# Existence of solution to the initial-boundary value problem for the dynamics capillarity equation

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# Darcy Law

Flow in a two phase porous medium is governed by the Darcy law

$$q_\alpha = -K_\alpha(S)\left(\nabla p_\alpha + \rho_\alpha g e_z\right), \quad \alpha = o, w,$$

where $K_\alpha$ is a permeability matrix depending on the oil saturation $S = S_o$ ($S_w = 1 - S_o$ is the water saturation).
$p_\alpha$ is pressure of the phase $\alpha$ and $\rho_\alpha$ is density of the phase $\alpha$.
We also need the conservation of mass

$$\partial_t(\rho_\alpha S_\alpha) + \mathrm{div}(q_\alpha S_\alpha) = 0.$$

# Capillarity

By $p_c$ we denote the capillary pressure which is equal to the difference of pressure of the two phases

$$p_c(S) = p_o - p_w.$$

If it is assumed to be (almost) constant we reach to the Buckley-Leverett equations (scalar conservation law):

$$\partial_t S + \partial_x f(S) = 0 \quad \ldots \quad [\mathrm{BL}]$$

for appropriate $f$.

If we assume that the $p_c$ is "static" (independent of $t$-derivative of $S$), then usually $p_c = p_c(S)$ and the Buckley-Leverett equation has additional second order parabolic ("diffusion") term.

However, both of the models appeared to give results inconsistent with certain experiments. Therefore, a non-static capillary pressure is proposed.

# Non-static capillary pressure

Gray and Hassanisadeh proposed

$$p_o - p_w = p_c(S) - L(S)\partial_t S$$

for appropriate damping term $L(S)$. After linearization and standard manipulations leading to the Buckle-Leverett type equation, one reaches to the equation that we are interested in

$$\begin{aligned} u_t + div(f(u)) &= A\triangle u + B\partial_t \triangle u, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \\ u|_{t=0} &= u_0(x) \in H_0^3(\mathbb{R}^d) \\ u|_{|x|\to\infty} &= 0. \end{aligned}$$

for a $C^1$-function $f$, where $A, B, T > 0$ are constants.

# Diffusion-dispersion limit problem

Using the self similar solution to the approximation of the form (we call it vanishing capillarity approximation)

$$\partial_t S_{\varepsilon,\delta} + \partial_x f(S_{\varepsilon,\delta}) = \varepsilon \Delta S_{\varepsilon,\delta} + \tau\delta\partial_t \Delta S_{\varepsilon,\delta}$$

with the corresponding Riemann initial data

$$S_{\varepsilon,\delta}(0,x) = \begin{cases} S_L, & x < 0 \\ S_R, & x > 0 \end{cases}$$

van Dujin, Peletier and Pop, obtained in the limit solution to the Buckley-Leverett which is non-monotonic and which up to some extent explains experimental results.

In general, a similar type of problem:

$$\partial_t S_{\varepsilon,\delta} + \partial_x f(S_{\varepsilon,\delta}) = \varepsilon \Delta S_{\varepsilon,\delta} + \tau\delta \sum_{i=1}^{d} \partial_{x_i x_i x_i} S_{\varepsilon,\delta}$$

was used to construct non-classical shocks of scalar conservation laws.
However, for neither of approximations, existence of a initial or initial-boundary value problem is unknown. Here, we are going to show that the dynamics capillarity approximation with corresponding initial and boundary conditions admits a unique solution.

# The proof strategy

We shall use the Leray-Schauder fixed point theorem.

- We first fix non-linear part of the equation to get

$$u_t + div(f(v)) = A\triangle u + B\partial_t\triangle u. \tag{1}$$

- We define the mapping $\mathcal{T} : L^2([0,T); H^1_0(\mathbb{R}^d)) \to L^2([0,T); H^1_0(\mathbb{R}^d))$ for a fixed $T \in (0,1)$:

$$\mathcal{T}(v) = u \tag{2}$$

given for the fixed function $v \in L^2([0,T); H^1_0(\mathbb{R}^d))$.

- We prove that the mapping $\mathcal{T}$ admits the fixed point.

We use the following variant of the fixed point theorem.

**Theorem**

*Let $\mathcal{T}$ be a compact mapping of Banach space $\mathcal{B}$ onto itself and suppose that there exists a constant $C$ such that*

$$\|u\|_{\mathcal{B}} \leq C \tag{3}$$

*for all $u \in \mathcal{B}$ and $\sigma \in [0, 1]$ satisfying $u = \sigma \mathcal{T} u$. Then $\mathcal{T}$ has a fixed point, that is, $\mathcal{T} u = u$ for some $x \in \mathcal{B}$.*

We reach necessary estimates essentially through the following lemma.

## Lemma

*The function* $u \in L^2([0,T); H_0^1(\mathbb{R}^d))$ *satisfying* (1) *satisfies for almost every* $t \in \mathbb{R}_+$

$$e^{-t}\int_0^t div(f(v(s,x)))e^s ds$$

$$= \triangle u(t,x) - u(t,x) + e^{-t}(u_0(x) - \triangle u_0(x)) + e^{-t}\int_0^t u(s,x)e^s ds. \tag{4}$$

Equation (4) is essentially a Poisson equation which has very nice properties and the integral terms make no particular problems. From there, we easily obtain all the necessary estimates. Remark, in particular, that due to zero boundary conditions, we can explicitly express $u$ from

$$\begin{aligned} \triangle u(t,x) - u(t,x) = e^{-t}\int_0^t div(f(v(s,x)))e^s ds \\ + e^{-t}(u_0(x) + \triangle u_0(x)) - e^{-t}\int_0^t u(s,x)e^s ds = F(t,x), \end{aligned} \tag{5}$$

by applying the Fourier transform ^ and the inverse Fourier transform $\mathcal{F}^{-1}$

$$u(t,x) = \mathcal{F}^{-1}\left(\frac{\hat{F}(t,\xi)}{1+|\xi|^2}\right)(x).$$

# The End

*Thank you for listening.*