Defect distributions related to classes of Λ-type weights

Ivana Vojnović

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Introduction

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- H-measures, micro-local defect measures (Tartar, Gérard, around 1990.) - L^2 space
- H-distributions (Antonić, Mitrović, 2011.) - $L^p - L^q$ spaces, $p = \dfrac{q}{q-1}$, $1 < p < \infty$

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Existence of H-measure (Tartar, [7])

There exists a subsequence $(u_{n'})_{n'}$ and a complex Radon measure μ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ s. t. for all $\varphi_1(x), \varphi_2(x) \in C_0(\mathbb{R}^d)$, $\psi(\xi) \in C(\mathbb{S}^{d-1})$ we have that

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'})(\xi)\overline{\mathcal{F}(\varphi_2 u_{n'})}(\xi)\psi\left(\frac{\xi}{|\xi|}\right)d\xi$$

$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \varphi_1(x)\overline{\varphi_2}(x)\psi(\xi)d\mu(x,\xi) = \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle$$

- If $u_n \to 0$ in $L^2(\mathbb{R}^d)$, then $\mu = 0$.
- If $\mu = 0$, then $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^d)$.

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- $\sum_{i=1}^{d} \partial_{x_i}(A_i(x)u_n(x)) = f_n(x) \to 0$ in $W^{-1,2}_{loc}(\mathbb{R}^d)$, $A_i \in C_0(\mathbb{R}^d)$

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Localisation principle for H-measures

$$P(x,\xi)\mu(x,\xi) = \sum_{j=1}^{d} A_j(x)\xi_j \, \mu(x,\xi) = 0, \text{i.e. supp } \mu \subset chP$$

H-distributions, $W^{-k,p} - W^{k,q}$, $1 < p < \infty$

Theorem

If a sequence $u_n \rightharpoonup 0$ weakly in $W^{-k,p}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ weakly in $W^{k,q}(\mathbb{R}^d)$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a distribution $\mu \in \mathcal{S}\mathcal{E}'(\mathbb{R}^d \times \mathbb{S}^{d-1})$ such that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in C^\kappa(\mathbb{S}^{d-1})$, $\kappa = [d/2] + 1$,

$$\lim_{n' \to \infty} \langle \varphi_1 u_{n'} , \mathcal{A}_\psi(\varphi_2 v_{n'}) \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \psi \rangle.$$

Theorem

Let $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbb{R}^d)$. If for every sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^d)$ the corresponding H-distribution is zero, then for every $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ strongly in $W^{-k,p}(\mathbb{R}^d)$, $n \to \infty$.

Unbounded multipliers

- For weakly convergent sequences in $W^{-k,p} - W^{k,q}$ spaces multiplier (symbol) ψ is a bounded function
- $\psi \in C(\mathbb{S}^{d-1})$ or $\psi \in C^{\kappa}(\mathbb{S}^{d-1})$
- Let $m \in \mathbb{R}$, $q \in [1, \infty]$, $N \in \mathbb{N}_0$. Then we consider the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{s^m_{q,N}} := \max_{|\alpha| \leq N} \|\partial^\alpha_\xi \psi(\xi) \langle \xi \rangle^{-m+|\alpha|}\|_{L^q} < \infty.$$

- $T_\psi(u)(x) = \mathcal{A}_\psi u(x) := \displaystyle\int_{\mathbb{R}^d} e^{ix\xi} \psi(\xi) \hat{u}(\xi) đ\xi$

With $(s^m_{\infty,N})_0 \subset s^m_{\infty,N}$ we denote the class of multipliers such that $\psi \in (s^m_{\infty,N})_0$ means that

$$\lim_{n\to\infty} \sup_{|\xi|\geq n} \frac{|\partial^\alpha_\xi \psi(\xi)|}{\langle \xi \rangle^{m-|\alpha|}} = 0, \quad \text{for all } |\alpha| \leq N.$$

Separability of symbol classes is important in the construction of H-distributions. The following results hold.

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The space $((s^m_{\infty,N+1})_0, |\cdot|_{s^m_{\infty,N}})$ *is separable.*

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- The Bessel potential space $H^p_s(\mathbb{R}^d)$, $1 \leq p < \infty$, $s \in \mathbb{R}$, is defined as a space of all $u \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\mathcal{A}_{\langle\xi\rangle^s} u := \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}u) \in L^p(\mathbb{R}^d).$$

- $\|u\|_{H^p_s} = \|\mathcal{A}_{\langle\xi\rangle^s} u\|_{L^p}$
- $(H^p_s(\mathbb{R}^d))' = H^q_{-s}(\mathbb{R}^d)$, $1 < p < \infty$
- $H^p_s(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$ for $s \in \mathbb{N}_0$ and $1 \leq p < \infty$

H-distributions with multiplier $\psi \in s^m_{\infty,N}$, $N \geq 3d+5$

Theorem

Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ u $H^q_m(\mathbb{R}^d)$, $m \in \mathbb{R}$. Then, up to subsequences, there exists a distribution $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^m_{\infty,N+1})_0)'$ such that for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in (s^m_{\infty,N+1})_0$,

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n)} \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \otimes \psi \rangle.$$

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Theorem

Let $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ u $H^q_{m+s}(\mathbb{R}^d)$, $m \in \mathbb{R}$. Then, up to subsequences, there exists a distribution $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^m_{\infty,N+1})_0)'$ such that for all $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in (s^m_{\infty,N+1})_0$,

$$\lim_{n\to\infty} \langle \varphi_1 u_n, \overline{\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n)} \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \otimes \psi \rangle.$$

Theorem

Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, $m \in \mathbb{R}$. If for every sequence $v_n \rightharpoonup 0$ in $H^q_m(\mathbb{R}^d)$ it holds that

$$\lim_{n\to\infty} \langle u_n, \mathcal{A}_{\langle\xi\rangle^m}(\varphi v_n)\rangle = 0,$$

then for every $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$, $n \to \infty$.

- $u_n \rightharpoonup 0$ in L^p, $1 < p < \infty$
- $$\sum_{|\alpha| \leq k} A_\alpha(x) \partial^\alpha u_n(x) = g_n(x), \tag{1}$$

  where $A_\alpha \in \mathcal{S}(\mathbb{R}^d)$ and $(g_n)_n$ is a sequence of temperate distributions such that

  $$\varphi g_n \to 0 \text{ in } H^p_{-k}, \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2}$$

Theorem

Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, satisfies (1), (2) and $\psi \in s^m_{\infty,N}$. Then, for any $v_n \rightharpoonup 0$ in $H^q_m(\mathbb{R}^d)$, the corresponding distribution $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$ satisfies

$$\sum_{|\alpha|\leq k} A_\alpha(x)\mu_{\frac{\xi^\alpha}{\langle\xi\rangle^k}\psi} = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{3}$$

Moreover, if $\psi = \langle\xi\rangle^m$ and (3) implies $\mu_{\langle\xi\rangle^m} = 0$ we have the strong convergence $\theta u_n \to 0$, in $L^p(\mathbb{R}^d)$, for every $\theta \in \mathcal{S}(\mathbb{R}^d)$.

H-distributions - different weights

Definition

Positive function $\Lambda \in C^\infty(\mathbb{R}^d)$ is a weight function if the following assumptions are satisfied:

1. There exist positive constants $1 \leq \mu_0 \leq \mu_1$ and $c_0 < c_1$ such that

$$c_0 \langle \xi \rangle^{\mu_0} \leq \Lambda(\xi) \leq c_1 \langle \xi \rangle^{\mu_1}, \quad \xi \in \mathbb{R}^d;$$

2. There exists $\omega \geq \mu_1$ such that for any $\alpha \in \mathbb{N}_0^d$ and $\gamma \in \mathbb{K} \equiv \{0, 1\}^d$

$$|\xi^\gamma \partial^{\alpha+\gamma} \Lambda(\xi)| \leq C_{\alpha,\gamma} \Lambda(\xi)^{1-\frac{1}{\omega}|\alpha|}, \quad \xi \in \mathbb{R}^d.$$

Constant ω is called the order of Λ.

Examples

1.
$$\Lambda(\xi) = \sqrt{1 + \sum_{i=1}^{d} \xi_i^{2m_i}}, \ \xi \in \mathbb{R}^d,$$
where $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$ and $\min_{1 \leq i \leq d} m_i \geq 1$.

2. In particular, $\langle \xi \rangle = \left(1 + \sum_{i=1}^{d} \xi_i^2\right)^{\frac{1}{2}}$ is a weight function.

More general weights are defined by

$$\Lambda_{\mathcal{P}}(\xi) = \Big(\sum_{\alpha \in V(\mathcal{P})} \xi^{2\alpha} \Big)^{\frac{1}{2}}, \ \xi \in \mathbb{R}^d,$$

where $\mathcal{P}$ is a given complete polyhedron with the set of vertices $V(\mathcal{P})$. A complete polyhedron is a convex polyhedron $\mathcal{P} \subset (\mathbb{R}_+ \cup \{0\})^d$ with the following properties: $V(\mathcal{P}) \subset \mathbb{N}_0^d$, $0 \in V(\mathcal{P})$, $V(\mathcal{P}) \neq \{0\}$, $N_0(\mathcal{P}) = \{e_1, \dots, e_d\}$ and $N_1(\mathcal{P}) \subset \mathbb{R}_+^d$. Here

$$\mathcal{P} = \{z \in \mathbb{R}^d : \nu \cdot z \geq 0, \ \forall \nu \in N_0(\mathcal{P})\} \cap \{z \in \mathbb{R}^d : \nu \cdot z \leq 1, \ \nu \in N_1(\mathcal{P})\}$$

and $N_0(\mathcal{P})$ and $N_1(\mathcal{P}) \subset \mathbb{R}^d$ are finite sets such that for all $\nu \in N_0(\mathcal{P})$, $|\nu| = 1$.

Definition

Let $m \in \mathbb{R}, \rho \in (0, 1/\omega]$, $N \in \mathbb{N}_0$. Then $s^{m,N}_{\rho,\Lambda}(\mathbb{R}^d)$ is the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{s^{m,N}_{\rho,\Lambda}} := \max_{|\gamma|:\gamma\in\mathbb{K}} \max_{|\alpha|\le N} \sup_{\xi\in\mathbb{R}^d} |\xi^\gamma \partial_\xi^{\alpha+\gamma}\psi(\xi)|\Lambda(\xi)^{-m+\rho|\alpha|} < \infty.$$

If $\rho = 1/\omega$, then we denote $s^{m,N}_{\Lambda} = s^{m,N}_{1/\omega,\Lambda}$.

We denote by $(s^{m,N}_{\rho,\Lambda})_0 \subset s^{m,N}_{\rho,\Lambda}$ the space of multipliers $\psi \in (s^{m,N}_{\rho,\Lambda})_0$ such that for all $|\alpha| \le N,\ \gamma \in \mathbb{K}$

$$\lim_{n\to\infty} \sup_{|\xi|\ge n} \frac{|\xi^\gamma \partial^{\alpha+\gamma}\psi(\xi)|}{\Lambda(\xi)^{m-\rho|\alpha|}} = 0.$$

Theorem

Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ in $H^{m,q}_\Lambda(\mathbb{R}^d)$, $m \in \mathbb{R}$, $\rho = 1/\omega$. Then, up to subsequences, there exists a distribution $\mu \in (\mathcal{S}(\mathbb{R}^d)\hat{\otimes}(s^{m,N+1}_\Lambda)_0)'$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all $\psi \in (s^{m,N+1}_\Lambda)_0$,

$$\lim_{n\to\infty} \langle u_n, \overline{\mathcal{A}_{\bar{\psi}}(\varphi v_n)}\rangle = \langle \mu, \bar{\varphi} \otimes \psi\rangle.$$

Let $\mathcal{P}$ be complete polyhedron in $\mathbb{R}^d$ with set of vertices $V(\mathcal{P})$ and $\Lambda = \Lambda_{\mathcal{P}}$. Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, $1 < p < \infty$, such that the following sequence of equations is satisfied

$$p(x, D)u(x) = \sum_{\alpha \in V(\mathcal{P})} a_\alpha(x) D^{2\alpha} u_n(x) = f_n(x), \tag{4}$$

where $a_\alpha(x) \in C_b^\infty(\mathbb{R}^d)$, and $(f_n)_n$ is a sequence of temperate distributions such that

$$\varphi f_n \to 0 \text{ in } H_\Lambda^{-2,p}(\mathbb{R}^d), \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{5}$$

Theorem

Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, satisfies (4), (5) and $\psi \in s_\Lambda^{m,N}$. Then, for any $v_n \rightharpoonup 0$ in $H_\Lambda^{m,q}(\mathbb{R}^d)$ and the corresponding distribution μ, there holds

$$\sum_{\alpha \in V(\mathcal{P})} a_\alpha(x) \mu_{\frac{\psi \xi^{2\alpha}}{\Lambda(\xi)^2}} = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{6}$$

Moreover, let $\psi = \Lambda(\xi)^m$ and the equality in (6) implies that $\mu_\psi = 0$. Then we have the strong convergence $\theta u_n \to 0$ in $L^p(\mathbb{R}^d)$, for every $\theta \in \mathcal{S}(\mathbb{R}^d)$.

Further work

$z = (x, \xi) \in \mathbb{R}^{2d}$

Definition (Morando, Garello)

We denote by $M\Gamma^m_{\rho,\Lambda}$, $m \in \mathbb{R}$, $\rho \in (0, 1/\omega]$ the class of all the functions $a(z) \in \mathbb{R}^{2d}$ such that

$$z^\gamma \partial^{\alpha+\gamma} a(z) \leq c_{\alpha,\gamma} \Lambda(z)^{m-\rho|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^{2d}$ and $\gamma \in \{0, 1\}^{2d}$.

Theorem

Any operator $A \in ML^0_{\rho,\Lambda}$ extends to a bounded operator from $L^p(\mathbb{R}^d)$ to itself, $1 < p < \infty$.

Let $\Lambda(x,\xi)$ be a weight function, $s \in \mathbb{R}$, $1 < p < \infty$. We denote by $H_\Lambda^{s,p}$ the space of all $u \in S'(\mathbb{R}^d)$ such that $\Lambda^s(x,D)u \in L^p(\mathbb{R}^d)$, where

$$\Lambda^s(x,D)u := \int e^{ix\cdot\xi}\Lambda(x,\xi)^s \mathcal{F}u(\xi)d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

Norm on $H_\Lambda^{s,p}$ is given by

$$\|u\|_{s,p,\Lambda} = \|\Lambda^s(x,D)u\|_{L^p} + \|R_s u\|_{L^p}.$$

Theorem

If $T \in ML^m_{\rho,\Lambda}$, then

$$T : H_\Lambda^{s+m,p} \to H_\Lambda^{s,p},$$

is a bounded operator, where $s \in \mathbb{R}$, $1 < p < \infty$.

Example

We consider linear differential operator

$$P(x, D) = -\Delta + V(x),$$

where $V(x) = \sum_{\alpha \in \mathcal{R}} a_\alpha x^\alpha$, $a_\alpha \in \mathbb{C}$ and $\mathcal{R}$ is a complete polyhedron in $\mathbb{R}^d$.
The symbol $p(x, \xi) = |\xi|^2 + V(x)$ is in the class $M\Gamma^1_{\rho,\lambda}$, where $\lambda = \lambda_{\mathcal{P}}$, for complete polyhedron $\mathcal{P}$ defined as the convex hull of $\{(\alpha, 0) : \alpha \in V(\mathcal{R})\} \cup \{(0, 2e_j) : j = 1, 2, \ldots d\}$. Precisely,

$$\lambda(x, \xi) = \sqrt{\sum_{\alpha \in V(\mathcal{R})} x^{2\alpha} + \sum_{j=1}^{d} \xi_j^4}.$$

$P(x, D)u_n = f_n$

1. Existence of defect distributions with symbols in $M\Gamma^m_{\rho,\Lambda}$?
2. Localisation principle?
3. Compactness of commutator?

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