Homogenization of Kirchhoff-Love plate equation

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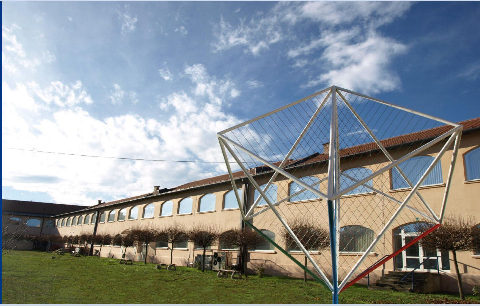

Joint work with **Jelena Jankov, Marko Vrdoljak**

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[89TH GAMM ANNUAL MEETING, MUNICH] March, 2018

Kirchhoff-Love plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

- $\Omega \subseteq \mathbb{R}^d$ bounded domain ($d = 2 \ldots$ plate)
- $f \in H^{-2}(\Omega)$ external load
- $u \in H_0^2(\Omega)$ vertical displacement of the plate
- $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{\mathbf{N} \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\, \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \alpha \mathbf{S} : \mathbf{S} \text{ and } \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geq \frac{1}{\beta}\mathbf{S} : \mathbf{S} \text{ a.e. } \mathbf{x}\}$ describes properties of material of the given plate

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H-convergence

Definition

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

coverges weakly to a limit u in $H_0^2(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $L^2(\Omega; \mathrm{Sym})$.

Theorem

Let $(\mathbf{M}^n)$ be a sequence in $\mathfrak{M}_2(\alpha, \beta; \Omega)$. Then there is a subsequence $(\mathbf{M}^{n_k})$ and a tensor function $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ such that $(\mathbf{M}^{n_k})$ H-converges to $\mathbf{M}$.

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Compactness

Antonić, Balenović, 1999.

Zikov, Kozlov, Oleinik, Ngoan, 1979.

Theorem (Compactness by compensation)

Let the following convergences be valid:

$$w^n \rightharpoonup w^\infty \quad \text{in} \quad \mathrm{H}^2_{\mathrm{loc}}(\Omega)\,,$$
$$\mathbf{D}^n \rightharpoonup \mathbf{D}^\infty \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathrm{Sym})\,,$$

with an additional assumption that the sequence $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ *is contained in a precompact (for the strong topology) set of the space* $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$*. Then we have*

$$\nabla\nabla w^n : \mathbf{D}^n \xrightharpoonup{*} \nabla\nabla w^\infty : \mathbf{D}^\infty$$

in the space of Radon measures.

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Properties

- Locality of the H-convergence
- Irrelevance of boundary conditions
- Energy convergence
- Ordering property
- Metrizability
- Corrector results
- Smooth dependence of H-limit on a parameter
- H-limit of periodic sequence
- Small-amplitude homogenization

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Definition of correctors

Definition

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$. Let $(w_n^{ij})_{1\leq i,j\leq d}$ be a family of test functions satisfying

$$w_n^{ij} \rightharpoonup \frac{1}{2}x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)$$

$$\mathbf{M}^n \nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$$

$$\operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla w_n^{ij}) \to \cdot \quad \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega).$$

The sequence of tensors $\mathbf{W}^n$ defined with $\mathbf{W}^n_{ijkm} = [\nabla\nabla w_n^{km}]_{ij}$ is called a sequence of correctors.

Uniqueness of correctors

Theorem

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $\mathbf{M}$. A sequence of correctors $(\mathbf{W}^n)$ is unique in the sense that, if there exist two sequences of correctors $(\mathbf{W}^n)$ and $(\tilde{\mathbf{W}}^n)$, their difference $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ converges strongly to zero in $L^2_{\rm loc}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Corrector result

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *which H-converges to* $\mathbf{M}$*. For* $f \in H^{-2}(\Omega)$*, let* (u_n) *be the solution of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f & \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,, \end{cases}$$

and let u *be the weak limit of* (u_n) *in* $H_0^2(\Omega)$*, i.e., the solution of the homogenized equation*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f & \text{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Then, $r_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u \to 0$ *strongly in* $L^1_{\mathrm{loc}}(\Omega; Sym)$*.*

Small-amplitude homogenization

$$\mathbf{M}_p^n(\mathbf{x}) := \mathbf{A}_0 + p\mathbf{B}^n(\mathbf{x}),\ p \in \mathbf{R}$$

$$\mathbf{M}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2),\ p \in \mathbf{R}$$

If $p \mapsto \mathbf{M}_n^p$ is a C^k mapping (for any $n \in \mathbf{N}$) from some subset of $\mathbf{R}$ to $L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, what can we say about $p \mapsto \mathbf{M}_p$?

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Smoothness with respect to a parameter

Theorem

Let $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*, where* $P \subseteq \mathbf{R}$ *is an open set. Assume that (for some* $k \in \mathbf{N}_0$*) a mapping* $p \mapsto \mathbf{M}^n(\cdot, p)$ *is of class* C^k *from* P *to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives (up to order* k*) being equicontinuous on every compact set* $K \subseteq P$*:*

$$(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \leq k)$$
$$|p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} < \varepsilon.$$

Then there is a subsequence $(\mathbf{M}^{n_k})$ *such that for every* $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

and $p \mapsto \mathbf{M}(\cdot, p)$ *is a* C^k *mapping from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*.*

Periodic case

- $Y = [0,1]^d$, $\mathbf{M} \in \mathrm{L}^\infty_\#(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) \cap \mathfrak{M}_2(\alpha, \beta; Y)$
- $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), \mathbf{x} \in \Omega \subseteq \mathbf{R}^d$ (open and bounded)
- $H^2_\#(Y) := \{f \in H^2_{\mathrm{loc}}(\mathbf{R}^d) \,\mathrm{such\ that\ f\ is\ Y-periodic}\}$ with the norm $\|\cdot\|_{H^2(Y)}$
- $H^2_\#(Y)/\mathbf{R}$ equipped with the norm $\|\nabla\nabla\cdot\|_{L^2(Y)}$
- $\mathbf{E}_{ij}, 1 \leq i, j \leq d$ are $M_{d\times d}$ matrices defined as

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \text{if } i = j = k = l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

H-limit of a periodic sequence

Theorem

Let $(\mathbf{M}^n)$ *be a sequence of tensors defined by* $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), x \in \Omega$. *Then* $(\mathbf{M}^n)$ *H-converges to a constant tensor* $\mathbf{M}^* \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *defined as*

$$m^*_{klij} = \int_Y \mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\mathbf{x}))\, d\mathbf{x},$$

where $(w_{ij})_{1\leq i,j\leq d}$ *is the family of unique solutions in* $H^2_\#(Y)/\mathbf{R}$ *of boundary value problems*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x}))) = 0 \text{ in Y} \\ \mathbf{x} \to w_{ij}(\mathbf{x}) \quad \text{is } Y\text{-periodic.} \end{cases}$$

Small-amplitude assumptions

Theorem

Let $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym};\mathrm{Sym})$ *be a constant coercive tensor,* $P \subseteq \mathbf{R}$ *an open set,* $\mathbf{B}^n(\mathbf{x}) := \mathbf{B}(n\mathbf{x})$, $\mathbf{x} \in \Omega$, *where* $\Omega \subseteq \mathbf{R}^d$ *is a bounded, open set, and* $\mathbf{B}$ *is a Y-periodic,* L^∞ *tensor function, satisfying* $\int_Y \mathbf{B}(\mathbf{x})\,d\mathbf{x} = 0$.

Then

$$\mathbf{M}_p^n(\mathbf{x}) := \mathbf{A}_0 + p\mathbf{B}^n(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

H-converges (for any $p \in P$*) to a tensor*

$$\mathbf{M}_p := \mathbf{A}_0 + p\mathbf{B}_0 + p^2\mathbf{C}_0 + o(p^2)$$

with coefficients $\mathbf{B}_0 = 0$ *and*

Small-amplitude limit

$$
\begin{aligned}
\mathbf{C}_0\mathbf{E}_{mn} : \mathbf{E}_{rs} = (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{mn}_{-\mathbf{k}}\mathbf{B}_{\mathbf{k}}(\mathbf{k}\otimes\mathbf{k}) : \mathbf{E}_{rs} + \\
+ (2\pi i)^4 \sum_{\mathbf{k}\in J} a^{mn}_{\mathbf{k}} a^{rs}_{-\mathbf{k}}\mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : \mathbf{k}\otimes\mathbf{k} + \\
+ (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{rs}_{-\mathbf{k}}\mathbf{B}_{\mathbf{k}}\mathbf{E}_{mn} : \mathbf{k}\otimes\mathbf{k}\,.
\end{aligned}
$$

where $m, n, r, s \in \{1, 2, \cdots, d\}$, $J := \mathbf{Z}^d/\{0\}$, and

$$
a^{mn}_{\mathbf{k}} = -\frac{\mathbf{B}_{\mathbf{k}}\mathbf{E}_{mn}\mathbf{k}\cdot\mathbf{k}}{(2\pi i)^2\mathbf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k}\in J, \quad m, n \in \{1, 2, \cdots, d\}
$$

and $\mathbf{B}_{\mathbf{k}}$, $\mathbf{k}\in J$, are Fourier coefficients of function **B**.

Now what?

- Small-amplitude homogenization - non-periodic case
- composite materials for plates
- lamination formula, higher-order laminates
- Hashin-Shtrikman bounds
- G-closure problem
- Optimal design of plates

Thank you for your attention!

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