Variant of optimality criteria method for multiple state optimal design problems

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Optimal design problem

- Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded and $f \in \mathrm{H}^{-1}(\Omega)$. We consider stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{cases}$$

- We assume that Ω is a mixture of two isotropic materials with conductivities $0 < \alpha < \beta$, i.e.

$$\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}, \quad \text{where} \quad \chi \in \mathrm{L}^\infty(\Omega;\{0,1\})$$

and that the amount of the first material is given by $q_\alpha = \int_\Omega \chi\, d\mathbf{x}$. Then, the **multiple state optimal design problem** is

$$\begin{cases} J(\chi) = \displaystyle\int_\Omega \chi(\mathbf{x}) g_\alpha(\mathbf{x},\mathsf{u}) + (1-\chi(\mathbf{x})) g_\beta(\mathbf{x},\mathsf{u})\, d\mathbf{x} \longrightarrow \min\,, \\ \chi \in \mathrm{L}^\infty(\Omega;\{0,1\})\,, \ \displaystyle\int_\Omega \chi\, d\mathbf{x} = q_\alpha\,, \end{cases} \tag{1}$$

where $\mathsf{u} = (u_1,\ldots,u_m)$ is the state function determined by

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1,\ldots,m$$

with $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$ and $f_i \in \mathrm{H}^{-1}(\Omega)$, while g_α, g_β are Caratheodory functions which satisfies growth condition

$$g_j(x,u) \le a|u|^s + b(x), \qquad j = \alpha,\beta,$$

for some $a > 0$, $b \in \mathrm{L}^1(\Omega)$ and $1 \le s < \frac{2d}{d-2}$, $d \ge 3$.

- Interpretations:
  - Minimize the amount of heat kept inside body
  - Maximize the flow rate of two viscous immiscible fluids through pipe

Relaxed problem

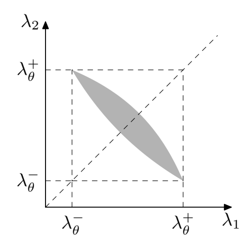
- Problem (1) does not have classical solution so we need to relax it. We use relaxation by the homogenization method introduced by Murat and Tartar in order to get problem

$$\left\{ \begin{array}{l} J(\theta,\mathbf{A}) = \displaystyle\int_\Omega (\theta(\mathbf{x}) g_\alpha(\mathbf{x},\mathsf{u}) + (1-\theta(\mathbf{x})) g_\beta(\mathbf{x},\mathsf{u}))\, d\mathbf{x} \longrightarrow \min \\ (\theta,\mathbf{A}) \in \{(\theta,\mathbf{A}) \in \mathrm{L}^\infty(\Omega;[0,1]\times \mathrm{M}_d(\mathbf{R})) : \mathbf{A} \in \mathcal{K}(\theta)\ a.e.\}\,, \ \displaystyle\int_\Omega \theta\, d\mathbf{x} = q_\alpha\,, \\ \mathsf{u} = (u_1,u_2,\ldots,u_m), \text{ where } u_i,\ i = 1,\ldots,m \text{ satisfies} \\ \qquad \begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega)\,. \end{cases} \end{array} \right. \tag{2}$$

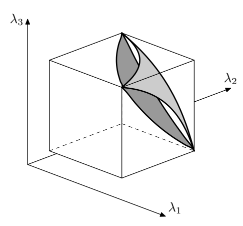
Set $\mathcal{K}(\theta)$ is given in terms of eigenvalues of matrix $\mathbf{A}$ (Murat & Tartar; Lurie & Cherkaev):

$$\begin{aligned} \lambda_\theta^- \le \ \lambda_j \ &\le \lambda_\theta^+ \quad j = 1,\ldots,d \\ \sum_{j=1}^d \frac{1}{\lambda_j - \alpha} &\le \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha} \\ \sum_{j=1}^d \frac{1}{\beta - \lambda_j} &\le \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+}\,, \\ \lambda_\theta^+ &= \theta\alpha + (1-\theta)\beta, \\ \frac{1}{\lambda_\theta^-} &= \frac{\theta}{\alpha} + \frac{1-\theta}{\beta} \end{aligned}$$

2D:



3D:



- Let us introduce adjoint states $p_1,\ldots,p_m$ as solutions of

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla p_i) = \theta\dfrac{\partial g_\alpha}{\partial u_i}(\cdot,\mathsf{u}) + (1-\theta)\dfrac{\partial g_\beta}{\partial u_i}(\cdot,\mathsf{u}) \\ p_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1,\ldots,m\,.$$

Result

Theorem 1. Let $(\theta^*,\mathbf{A}^*)$ be a minimizer of objective functional $J(\theta,\mathbf{A})$ in (2) with states u_i^* and corresponding adjoint states p_i^*. We introduce symmetric matrix

$$\mathbf{N} = \mathrm{Sym}\sum_{i=1}^m \sigma_i^* \otimes \tau_i^*,$$

for $\sigma_i^* = \mathbf{A}\nabla u_i^*$, $\tau_i^* = \mathbf{A}\nabla p_i^*$ and define function $g(\theta,\mathbf{N}) = \min_{\mathbf{A}\in\mathcal{K}(\theta)} (\mathbf{A}^{-1} : \mathbf{N})$. Then

$$(\mathbf{A}^*)^{-1}(\mathbf{x}) : \mathbf{N}(\mathbf{x}) = g(\theta^*(\mathbf{x}),\mathbf{N}(\mathbf{x})), \qquad \text{a.e. } x \in \Omega.$$

Moreover, if we define function

$$R(\mathbf{x}) = g_\alpha(\mathbf{x},\mathsf{u}) - g_\beta(\mathbf{x},\mathsf{u}) + l + \frac{\partial g}{\partial\theta}(\theta^*(\mathbf{x}),\mathbf{N}^*(\mathbf{x}))$$

the optimal θ^* satisfies (a.e. on Ω)

$$\begin{aligned} \theta^*(\mathbf{x}) &= 0 & &\text{if } R(\mathbf{x}) > 0 \\ \theta^*(\mathbf{x}) &= 1 & &\text{if } R(\mathbf{x}) < 0 \\ 0 \le \theta^*(\mathbf{x}) &\le 1 & &\text{if } R(\mathbf{x}) = 0\,. \end{aligned}$$

Teorem 2. (d=3) For given $\theta \in [0,1]$ and matrix $\mathbf{N}$ with eigenvalues $\eta_1 \ge \eta_2 \ge \eta_3$ we have

A. If $\eta_3 = 0$, then $\dfrac{\partial g}{\partial\theta}(\theta,\mathbf{N}) = \dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2)$.

B. If $\eta_3 > 0$, then

$$\frac{\partial g}{\partial\theta}(\theta,\mathbf{N}) = \begin{cases} \beta^{-1}(\beta-\alpha)(\alpha+2\beta)\left(\dfrac{\sqrt{\eta_1}+\sqrt{\eta_2}+\sqrt{\eta_3}}{2\theta(\alpha-\beta)+\alpha+2\beta}\right)^2, & \theta < \theta_1^B, \\ \beta^{-1}(\beta^2-\alpha^2)\left(\dfrac{\sqrt{\eta_2}+\sqrt{\eta_3}}{\theta(\alpha-\beta)+\alpha+\beta}\right)^2 + \eta_1\dfrac{(\beta-\alpha)}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_1^B \le \theta < \theta_2^B, \\ (\alpha^{-1}-\beta^{-1})\eta_3 + \dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \ge \theta_2^B, \end{cases}$$

where $\theta_1^B = 1 - \dfrac{\alpha(2\sqrt{\eta_1}-\sqrt{\eta_2}-\sqrt{\eta_3})}{(\beta-\alpha)(\sqrt{\eta_2}+\sqrt{\eta_3}-\sqrt{\eta_1})}$ and $\theta_2^B = 1 - \dfrac{\alpha(\sqrt{\eta_2}-\sqrt{\eta_3})}{(\beta-\alpha)\sqrt{\eta_3}}$.

C. Let $\eta_3 < 0$. If η_2 and η_3 are negative as well then

$$\frac{\partial g}{\partial\theta}(\theta,\mathbf{N}) = \begin{cases} -\alpha^{-1}(\beta-\alpha)(2\alpha+\beta)\left(\dfrac{\sqrt{-\eta_1}+\sqrt{-\eta_2}+\sqrt{-\eta_3}}{2\theta(\alpha-\beta)+3\beta}\right)^2, & \theta > \theta_1^C, \\ -\alpha^{-1}(\beta^2-\alpha^2)\left(\dfrac{\sqrt{-\eta_2}+\sqrt{-\eta_3}}{\theta(\alpha-\beta)+2\beta}\right)^2 + \eta_1\dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}, & \theta_2^C < \theta \le \theta_1^C, \\ (\alpha^{-1}-\beta^{-1})\eta_3 + \dfrac{\beta-\alpha}{(\theta\alpha+(1-\theta)\beta)^2}(\eta_1+\eta_2), & \theta \le \theta_2^C\,. \end{cases}$$

where $\theta_1^C = \dfrac{\beta(\sqrt{-\eta_2}+\sqrt{-\eta_3}-2\sqrt{-\eta_1})}{(\beta-\alpha)(\sqrt{-\eta_2}+\sqrt{-\eta_3}-\sqrt{-\eta_1})}$ and $\theta_2^C = \dfrac{\beta(\sqrt{-\eta_3}-\sqrt{-\eta_2})}{(\beta-\alpha)\sqrt{-\eta_3}}$.

If $\eta_2 < 0$ and $\eta_1 \ge 0$, then θ_1^C is not defined so we can express $\dfrac{\partial g}{\partial\theta}(\theta,\mathbf{N})$ by the formulae given above, but omitting its first case and the assumption $\theta \le \theta_1^C$ in the second case.

If $\eta_2 \ge 0$ then both θ_1^C and θ_2^C are not defined, and $\dfrac{\partial g}{\partial\theta}(\theta,\mathbf{N})$ is given by the formula given in the third case above, for any $\theta \in [0,1]$.

Optimality criteria method

- Using Teorem 1. we can develop a variant of optimality criteria method for numerical solution of problem (2). The algorithm is following:

Algorithm. Take some initial θ^0 and $\mathbf{A}^0$. For $k \in \mathbf{N}$:

1. Calculate u_i^k, $i = 1,\ldots,m$, the solution of $\begin{cases} -\mathsf{div}\,(\mathbf{A}^k\nabla u_i) = f_i \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{cases}$
2. Calculate p_i^k, $i = 1,\ldots,m$, the solution of $\begin{cases} -\mathsf{div}\,(\mathbf{A}^k\nabla p_i) = \theta^k\dfrac{\partial g_\alpha}{\partial u_i}(\cdot,\mathsf{u}) + (1-\theta^k)\dfrac{\partial g_\beta}{\partial u_i}(\cdot,\mathsf{u}) \\ p_i \in \mathrm{H}^1_0(\Omega), \end{cases}$

   $\sigma_i^k = \mathbf{A}^k\nabla u_i^k$, $\tau_i^k = \mathbf{A}^k\nabla p_i^k$ and $\mathbf{N}^k = \mathrm{Sym}\sum_{i=1}^m(\sigma_i^k \otimes \tau_i^k)$
3. For $\mathbf{x} \in \Omega$, let $\theta^{k+1}(\mathbf{x})$ be the zero of function

$$\theta \mapsto g_\alpha(\mathbf{x},\mathsf{u}^k(\mathbf{x})) - g_\alpha(\mathbf{x},\mathsf{u}^k(\mathbf{x})) + l + \frac{\partial g}{\partial\theta}(\theta,\mathbf{N}^k(\mathbf{x}))\,,$$

   if a zero doesn't exist, take 0 (or 1) in case the function is positive (or negative).
4. Calculate $(\mathbf{A}^{k+1})^{-1}(\mathbf{x})$ as the minimizer of a function $g(\theta^{k+1},\mathbf{N}^k) = \min_{\mathbf{A}\in\mathcal{K}(\theta^{k+1})}(\mathbf{A}^{-1} : \mathbf{N}^k)$.
5. Stopping criteria: $\|\theta^{k+1} - \theta^k\|_{L^1(\Omega)} < \varepsilon$.

References

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