Defect distributions related to classes of \( \Lambda \)-type weights

Ivana Vojnović

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Introduction

- H-measures, micro-local defect measures (Tartar, Gérard, around 1990.) - $L^2$ space
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- H-measures, micro-local defect measures (Tartar, Gérard, around 1990.) - $L^2$ space
- H-distributions (Antonić, Mitrović, 2011.) - $L^p - L^q$ spaces, $p = \frac{q}{q-1}, 1 < p < \infty$
• $u_n \rightarrow 0$ in $L^2(\mathbb{R}^d)$, $n \rightarrow \infty$
• $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$, $n \to \infty$

Existence of $H$-measure (Tartar, [7])

There exists a subsequence $(u_{n'})_{n'}$ and a complex Radon measure $\mu$ on $\mathbb{R}^d \times S^{d-1}$ s. t. for all $\varphi_1(x), \varphi_2(x) \in C_0(\mathbb{R}^d)$, $\psi(\xi) \in C(S^{d-1})$ we have that

$$
\lim_{n' \to \infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'})(\xi) \mathcal{F}(\overline{\varphi_2 u_{n'}})(\xi) \psi\left(\frac{\xi}{|\xi|}\right) d\xi
$$

$$
= \int_{\mathbb{R}^d \times S^{d-1}} \varphi_1(x) \overline{\varphi_2(x)} \psi(\xi) d\mu(x, \xi) = \langle \mu, \varphi_1 \overline{\varphi_2} \psi \rangle
$$
• If $u_n \to 0$ in $L^2(\mathbb{R}^d)$, then $\mu = 0$.
• If $\mu = 0$, then $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^d)$. 
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• $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$

• $\sum_{i=1}^d \partial x_i (A_i(x) u_n(x)) = f_n(x) \to 0$ in $W^{-1,2}_{loc}(\mathbb{R}^d)$, $A_i \in C_0(\mathbb{R}^d)$
• If \( u_n \to 0 \) in \( L^2(\mathbb{R}^d) \), then \( \mu = 0 \).
• If \( \mu = 0 \), then \( u_n \to 0 \) in \( L^2_{loc}(\mathbb{R}^d) \).

\[
\sum_{i=1}^{d} \partial_{x_i}(A_i(x)u_n(x)) = f_n(x) \to 0 \text{ in } W^{-1,2}_{loc}(\mathbb{R}^d), \ A_i \in C_0(\mathbb{R}^d)
\]

Localisation principle for H-measures

\[
P(x, \xi)\mu(x, \xi) = \sum_{j=1}^{d} A_j(x)\xi_j \mu(x, \xi) = 0, \text{ i.e. supp } \mu \subset \text{chP}
\]
H-distributions, $W^{-k,p} - W^{k,q}$, $1 < p < \infty$

**Theorem**

If a sequence $u_n \rightharpoonup 0$ weakly in $W^{-k,p}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ weakly in $W^{k,q}(\mathbb{R}^d)$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a distribution $\mu \in \mathcal{S}'(\mathbb{R}^d \times S^{d-1})$ such that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in C^\kappa(S^{d-1})$, $\kappa = [d/2] + 1$,  

$$\lim_{n' \to \infty} \langle \varphi_1 u_{n'}, A_\psi(\varphi_2 v_{n'}) \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \psi \rangle.$$
Theorem
Let $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbb{R}^d)$. If for every sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^d)$ the corresponding H-distribution is zero, then for every $\theta \in S(\mathbb{R}^d)$, $\theta u_n \rightarrow 0$ strongly in $W^{-k,p}(\mathbb{R}^d)$, $n \rightarrow \infty$. 
Unbounded multipliers

- For weakly convergent sequences in $W^{-k,p} = W^{k,q}$ spaces, multiplier (symbol) $\psi$ is a bounded function.

- $\psi \in C(S^{d-1})$ or $\psi \in C^\kappa(S^{d-1})$.

- Let $m \in \mathbb{R}$, $q \in [1, \infty]$, $N \in \mathbb{N}_0$. Then we consider the space of all $\psi \in C^N(\mathbb{R}^d)$ for which the norm

$$|\psi|_{s^m_{q,N}} := \max_{|\alpha| \leq N} \| \partial_\xi^\alpha \psi(\xi) \langle \xi \rangle^{-m + |\alpha|} \|_{L^q} < \infty.$$  

- $T_\psi(u)(x) = A_\psi u(x) := \int_{\mathbb{R}^d} e^{ix\xi} \psi(\xi) \hat{u}(\xi) d\xi$

With $(s^m_{\infty,N})_0 \subset s^m_{\infty,N}$ we denote the class of multipliers such that $\psi \in (s^m_{\infty,N})_0$ means that

$$\lim_{n \to \infty} \sup_{|\xi| \geq n} \frac{|\partial_\xi^\alpha \psi(\xi)|}{\langle \xi \rangle^{|m-|\alpha|}} = 0, \quad \text{for all } |\alpha| \leq N.$$
Separability of symbol classes is important in the construction of H-distributions. The following results hold.

**Theorem**

The space \(((s^m_{\infty,N+1})_0, \cdot |s^m_{\infty,N})\) is separable.
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**Theorem**

*The space \((((s^m_{\infty}, N+1)_0, \cdot |s^m_{\infty}, N))\) is separable.*

- The Bessel potential space \(H^p_s(\mathbb{R}^d), 1 \leq p < \infty, s \in \mathbb{R}\), is defined as a space of all \(u \in S'(\mathbb{R}^d)\) such that

  \[A_\langle \xi \rangle^s u := \mathcal{F}^{-1}((1 + |\xi|^2)^{s/2} \mathcal{F}u) \in L^p(\mathbb{R}^d)\].

- \(\|u\|_{H^p_s} = \|A_\langle \xi \rangle^s u\|_{L^p}\)
- \((H^p_s(\mathbb{R}^d))' = H^q_{-s}(\mathbb{R}^d), 1 < p < \infty\)
- \(H^p_s(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)\) for \(s \in \mathbb{N}_0\) and \(1 \leq p < \infty\)
Theorem
Let \( u_n \rightharpoonup 0 \) in \( L^p(\mathbb{R}^d) \) and \( v_n \rightharpoonup 0 \) in \( H^q_m(\mathbb{R}^d) \), \( m \in \mathbb{R} \). Then, up to subsequences, there exists a distribution \( \mu \in \left( S(\mathbb{R}^d) \hat{\otimes} (s^m_{\infty,N+1})_0 \right)' \) such that for all \( \varphi_1, \varphi_2 \in S(\mathbb{R}^d) \) and all \( \psi \in (s^m_{\infty,N+1})_0 \),

\[
\lim_{n \to \infty} \langle \varphi_1 u_n, A_{\bar{\psi}}(\varphi_2 v_n) \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \otimes \psi \rangle.
\]
H-distributions with multiplier $\psi \in s^m_{\infty,N}, N \geq 3d + 5$

**Theorem**

Let $u_n \rightharpoonup 0$ in $H^p_{-s}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ in $H^q_{m+s}(\mathbb{R}^d)$, $m \in \mathbb{R}$. Then, up to subsequences, there exists a distribution $\mu \in (S(\mathbb{R}^d) \hat{\otimes} (s^m_{\infty,N+1})_0)'$ such that for all $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$ and all $\psi \in (s^m_{\infty,N+1})_0$,

$$\lim_{n \to \infty} \langle \varphi_1 u_n, A_\psi(\varphi_2 v_n) \rangle = \langle \mu, \varphi_1 \varphi_2 \otimes \psi \rangle.$$
Theorem
Let \( u_n \rightharpoonup 0 \) in \( L^p(\mathbb{R}^d) \), \( m \in \mathbb{R} \). If for every sequence \( v_n \rightharpoonup 0 \) in \( H^q_m(\mathbb{R}^d) \) it holds that
\[
\lim_{n \to \infty} \langle u_n, A_{\langle \xi \rangle m}(\varphi v_n) \rangle = 0,
\]
then for every \( \theta \in \mathcal{S}(\mathbb{R}^d) \), \( \theta u_n \rightharpoonup 0 \) strongly in \( L^p(\mathbb{R}^d) \), \( n \to \infty \).
- $u_n \to 0$ in $L^p$, $1 < p < \infty$
- 
\[
\sum_{|\alpha| \leq k} A_\alpha(x) \partial^\alpha u_n(x) = g_n(x),
\]

where $A_\alpha \in S(\mathbb{R}^d)$ and $(g_n)_n$ is a sequence of temperate distributions such that

\[
\varphi g_n \to 0 \text{ in } H^p_{-k}, \text{ for every } \varphi \in S(\mathbb{R}^d).
\]
Theorem

Let \( u_n \rightharpoonup 0 \) in \( L^p(\mathbb{R}^d) \), satisfies (1), (2) and \( \psi \in s^m_{\infty,N} \). Then, for any \( v_n \rightharpoonup 0 \) in \( H^q_{m}(\mathbb{R}^d) \), the corresponding distribution \( \mu_{\psi} \in S'(\mathbb{R}^d) \) satisfies

\[
\sum_{|\alpha| \leq k} A_{\alpha}(x) \mu \frac{\xi^\alpha}{\langle \xi \rangle_k} \psi = 0 \text{ in } S'(\mathbb{R}^d).
\]  

(3)

Moreover, if \( \psi = \langle \xi \rangle^m \) and (3) implies \( \mu_{\langle \xi \rangle^m} = 0 \) we have the strong convergence \( \theta u_n \rightharpoonup 0 \), in \( L^p(\mathbb{R}^d) \), for every \( \theta \in S(\mathbb{R}^d) \).
H-distributions - different weights

**Definition**

Positive function $\Lambda \in C^\infty(\mathbb{R}^d)$ is a weight function if the following assumptions are satisfied:

1. There exist positive constants $1 \leq \mu_0 \leq \mu_1$ and $c_0 < c_1$ such that

   $$c_0 \langle \xi \rangle^{\mu_0} \leq \Lambda(\xi) \leq c_1 \langle \xi \rangle^{\mu_1}, \quad \xi \in \mathbb{R}^d;$$

2. There exists $\omega \geq \mu_1$ such that for any $\alpha \in \mathbb{N}_0^d$ and $\gamma \in K \equiv \{0, 1\}^d$

   $$|\xi^\gamma \partial^{\alpha+\gamma} \Lambda(\xi)| \leq C_{\alpha, \gamma} \Lambda(\xi)^{1-\frac{1}{\omega}} |\alpha|, \quad \xi \in \mathbb{R}^d.$$

Constant $\omega$ is called the order of $\Lambda$. 
Examples

1. \[ \Lambda(\xi) = \sqrt{1 + \sum_{i=1}^{d} \xi_i^{2m_i}}, \quad \xi \in \mathbb{R}^d, \]
   where \( m = (m_1, \ldots, m_d) \in \mathbb{N}^d \) and \( \min_{1 \leq i \leq d} m_i \geq 1 \).

2. In particular, \( \langle \xi \rangle = \left(1 + \sum_{i=1}^{d} \xi_i^2 \right)^{\frac{1}{2}} \) is a weight function.
More general weights are defined by

$$\Lambda_\mathcal{P}(\xi) = \left( \sum_{\alpha \in V(\mathcal{P})} \xi^{2\alpha} \right)^{\frac{1}{2}}, \, \xi \in \mathbb{R}^d,$$

where $\mathcal{P}$ is a given complete polyhedron with the set of vertices $V(\mathcal{P})$. A complete polyhedron is a convex polyhedron $\mathcal{P} \subset (\mathbb{R}_+ \cup \{0\})^d$ with the following properties: $V(\mathcal{P}) \subset \mathbb{N}_0^d$, $0 \in V(\mathcal{P})$, $V(\mathcal{P}) \neq \{0\}$, $N_0(\mathcal{P}) = \{e_1, \ldots, e_d\}$ and $N_1(\mathcal{P}) \subset \mathbb{R}_+^d$. Here

$$\mathcal{P} = \{z \in \mathbb{R}^d : \nu \cdot z \geq 0, \, \forall \nu \in N_0(\mathcal{P})\} \cap \{z \in \mathbb{R}^d : \nu \cdot z \leq 1, \, \nu \in N_1(\mathcal{P})\}$$

and $N_0(\mathcal{P})$ and $N_1(\mathcal{P}) \subset \mathbb{R}^d$ are finite sets such that for all $\nu \in N_0(\mathcal{P})$, $|\nu| = 1$. 
Definition

Let \( m \in \mathbb{R}, \rho \in (0, 1/\omega], N \in \mathbb{N}_0 \). Then \( s_{\rho, \Lambda}^{m, N}(\mathbb{R}^d) \) is the space of all \( \psi \in C^N(\mathbb{R}^d) \) for which the norm

\[
|\psi|_{s_{\rho, \Lambda}^{m, N}} := \max_{|\gamma| : \gamma \in K} \max_{|\alpha| \leq N} \sup_{\xi \in \mathbb{R}^d} |\xi^\gamma \partial^\alpha + \gamma \psi(\xi)| \Lambda(\xi)^{-m+\rho|\alpha|} < \infty.
\]

If \( \rho = 1/\omega \), then we denote \( s_{\Lambda}^{m, N} = s_{1/\omega, \Lambda}^{m, N} \).

We denote by \( (s_{\rho, \Lambda}^{m, N})_0 \subset s_{\rho, \Lambda}^{m, N} \) the space of multipliers \( \psi \in (s_{\rho, \Lambda}^{m, N})_0 \) such that for all \( |\alpha| \leq N, \gamma \in K \)

\[
\lim_{n \to \infty} \sup_{|\xi| \geq n} \frac{|\xi^\gamma \partial^\alpha + \gamma \psi(\xi)|}{\Lambda(\xi)^{m-\rho|\alpha|}} = 0.
\]
Theorem

Let \( u_n \to 0 \) in \( L^p(\mathbb{R}^d) \) and \( v_n \to 0 \) in \( H^{m,q}_\Lambda(\mathbb{R}^d) \), \( m \in \mathbb{R} \), \( \rho = 1/\omega \). Then, up to subsequences, there exists a distribution \( \mu \in (S(\mathbb{R}^d) \hat{\otimes} (s^{m,N+1}_\Lambda)_0)' \) such that for all \( \varphi \in S(\mathbb{R}^d) \) and all \( \psi \in (s^{m,N+1}_\Lambda)_0 \),

\[
\lim_{n \to \infty} \langle u_n, A_{\bar{\varphi}}(\varphi v_n) \rangle = \langle \mu, \bar{\varphi} \otimes \psi \rangle.
\]
Let $\mathcal{P}$ be complete polyhedron in $\mathbb{R}^d$ with set of vertices $V(\mathcal{P})$ and $\Lambda = \Lambda_{\mathcal{P}}$. Let $u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, $1 < p < \infty$, such that the following sequence of equations is satisfied

$$p(x, D)u(x) = \sum_{\alpha \in V(\mathcal{P})} a_\alpha(x) D^{2 \alpha} u_n(x) = f_n(x), \quad (4)$$

where $a_\alpha(x) \in C^\infty_b(\mathbb{R}^d)$, and $(f_n)_n$ is a sequence of temperate distributions such that

$$\varphi f_n \to 0 \text{ in } H_{\Lambda}^{-2,p}(\mathbb{R}^d), \text{ for every } \varphi \in S(\mathbb{R}^d). \quad (5)$$
**Theorem**

Let \( u_n \rightharpoonup 0 \) in \( L^p(\mathbb{R}^d) \), satisfies (4), (5) and \( \psi \in s_{\Lambda}^{m,N} \). Then, for any \( v_n \rightharpoonup 0 \) in \( H^{m,q}_\Lambda(\mathbb{R}^d) \) and the corresponding distribution \( \mu \), there holds

\[
\sum_{\alpha \in V(\mathcal{P})} a_\alpha(x) \frac{\mu_{\psi_\xi^2 \alpha}}{\Lambda(\xi)^2} = 0 \quad \text{in} \quad S'(\mathbb{R}^d). \tag{6}
\]

Moreover, let \( \psi = \Lambda(\xi)^m \) and the equality in (6) implies that \( \mu_\psi = 0 \). Then we have the strong convergence \( \theta u_n \to 0 \) in \( L^p(\mathbb{R}^d) \), for every \( \theta \in S(\mathbb{R}^d) \).
Further work

\[ z = (x, \xi) \in \mathbb{R}^{2d} \]

**Definition (Morando, Garello)**

We denote by \( M_{\Gamma}^{m,\Lambda} \), \( m \in \mathbb{R}, \rho \in (0, 1/\omega] \) the class of all the functions \( a(z) \in \mathbb{R}^{2d} \) such that

\[
z^{\gamma} \partial^{\alpha+\gamma} a(z) \leq c_{\alpha,\gamma} \Lambda(z)^{m-\rho|\alpha|}
\]

for all \( \alpha \in \mathbb{N}_0^{2d} \) and \( \gamma \in \{0, 1\}^{2d} \).

**Theorem**

Any operator \( A \in M\mathcal{L}_{0,\Lambda}^{0,\rho} \) extends to a bounded operator from \( L^p(\mathbb{R}^d) \) to itself, \( 1 < p < \infty \).
Let $\Lambda(x, \xi)$ be a weight function, $s \in \mathbb{R}$, $1 < p < \infty$. We denote by $H^s_{\Lambda} \in \mathcal{ML}^m_{\rho, \Lambda}$ the space of all $u \in S'(\mathbb{R}^d)$ such that $\Lambda^s(x, D)u \in L^p(\mathbb{R}^d)$, where

$$\Lambda^s(x, D)u := \int e^{i x \cdot \xi} \Lambda(x, \xi)^s \mathcal{F} u(\xi) d\xi, \quad u \in S(\mathbb{R}^d).$$

Norm on $H^s_{\Lambda}$ is given by

$$\|u\|_{s, p, \Lambda} = \|\Lambda^s(x, D)u\|_{L^p} + \|R_s u\|_{L^p}.$$

**Theorem**

If $T \in \mathcal{ML}^m_{\rho, \Lambda}$, then

$$T : H^s_{\Lambda} \rightarrow H^s_{\Lambda},$$

is a bounded operator, where $s \in \mathbb{R}$, $1 < p < \infty$. 
Example

We consider linear differential operator

\[ P(x, D) = -\Delta + V(x), \]

where \( V(x) = \sum_{\alpha \in \mathcal{R}} a_\alpha x^\alpha, \ a_\alpha \in \mathbb{C} \) and \( \mathcal{R} \) is a complete polyhedron in \( \mathbb{R}^d \).

The symbol \( p(x, \xi) = |\xi|^2 + V(x) \) is in the class \( M_{\rho, \lambda}^1 \), where \( \lambda = \lambda_{\mathcal{P}} \), for complete polyhedron \( \mathcal{P} \) defined as the convex hull of \( \{(\alpha, 0) : \alpha \in V(\mathcal{R})\} \cup \{(0, 2e_j) : j = 1, 2, \ldots d\} \). Precisely,

\[
\lambda(x, \xi) = \sqrt{\sum_{\alpha \in V(\mathcal{R})} x^{2\alpha} + \sum_{j=1}^{d} \xi_j^4}.
\]
$P(x, D)u_n = f_n$

1. Existence of defect distributions with symbols in $M_{\rho, \Lambda}^m$?
2. Localisation principle?
3. Compactness of commutator?
References I


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