Explicit solutions in optimal design problems for stationary diffusion equation

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WeConMApp

Joint work with Marko Vrdoljak

Outline

Compliance optimization, composite materials and relaxation

Multiple states - spherically symmetric case

Examples

Optimal design problem (single state)

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For given Ω, α, β, q_α and f we want to find such material $\mathbf{A}$ which minimizes the compliance functional (total amount of heat/electrical energy dissipated in Ω):

$$J(\chi) = \int_\Omega f(\mathbf{x})u(\mathbf{x})d\mathbf{x} = \int_\Omega \mathbf{A}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla u(\mathbf{x}) \, d\mathbf{x} \longrightarrow \min,$$

where u is the solution of the state equation (1).

Relaxation by homogenisation

$$\begin{array}{rcl} \chi \in \mathrm{L}^\infty(\Omega;\{0,1\}) & \cdots & \theta \in \mathrm{L}^\infty(\Omega;[0,1]) \\ \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I} & & \mathbf{A} \in \mathcal{K}(\theta) \quad \text{a.e. on } \Omega \\ \text{classical material} & & \text{composite material - relaxation} \end{array}$$

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Definition

A sequence of matrix functions $\mathbf{A}^\varepsilon$ is said to H-converge to $\mathbf{A}^*$ if for every f the sequence of solutions of

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}^\varepsilon \nabla u_\varepsilon) = f \\ u_\varepsilon \in \mathrm{H}^1_0(\Omega) \end{cases}$$

satisfies $u_\varepsilon \rightharpoonup u$ in $\mathrm{H}^1_0(\Omega)$, $\mathbf{A}^\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbf{A}^* \nabla u$ in $\mathrm{L}^2(\Omega; \mathbf{R}^d)$, where u is the solution of the homogenised equation

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}^* \nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega)\,. \end{cases}$$

Composite material

Definition

If a sequence of characteristic functions $\chi_\varepsilon \in \mathrm{L}^\infty(\Omega; \{0, 1\})$ *and conductivities*

$$\mathbf{A}^\varepsilon(x) = \chi_\varepsilon(x)\alpha\mathbf{I} + (1 - \chi_\varepsilon(x))\beta\mathbf{I}$$

satisfy $\chi_\varepsilon \rightharpoonup \theta$ *weakly* $*$ *and* $\mathbf{A}^\varepsilon$ *H-converges to* $\mathbf{A}^*$*, then it is said that* $\mathbf{A}^*$ *is homogenised tensor of two-phase composite material with proportions* θ *of first material and microstructure defined by the sequence* (χ_ε)*.*

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Example – simple laminates: if χ_ε depend only on x_1, then

$$\mathbf{A}^* = diag(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+, \dots, \lambda_\theta^+)\,,$$

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$$\lambda_\theta^+ = \theta\alpha + (1 - \theta)\beta\,, \qquad \frac{1}{\lambda_\theta^-} = \frac{\theta}{\alpha} + \frac{1 - \theta}{\beta}\,.$$

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Set of all composites:

$$\mathcal{A} := \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]\times\mathrm{M}_d(\mathbf{R})) : \int_\Omega \theta\, d\mathbf{x} = q_\alpha\,,\ \mathbf{A} \in \mathcal{K}(\theta) \text{ a.e.}\}$$

Effective conductivities – set $\mathcal{K}(\theta)$

G–closure problem: for given θ find all possible homogenised (effective) tensors $\mathbf{A}^*$

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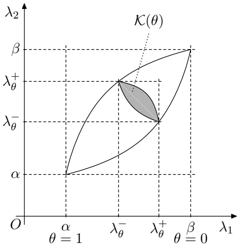

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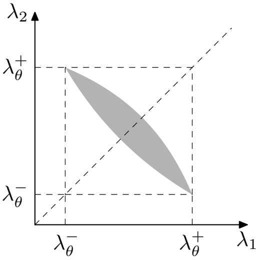

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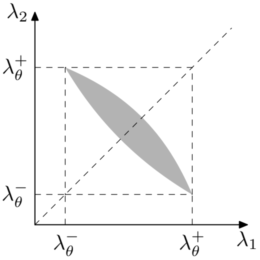

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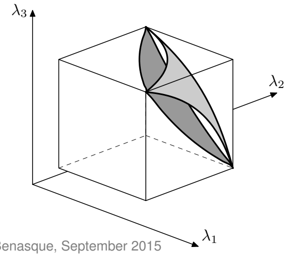

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$\min_{\mathcal{A}} J$ is a proper relaxation of $\min_{\mathrm{L}^\infty(\Omega;\{0,1\})} I$

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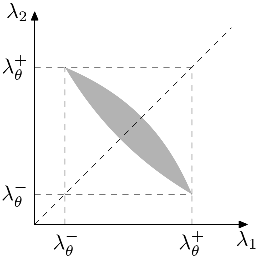

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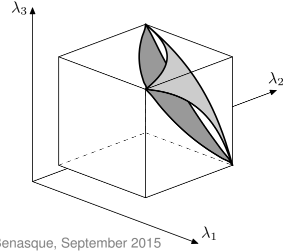

Multiple state optimal design problem

State equations

$$\begin{cases} -\mathrm{div}\left(\mathbf{A}\nabla u_i\right) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega) \end{cases} \qquad i = 1, \ldots, m$$

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for some given weights $\mu_i > 0$.

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for some given weights $\mu_i > 0$. Proper relaxation:

$$J(\theta, \mathbf{A}) = \sum_{i=1}^m \mu_i \int_\Omega f_i u_i \, d\mathbf{x} \to \min \qquad \text{on}$$

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In spherically symmetric case the simpler relaxation can be done!

Relaxed designs

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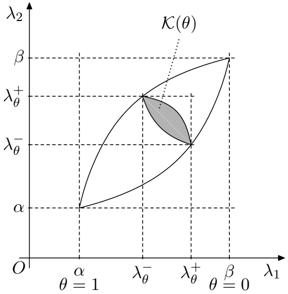

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Further relaxation:

$\mathcal{B}$ $\quad \ldots \quad \int_\Omega \theta \, d\mathbf{x} = q_\alpha$

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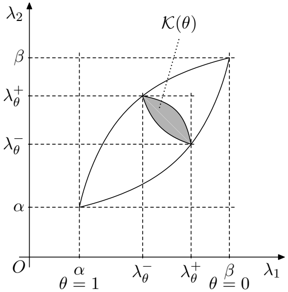

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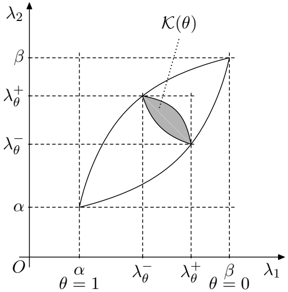
$\mathcal{B}$ is convex and compact and J is continuous on $\mathcal{B}$, so there is a solution of $\min_{\mathcal{B}} J$.



Equivalence of $\min_{\mathcal{B}} J$ and $\min_{\mathcal{T}} I$

Theorem

- *There is unique* $\mathrm{u}^* \in \mathrm{H}^1_0(\Omega; \mathbf{R}^m)$ *which is the state for every solution of* $\min_{\mathcal{B}} J$ *and* $\min_{\mathcal{T}} I$*.*

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- *If* $(\theta^*, \mathbf{A}^*)$ *is an optimal design for the problem* $\min_{\mathcal{B}} J$*, then* θ^* *is optimal design for* $\min_{\mathcal{T}} I$.
- *Conversely, if* θ^* *is a solution of optimal design problem* $\min_{\mathcal{T}} I$*, then any* $(\theta^*, \mathbf{A}^*) \in \mathcal{B}$ *satisfying* $\mathbf{A}^* \nabla u_i^* = \lambda_+(\theta^*) \nabla u_i^*$ *almost everywhere on* Ω *(e.g.* $\mathbf{A}^* = \lambda_+(\theta^*)\mathbf{I}$*) is an optimal design for the problem* $\min_{\mathcal{B}} J$.

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- *If* $m < d$*, then there exists minimizer* $(\theta^*, \mathbf{A}^*)$ *for* J *on* $\mathcal{B}$*, such that* $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$*, and thus it is also minimizer for* J *on* $\mathcal{A}$*.*

Simpler relaxation in case of spherical symmetry

Theorem

Let $\Omega \subseteq \mathbf{R}^d$ be spherically symmetric, and let the right-hand sides $f_i = f_i(r)$, $r \in \omega$, $i = 1, \ldots, m$ be a radial function. Then there exists a minimizer $(\theta^*, \mathbf{A}^*)$ of the optimal design problem $\min_{\mathcal{A}} J$ which is a radial function.

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a) *For any minimizer θ of functional I over $\mathcal{T}$, let us define a radial function $\theta^* : \Omega \longrightarrow \mathbf{R}$ as the average value over spheres of θ: for $r \in \omega$ we take*

$$\theta^*(r) := \fint_{\partial B(\mathbf{0},r)} \theta \, dS \,,$$

where S denotes the surface measure on a sphere. Then θ^* is also minimizer for I over $\mathcal{T}$.

Simpler relaxation in case of spherical symmetry...cont.

Theorem

b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^*$ as a simple laminate with layers orthogonal to a radial direction $\mathbf{e}_r$ and local proportion of the first material θ^*. To be specific, we can define $\mathbf{A}^* : \Omega \longrightarrow \mathrm{M}_d(\mathbf{R})$ in the following way:*

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Simpler relaxation in case of spherical symmetry...cont.

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b) *For any radial minimizer θ^* of I over $\mathcal{T}$, let us define $\mathbf{A}^*$ as a simple laminate with layers orthogonal to a radial direction $\mathbf{e}_r$ and local proportion of the first material θ^*. To be specific, we can define $\mathbf{A}^* : \Omega \longrightarrow \mathrm{M}_d(\mathbf{R})$ in the following way:*

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Moreover, $(\theta^*, \mathbf{A}^*) \in \mathcal{A}$, and thus it is also a solution for $\min_{\mathcal{A}} J$.

Optimality conditions for $\min_{\mathcal{T}} I$

Lemma

$\theta^* \in \mathcal{T}$ is a solution $\min_{\mathcal{T}} I$ if and only if there exists a Lagrange multiplier $c \geq 0$ such that

$$\begin{aligned}
\theta^* \in \langle 0, 1\rangle \quad &\Rightarrow \quad \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 = c\,, \\
\theta^* = 0 \quad &\Rightarrow \quad \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 \geq c\,, \\
\theta^* = 1 \quad &\Rightarrow \quad \sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 \leq c\,,
\end{aligned}$$

or equivalently

$$\begin{aligned}
\sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 > c \quad &\Rightarrow \quad \theta^* = 0\,, \\
\sum_{i=1}^{m} \mu_i |\nabla u_i^*|^2 < c \quad &\Rightarrow \quad \theta^* = 1\,.
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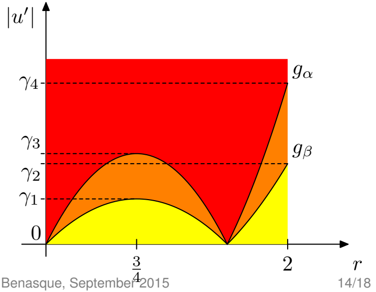

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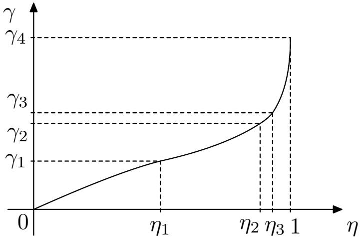

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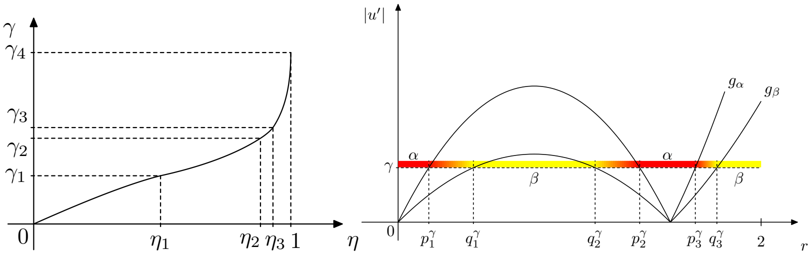

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Solving state equation

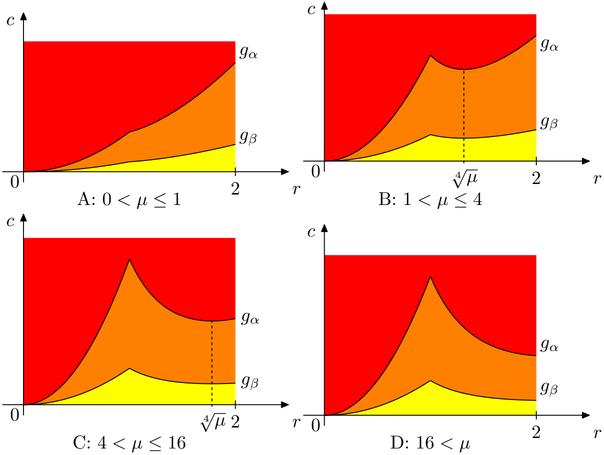
$$u_i'(r) = \frac{\psi_i(r)}{\theta(r)\alpha + (1 - \theta(r))\beta}\ ,\ i = 1, 2\ ,$$

with

$$\psi_1(r) = \begin{cases} -\dfrac{r}{2}\,, & 0 \leq r < 1\,, \\ -\dfrac{1}{2r}\,, & 1 \leq r \leq 2\,, \end{cases} \quad \text{and} \quad \psi_2(r) = -\frac{r}{2}\,.$$

Similarly as in the first example: $\psi := \mu\psi_1^2 + \psi_2^2$, $g_\alpha := \frac{\psi}{\alpha^2}$, $g_\beta := \frac{\psi}{\beta^2}$.

Geometric interpretation of optimality conditions



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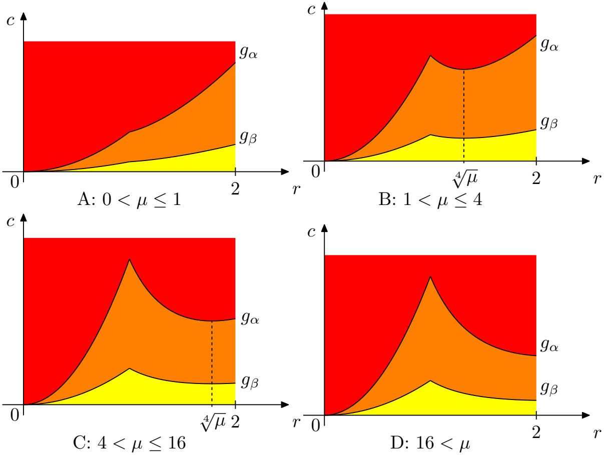
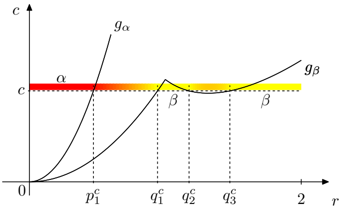


As before, Lagrange multiplier can be numerically calculated from corresponding algebraic equation $⨍_\Omega \theta^* \, d\mathbf{x} = \eta$.

Optimal θ^* for case B



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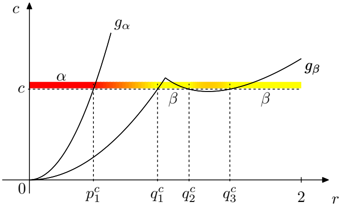

In orange region:

$$\theta^*(r) = \frac{1}{\beta - \alpha}\left(\beta - \sqrt{\frac{\psi(r)}{c}}\right)$$