta\, d\mathbf{x} = q_\alpha\,. \end{array} \right.$$

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Optimality condition

- Let $(\theta^*, \mathbf{A}^*)$ be optimal point of the relaxed problem and $\varepsilon \mapsto (\theta^\varepsilon, \mathbf{A}^\varepsilon)$ be a smooth path with $(\theta^0, \mathbf{A}^0) = (\theta^*, \mathbf{A}^*)$.
- If $\varepsilon \mapsto J(\theta^\varepsilon, \mathbf{A}^\varepsilon)$ is smooth, then necessary condition of optimality is

$$\delta J(\theta^*, \mathbf{A}^*) := \frac{d}{d\epsilon} J(\theta^\varepsilon, \mathbf{A}^\varepsilon)_{|_{\varepsilon=0}} \geq 0$$

Let us introduce adjoint states $p_1, \ldots, p_m$ as solutions of

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla p_i) = \theta \dfrac{\partial g_\alpha}{\partial u_i}(\cdot, \mathsf{u}) + (1-\theta)\dfrac{\partial g_\beta}{\partial u_i}(\cdot, \mathsf{u}) \\ p_i \in \mathrm{H}_0^1(\Omega) \end{cases} \qquad i = 1, \ldots, m\,.$$

Then

$$\delta J(\theta^*, \mathbf{A}^*) = \int_\Omega \delta\theta(\mathbf{x})[g_\alpha(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) - g_\beta(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) + l]\, d\mathbf{x} - \int_\Omega \sum_{i=1}^m \delta\mathbf{A}\nabla u_i^* \cdot \nabla p_i^*\, d\mathbf{x}\,,$$

for any admissible variation $(\delta\theta, \delta\mathbf{A}) := \left(\dfrac{d\theta^\varepsilon}{d\varepsilon}, \dfrac{d\mathbf{A}^\varepsilon}{d\varepsilon}\right)\Big|_{\varepsilon=0}$.

Optimality conditions

How to choose a smooth path?

1. Let us fix θ^*.
   a) Then set $\mathcal{K}(\theta^*)$ is convex and we can choose segment

$$\mathbf{A}^\varepsilon = \mathbf{A}^* + \varepsilon(\mathbf{A} - \mathbf{A}^*), \qquad \mathbf{A} \in \mathcal{K}(\theta^*)$$

$$\vdots$$

[L.Tartar, G. Allaire, M. Vrdoljak]

   b) We choose segment in terms of inverse matrices

$$(\mathbf{A}^\varepsilon)^{-1} = (\mathbf{A}^*)^{-1} + \varepsilon(\mathbf{A}^{-1} - (\mathbf{A}^*)^{-1}), \qquad \mathbf{A}^{-1} \in \tilde{\mathcal{K}}(\theta^*)$$

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Sets $\mathcal{K}(\theta)$ and $\tilde{\mathcal{K}}(\theta)$

One can show equvalence of the following sets

$\mathcal{K}(\theta)$ - Set of all symmetric matrices $\mathbf{A}$ with eigenvalues $\lambda_1, \ldots, \lambda_d$ which satisfies inequalities

$$\lambda_\theta^- \leq \lambda_j \leq \lambda_\theta^+, \quad j = 1, \ldots, d$$
$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \leq \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}$$
$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \leq \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+}$$

$\tilde{\mathcal{K}}(\theta)$ - Set of all symmetric matrices $\mathbf{A}^{-1}$ with eigenvalues $\nu_1, \ldots, \nu_d$ which satisfies inequalities

$$\nu_\theta^+ \leq \nu_j \leq \nu_\theta^-, \quad j = 1, \ldots, d$$
$$\sum_{j=1}^{d} \frac{1}{\alpha^{-1} - \nu_j} \leq \frac{1}{\alpha^{-1} - \nu_\theta^-} + \frac{d-1}{\alpha^{-1} - \nu_\theta^+}$$
$$\sum_{j=1}^{d} \frac{1}{\nu_j - \beta^{-1}} \leq \frac{1}{\nu_\theta^- - \beta^{-1}} + \frac{d-1}{\nu_\theta^+ - \beta^{-1}}$$

$$\mathbf{A} \in \mathcal{K}(\theta) \Leftrightarrow \mathbf{A}^{-1} \in \tilde{\mathcal{K}}(\theta)$$

Optimality conditions

From necessary condition of optimality

$$-\int_\Omega \sum_{i=1}^{m} \delta\mathbf{A}\nabla u_i^* \cdot \nabla p_i^* \, d\mathbf{x} \ \geq 0$$

then follows (for a.e. in Ω)

$$(\mathbf{A}^*)^{-1} : \mathbf{N} = \min_{\mathbf{A}\in\mathcal{K}(\theta^*)} \mathbf{A}^{-1} : \mathbf{N},$$

where $\mathbf{N} =$ Sym $\sum_{i=1}^{m} \sigma_i^* \otimes \tau_i^*$, $\sigma_i^* = \mathbf{A}^*\nabla u_i^*$ and $\tau_i^* = \mathbf{A}^*\nabla p_i^*$, $i = 1, \ldots, m$.

Theorem (von Neumann)

For symmetric matrices $\mathbf{A}$ and $\mathbf{M}$ following inequality is valid

$$A : M \leq \lambda(\mathbf{A}) \cdot \lambda(\mathbf{M}),$$

where $\lambda(\mathbf{A})$ and $\lambda(\mathbf{M})$ are vectors of eigenvalues of $\mathbf{A}$ and $\mathbf{M}$ in nondecreasing order. Equality holds if and only if $\mathbf{A}$ and $\mathbf{M}$ are diagonalizable in same basis.

$$\begin{cases} \displaystyle\sum_{i=1}^{d} \nu_i \eta_i \longrightarrow \min, \\ \nu_\theta^+ \leq \nu_j \leq \nu_\theta^- \ , \quad j = 1, \dots, d \\ \displaystyle\sum_{j=1}^{d} \frac{1}{\alpha^{-1} - \nu_j} \leq \frac{1}{\alpha^{-1} - \nu_\theta^-} + \frac{d-1}{\alpha^{-1} - \nu_\theta^+} \\ \displaystyle\sum_{j=1}^{d} \frac{1}{\nu_j - \beta^{-1}} \leq \frac{1}{\nu_\theta^- - \beta^{-1}} + \frac{d-1}{\nu_\theta^+ - \beta^{-1}} \end{cases}$$

where ν_i are increasing, while η_i decreasing for $i = 1..d$

Optimality conditions

2. Let us now consider variation of θ. We take smooth path $(\theta^\varepsilon, \mathbf{A}^\varepsilon)$ such that a.e. $\mathbf{x} \in \Omega$

$$(\mathbf{A}^\varepsilon)^{-1}(\mathbf{x}) : \mathbf{N}(\mathbf{x}) = \min_{\mathbf{A} \in \mathcal{K}(\theta^\varepsilon)} (\mathbf{A}^{-1}(\mathbf{x}) : \mathbf{N}(\mathbf{x}))$$

Again, from neccesary condition of optimality follows (a.e. $\mathbf{x} \in \Omega$)

$$\delta\theta(\mathbf{x}) \left(g_\alpha(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) - g_\beta(\mathbf{x}, \mathsf{u}^*(\mathbf{x})) + l + \frac{\partial g}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{N}(\mathbf{x})) \right) \geq 0.$$

Theorem

Let $(\theta^*, \mathbf{A}^*)$ *be minimizer of objective functional* $J(\theta, \mathbf{A})$. *We introduce symmetric matrix* $\mathbf{N} = \mathit{Sym}\ \sum_{i=1}^{m} \sigma_i^* \otimes \tau_i^*$, *for* $\sigma_i^* = \mathbf{A}\nabla u_i^*$, $\tau_i^* = \mathbf{A}\nabla p_i^*$ *and define function* $g(\theta, \mathbf{N}) = \min_{\mathbf{A}\in\mathcal{K}(\theta)} (\mathbf{A}^{-1} : \mathbf{N})$. *Then*

$$(\mathbf{A}^*)^{-1}(\mathbf{x}) : \mathbf{N}(\mathbf{x}) = g(\theta^*(\mathbf{x}), \mathbf{N}(\mathbf{x})), \qquad \textit{a.e}\, x \in \Omega.$$

Moreover, if we define function

$$R(\mathbf{x}) = g_\alpha(\mathbf{x}, \mathsf{u}) - g_\beta(\mathbf{x}, \mathsf{u}) + l + \frac{\partial g}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{N}^*(\mathbf{x}))$$

the optimal θ^* *satisfies (a. e. on* Ω*)*

$$\begin{array}{rl} \theta^*(\mathbf{x}) = 0 & \textit{if } R(\mathbf{x}) > 0 \\ \theta^*(\mathbf{x}) = 1 & \textit{if } R(\mathbf{x}) < 0 \\ 0 \le \theta^*(\mathbf{x}) \le 1 & \textit{if } R(\mathbf{x}) = 0\,. \end{array}$$

Algorithm

Take some initial θ^0 and $\mathbf{A}^0$. For k from 1 to N:

1. Calculate u_i^k, $i = 1, \ldots, m$, the solution of

$$\begin{cases} -\text{div}\,(\mathbf{A}^k \nabla u_i) = f_i \\ u \in \mathrm{H}_0^1(\Omega)\,. \end{cases}$$

2. Calculate p_i^k, $i = 1, \ldots, m$, the solution of

$$\begin{cases} -\text{div}\,(\mathbf{A}^k \nabla p_i) = \theta^k \dfrac{\partial g_\alpha}{\partial u_i}(\cdot, \mathsf{u}) + (1-\theta^k)\dfrac{\partial g_\beta}{\partial u_i}(\cdot, \mathsf{u}) \\ p_i \in \mathrm{H}_0^1(\Omega), \end{cases}$$

$$\sigma_i^k = \mathbf{A}^k \nabla u_i^k,\ \tau_i^k = \mathbf{A}^k \nabla p_i^k \text{ and } \mathbf{N}^k = \mathrm{Sym} \sum_{i=1}^m (\sigma_i^k \otimes \tau_i^k)$$

3. For $\mathbf{x} \in \Omega$, let $\theta^{k+1}(\mathbf{x})$ be the zero of function

$$\theta \mapsto g_\alpha(\mathbf{x}, \mathsf{u}^k(\mathbf{x})) - g_\alpha(\mathbf{x}, \mathsf{u}^k(\mathbf{x})) + l + \frac{\partial g}{\partial \theta}(\theta^k, \mathbf{N}^k(\mathbf{x}))\,,$$

   if a zero doesn't exist, take 0 (or 1) in case the function is positive (or < 0).
4. $(\mathbf{A}^{k+1})^{-1}(\mathbf{x})$ be the minimizer in the definition of $g(\theta^{k+1}(\mathbf{x}), \mathbf{N}^k(\mathbf{x}))$.

Theorem (d=3)

For given $\theta \in [0,1]$ *and matrix* $\mathbf{N}$ *with eigenvalues* $\eta_1 \geq \eta_2 \geq \eta_3$ *we have*

A. *If* $\eta_3 = 0$, *then* $\dfrac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \dfrac{\beta - \alpha}{(\theta\alpha + (1-\theta)\beta)^2}(\eta_1 + \eta_2)$.

B. *If* $\eta_3 > 0$, *then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} \beta^{-1}(\beta-\alpha)(\alpha+2\beta)\left(\dfrac{\sqrt{\eta_1}+\sqrt{\eta_2}+\sqrt{\eta_3}}{2\theta(\alpha-\beta)+\alpha+2\beta}\right)^2, & \theta < \theta_1^B, \\ \beta^{-1}(\beta^2-\alpha^2)\left(\dfrac{\sqrt{\eta_2}+\sqrt{\eta_3}}{\theta(\alpha-\beta)+\alpha+\beta}\right)^2 + \eta_1\dfrac{(\beta-\alpha)}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_1^B \leq \theta < \theta_2^B, \\ (\alpha^{-1}-\beta^{-1})\eta_3 + \dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \geq \theta_2^B, \end{cases}$$

where $\theta_1^B = 1 - \dfrac{\alpha(2\sqrt{\eta_1}-\sqrt{\eta_2}-\sqrt{\eta_3})}{(\beta-\alpha)(\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1})}$ *i* $\theta_2^B = 1 - \dfrac{\alpha(\sqrt{\eta_2}-\sqrt{\eta_3})}{(\beta-\alpha)\sqrt{\eta_3}}$.

C. *Let* $\eta_3 < 0$. *If* η_2 *and* η_3 *are negative as well then*

$$\frac{\partial g}{\partial \theta}(\theta, \mathbf{N}) = \begin{cases} -\alpha^{-1}(\beta-\alpha)(2\alpha+\beta)\left(\dfrac{\sqrt{-\eta_1}+\sqrt{-\eta_2}+\sqrt{-\eta_3}}{2\theta(\alpha-\beta)+3\beta}\right)^2, & \theta > \theta_1^C, \\ -\alpha^{-1}(\beta^2-\alpha^2)\left(\dfrac{\sqrt{-\eta_2}+\sqrt{-\eta_3}}{\theta(\alpha-\beta)+2\beta}\right)^2 + \eta_1\dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_2^C < \theta \leq \theta_1^C, \\ (\alpha^{-1}-\beta^{-1})\eta_3 + \dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \leq \theta_2^C. \end{cases}$$

where $\theta_1^C = \dfrac{\beta(\sqrt{-\eta_2}+\sqrt{-\eta_3}-2\sqrt{-\eta_1})}{(\beta-\alpha)(\sqrt{-\eta_2}+\sqrt{-\eta_3}-\sqrt{-\eta_1})}$ *and* $\theta_2^C = \dfrac{\beta(\sqrt{-\eta_3}-\sqrt{-\eta_2})}{(\beta-\alpha)\sqrt{-\eta_3}}$.

If $\eta_2 < 0$ *and* $\eta_1 \geq 0$, *then* θ_1^C *is not defined so we can express* $\dfrac{\partial g}{\partial \theta}(\theta, \mathbf{N})$ *by the formulae given above, but omitting its first case and the assumption* $\theta \leq \theta_1^C$ *in the second case.*

If $\eta_2 \geq 0$ *then both* θ_1^C *and* θ_2^C *are not defined, and* $\dfrac{\partial g}{\partial \theta}(\theta, \mathbf{N})$ *is given by the formula given in the third case above, for any* $\theta \in [0,1]$.

Thank you for your attention!