The Schwartz kernel theorem and distributions of finite order

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Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

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and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

can be extended to a continuous mapping $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$.

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Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|}\ ,$$

then for any $p \in \langle 1, \infty \rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty)\ .$$

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For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d_*$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

Existence of H-distributions

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The assumptions imply $u_n, v_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, resulting in a distribution μ of order zero (an unbounded Radon measure, not a general distribution).
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For vector-valued $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

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The H-distribution would correspond to a non-diagonal block for an H-measure.

A particular Nemyckiĭ operator

Canonical choice of $\mathrm{L}^{p'}$ sequence corresponding to an L^p, $p \in \langle 1, \infty \rangle$, sequence (u_n) is given by $v_n = \Phi_p(u_n)$, where Φ_p is an operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\Phi_p(u) = |u|^{p-2}u$.

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Φ_p is a nonlinear Nemytskiĭ operator, continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ and additionally we have the following bound

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It maps bounded sets in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ topology to bounded sets in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ topology. Hence for an L^p bounded sequence (u_n), we get that $(\Phi_p(u_n))$ is weakly precompact in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

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It is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

Example: concentration

$u \in \mathrm{L}^p_c(\mathbf{R}^d)$, and define $u_n(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x} - \mathbf{z}))$ for some $\mathbf{z} \in \mathbf{R}^d$.

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Indeed, the sequence is bounded, while for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$

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&= \frac{1}{n^{d/p'}} \int_{\mathbf{R}^d} u(\mathbf{y})\chi_{\mathrm{supp}\, u}(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\
&\leqslant \left(\frac{\mathsf{vol}(\mathrm{supp}\, u)}{n^d}\right)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi|.
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Passing to the limit, we get our claim.

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Actually, the H-distribution corresponding to sequences (u_n) and $(\Phi_p(u_n))$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a distribution on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle \nu, \psi \rangle = \int_{\mathbf{R}^d} u(\mathbf{x}) \overline{\mathcal{A}_{\bar\psi}(|u|^{p-2}u)(\mathbf{x})} d\mathbf{x}.$$

Functions of anisotropic smoothness

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By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \leqslant l$ and $|\boldsymbol{\beta}| \leqslant m$,

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$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} \, ,$$

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For a compact set $K \subseteq \Omega$ we define a subspace of $\mathrm{C}^{l,m}(\Omega)$

$$\mathrm{C}_K^{l,m}(\Omega) := \left\{ f \in \mathrm{C}^{l,m}(\Omega) : \; \mathrm{supp}\, f \subseteq K \right\} .$$

This subspace inherits the topology from $\mathrm{C}^{l,m}(\Omega)$, which is, when considered only on the subspace, a norm topology determined by

$$\|f\|_{l,m,K} := p_K^{l,m}(f) \; ,$$

and $\mathrm{C}_K^{l,m}(\Omega)$ is a Banach space (it can be identified with a proper subspace of $\mathrm{C}^{l,m}(K)$). However, if $m = \infty$ (or $l = \infty$), then we shall not get a Banach space, but a Fréchet space. As in the isotropic case, an increasing sequence of seminorms that makes $\mathrm{C}_{K_n}^{l,\infty}(\Omega)$ a Fréchet space is given by $(p_{K_n}^{l,k}), k \in \mathbf{N}_0$.

Functions of anisotropic smoothness (cont.)

We can also consider the space

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of all functions with compact support in $\mathrm{C}^{l,m}(\Omega)$, and equip it by a stronger topology than the one induced from $\mathrm{C}^{l,m}(\Omega)$: by the topology of *strict inductive limit*.

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More precisely, it can easily be checked that

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Of course, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega)$ is a continuous and dense imbedding.

Distributions of anisotropic order

Definition. A *distribution of order* l *in* $\mathbf{x}$ *and order* m *in* $\mathbf{y}$ is any linear functional on $\mathrm{C}_c^{l,m}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

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Clearly, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, with continuous and dense imbeddings, thus $\mathrm{C}_c^{l,m}(\Omega)$ is a normal space of distributions, hence its dual $\mathcal{D}'_{l,m}(\Omega)$ forms a subspace of $\mathcal{D}'(\Omega)$. If we equip it with a strong topology, it is even continuously imbedded in $\mathcal{D}'(\Omega)$.

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Lemma. *Let* X *and* Y *be* C^∞ *manifolds. For a linear functional* u *on* $\mathrm{C}_c^{l,m}(X \times Y)$, *the following statements are equivalent*

a) $u \in \mathcal{D}'_{l,m}(X \times Y)$,

b) $(\forall K \in \mathcal{K}(X \times Y))(\exists C > 0)(\forall \Psi \in \mathrm{C}_K^{l,m}(X \times Y)) \quad |\langle u, \Psi \rangle| \leqslant C p_K^{l,m}(\Psi)$. ∎

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Statement (b) of previous lemma implies:

$$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^{l}(X))(\forall \psi \in \mathrm{C}_L^{m}(Y)) \qquad |\langle u, \varphi \boxtimes \psi \rangle| \leqslant C p_K^{l}(\varphi) p_L^{m}(\psi) \, .$$

The reverse implication would have significantly greater practical use.

Tensor product of distributions

In order to better understand the properties of elements of $\mathcal{D}'_{l,m}(\Omega)$, we shall relate them to tensor products.
The first step is to consider the algebraic tensor product $\mathrm{C}_c^l(X) \boxtimes \mathrm{C}_c^m(Y)$, the vector space of all (finite) linear combinations of functions of the form $(\phi \boxtimes \psi)(\mathbf{x}, \mathbf{y}) := \phi(\mathbf{x})\psi(\mathbf{y})$. This is a vector subspace of $\mathrm{C}_c^{l,m}(X \times Y)$.

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Theorem. *Let X and Y be C^∞ manifolds, $u \in \mathcal{D}'_l(X)$ and $v \in \mathcal{D}'_m(Y)$. Then*

$$\Big(\exists! w \in \mathcal{D}'_{l,m}(X\times Y)\Big)\Big(\forall\varphi \in \mathrm{C}^l_c(X)\Big)\Big(\forall\psi \in \mathrm{C}^m_c(Y)\Big) \quad \langle w, \varphi\boxtimes\psi\rangle = \langle u, \varphi\rangle\langle v, \psi\rangle.$$

Furthermore, for any $\Phi \in \mathrm{C}^{l,m}_c(X \times Y)$, function $V : \mathbf{x} \mapsto \langle v, \Phi(\mathbf{x}, \cdot)\rangle$ is in $\mathrm{C}^l_c(X)$, while $U : \mathbf{y} \mapsto \langle u, \Phi(\cdot, \mathbf{y})\rangle$ is in $\mathrm{C}^m_c(Y)$, and we have that

$$\langle w, \Phi\rangle = \langle u, V\rangle = \langle v, U\rangle.$$

∎

Lemma. *If* $u \in \mathcal{D}'_{l,m}(X \times Y)$ *then, for any* $\psi \in \mathrm{C}^{l,m}(X \times Y)$*,* ψu *is a well defined distribution of order at most* (l, m)*.* ∎

Theorem. *Let* $u \in \mathcal{D}'_{l,m}(X \times Y)$ *and take* $F \subseteq X \times Y$ *relatively compact set such that* $\operatorname{supp} u \subseteq F$*. Then there exists unique linear functional* $\tilde{u}$ *on* $\mathcal{Q} := \{\varphi \in \mathrm{C}^{l,m}(X \times Y) : F \cap \operatorname{supp} \varphi \Subset X \times Y\}$ *such that*

a) $(\forall \varphi \in \mathrm{C}_c^{l,m}(X \times Y)) \quad \langle \tilde{u}, \varphi \rangle = \langle u, \varphi \rangle$*,*

b) $(\forall \varphi \in \mathrm{C}^{l,m}(X \times Y)) \quad F \cap \operatorname{supp} \varphi = \emptyset \quad \Longrightarrow \quad \langle \tilde{u}, \varphi \rangle = 0$*.*

The domain of $\tilde{u}$ *is largest for* $F = \operatorname{supp} u$*.* ∎

First conjecture

Let X, Y be C^∞ manifolds and u a linear functional on $\mathrm{C}_c^{l,m}(X \times Y)$. If $u \in \mathcal{D}'(X \times Y)$ and satisfies

$$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(X))(\forall \psi \in \mathrm{C}_L^\infty(Y)) \qquad |\langle u, \varphi \boxtimes \psi \rangle| \leqslant C p_K^l(\varphi) p_L^m(\psi) \ ,$$

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If it were true, then the H-distribution μ would belong to $\mathcal{D}'_{0,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it would be a distribution of order 0 in $\mathbf{x}$ and of order not more than κ in $\boldsymbol{\xi}$.

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Indeed, from the proof of the existence theorem, we already have $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ and the following bound with $\varphi := \varphi_1 \overline{\varphi_2}$:

$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leqslant C \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \,,$$

where C does not depend on φ and ψ.
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We need a more complicated result.

Schwartz kernel theorem

Theorem. *Let X and Y be two differentiable manifolds.*

a) *Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.*

b) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as above. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathrm{C}^\infty_c(X)$ and $\psi \in \mathrm{C}^\infty_c(Y)$*

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y)$. ∎

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We can assume X and Y to be open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$.
Indeed, first take an open covering of Y, consisting of chart domains, and a partition of unity (f_α) subordinate to that covering such that $\sum_\alpha f_\alpha(\mathbf{y}) = 1, \mathbf{y} \in H$ (note that the sum is finite).
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By [Gösser, Kunzinger & al., Chapter 3.1.4], we can identify distributions localised on chart domains with distributions on subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$. Thus, in what follows we shall assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$.

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We shall therefore show that there exists a constant $C > 0$ such that for any $\psi \in \mathrm{C}_H^m(Y)$ it holds

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for m finite, while for $m = \infty$ we should modify the above to

$$(\exists\, m' \in \mathbf{N})(\exists\, C > 0)(\forall\, \psi \in \mathrm{C}_H^\infty(Y)) \quad |\langle K_\varphi, \psi \rangle| \leqslant C \max_{|\boldsymbol{\beta}| \leqslant m'} \|\partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(H)} \,.$$

From kernel to operator (a)(cont.)

K is a distribution of anisotropic order on $X \times Y$:

$$(\forall\, M \in \mathcal{K}(X \times Y))(\exists \tilde{C} > 0)(\forall \Psi \in \mathrm{C}_c^{l,m}(X \times Y))$$
$$\operatorname{supp} \Psi \subseteq M \Longrightarrow |\langle K, \Psi \rangle| \leqslant \tilde{C} \max_{|\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} \Psi\|_{\mathrm{L}^\infty(M)} \,,$$

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by taking M to be of the form $L \times H$, with $L \in \mathcal{K}(X)$, and $\Psi = \varphi \boxtimes \psi$ such that $\operatorname{supp}\varphi \subseteq L$, we have

$$|\langle K_\varphi, \psi\rangle| = |\langle K, \varphi \boxtimes \psi\rangle| \leqslant \tilde{C} \max_{|\boldsymbol{\alpha}|\leqslant l, |\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\alpha}}\varphi \boxtimes \partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(L\times H)}$$
$$\leqslant \tilde{C} \max_{|\boldsymbol{\alpha}|\leqslant l} \|\partial^{\boldsymbol{\alpha}}\varphi\|_{\mathrm{L}^\infty(L)} \max_{|\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(H)} \leqslant C \max_{|\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\beta}}\psi\|_{\mathrm{L}^\infty(H)}\ ,$$

and therefore $K_\varphi \in \mathcal{D}'_m(Y)$.

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For continuity, take an arbitrary $L \in \mathcal{K}(X)$ and an arbitrary $\psi \in \mathrm{C}_c^m(Y)$. We need to show the existence of $\bar{C} > 0$ such that

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However, we have already shown that above: just take

$$\bar{C} = \tilde{C} \max_{|\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(H)} .$$

Therefore, the mapping $\varphi \mapsto K_\varphi$, from $\mathrm{C}_c^l(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.

From operator to kernel (b): uniqueness and overview

Let us first prove the uniqueness. By formula

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ ,$$

a continuous functional K on $\mathrm{C}_c^\infty(X) \boxtimes \mathrm{C}_c^\infty(Y)$ is defined. As it is defined on a dense subset of $\mathrm{C}_c^\infty(X \times Y)$, such K is uniquely determined on the whole $\mathrm{C}_c^\infty(X \times Y)$.

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The proof of existence will be divided into two steps. In the first step we assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$, and additionally, that the range of A is $\mathrm{C}(Y) \subseteq \mathcal{D}'_m(Y)$ (understood as distributions which can be identified with continuous functions). This will allow us to write explicitly the action of $A\varphi$ on a test function $\psi \in \mathrm{C}_c^m(Y)$, which will finally enable us to define the kernel K. In the second step, we shall use a partition of unity and the structure theorem of distributions to reduce the problem to the first step.

From operator to kernel (b): existence under additional assumptions

Additionally assume that X and Y are open and bounded subsets of euclidean spaces, and that for each $\varphi \in \mathrm{C}_c^l(X)$, $A\varphi \in \mathrm{C}(Y)$.

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Its action on a test function $\psi \in \mathrm{C}_c^m(Y)$ is given by

$$\langle A\varphi, \psi \rangle = \int_Y (A\varphi)(\mathbf{y})\psi(\mathbf{y})d\mathbf{y} \ .$$

Continuity of A implies that $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$ is continuous when the range is equipped with the weak $*$ topology inherited from $\mathcal{D}'_m(Y)$.

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Continuity of A implies that $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$ is continuous when the range is equipped with the weak $*$ topology inherited from $\mathcal{D}'_m(Y)$.

As the latter is a Hausdorff space, that operator has a closed graph, but this remains true even when we replace the topology on $\mathrm{C}(Y)$ by its standard Fréchet topology [Narici & Beckenstein, Exercise 14.101(a)], which is stronger.

From operator to kernel (b): existence under additional assumptions

Additionally assume that X and Y are open and bounded subsets of euclidean spaces, and that for each $\varphi \in \mathrm{C}_c^l(X)$, $A\varphi \in \mathrm{C}(Y)$.

Its action on a test function $\psi \in \mathrm{C}_c^m(Y)$ is given by

$$\langle A\varphi, \psi \rangle = \int_Y (A\varphi)(\mathbf{y})\psi(\mathbf{y})d\mathbf{y} \; .$$

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As the latter is a Hausdorff space, that operator has a closed graph, but this remains true even when we replace the topology on $\mathrm{C}(Y)$ by its standard Fréchet topology [Narici & Beckenstein, Exercise 14.101(a)], which is stronger.

Now we can apply the Closed graph theorem [Narici & Beckenstein, Theorem 14.3.4(b)], as $\mathrm{C}_c^l(X)$ is barreled, as a strict inductive limit of barreled spaces, to conclude that $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$ is continuous with usual strong topologies on its domain and range.

(b): existence under additional assumptions (cont.)

For $\mathbf{y} \in Y$ consider a linear functional $F_{\mathbf{y}} : \mathrm{C}_c^l(X) \longrightarrow \mathbf{C}$ defined by

$$F_{\mathbf{y}}(\varphi) = (A\varphi)(\mathbf{y}) \; .$$

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Since $A\varphi$ is a continuous function, $F_{\mathbf{y}}$ is well-defined and continuous as a composition of continuous mappings, thus a distribution in $\mathcal{D}_l'(X)$.

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Since $A\varphi$ is a continuous function, $F_{\mathbf{y}}$ is well-defined and continuous as a composition of continuous mappings, thus a distribution in $\mathcal{D}'_l(X)$.

Take a test function $\Psi \in \mathrm{C}_c^{l,0}(X \times Y)$, and fix its second variable (get a function from $\mathrm{C}_c^l(X)$) and apply $F_{\mathbf{y}}$; we are interested in the properties of this mapping:

$$\mathbf{y} \mapsto F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) = \Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y}) \ .$$

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$$\mathbf{y} \mapsto F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) = \Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y}) \ .$$

Clearly, it is well defined on Y, with a compact support contained in the projection $\pi_Y(\operatorname{supp} \Psi)$. Furthermore, we have:

$$\begin{aligned} \Big|F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\Big| = \Big|\Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y})\Big| &\leqslant \|A\Psi(\cdot, \mathbf{y})\|_{\mathrm{L}^\infty(\pi_Y(\operatorname{supp} \Psi))} \\ &\leqslant C\|\Psi(\cdot, \mathbf{y})\|_{\mathrm{C}^l_{\pi_Y(\operatorname{supp} \Psi)}(X)} \leqslant C\|\Psi\|_{\mathrm{C}^{l,0}_{\operatorname{supp} \Psi}(X \times Y)} \ . \end{aligned}$$

(b): existence under additional assumptions (cont.)

We show sequential continuity: take a sequence $\mathbf{y}_n \to \mathbf{y}$ in Y. Denote $H = \pi_X(\operatorname{supp}\Psi)$ and let $L \subseteq Y$ be a compact such that $\mathbf{y}_n, \mathbf{y} \in L$; Ψ is uniformly continuous on compact $H \times L$.

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This is also valid for $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\Psi$, where $|\boldsymbol{\alpha}| \leqslant l$, thus $\Psi(\cdot, \mathbf{y}_n) \longrightarrow \Psi(\cdot, \mathbf{y})$ in $\mathrm{C}_c^l(X)$.

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As A is continuous, the convergence is carried to $\mathrm{C}(Y)$, i.e. to uniform convergence on compacts of a sequence of functions $A\Psi(\cdot, \mathbf{y}_n)$ to $A\Psi(\cdot, \mathbf{y})$. In particular,$(A\Psi(\cdot, \mathbf{y}_n))(\bar{\mathbf{y}}) - (A\Psi(\cdot, \mathbf{y}))(\bar{\mathbf{y}})$ is arbitrary small independently of $\bar{\mathbf{y}} \in L$, for large enough n.

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On the other hand, $A\Psi(\cdot, \mathbf{y})$ is uniformly continuous, thus $(A\Psi(\cdot, \mathbf{y}))(\bar{\mathbf{y}}) - (A\Psi(\cdot, \mathbf{y}))(\mathbf{y})$ is small for large n, independetly of $\bar{\mathbf{y}} \in L$. In other terms, we have the required convergence

$$F_{\mathbf{y}_n}(\Psi(\cdot, \mathbf{y}_n)) \longrightarrow F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) .$$

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$$F_{\mathbf{y}_n}(\Psi(\cdot, \mathbf{y}_n)) \longrightarrow F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) \ .$$

A continuous function with compact support is summable, so we can define K on $\mathrm{C}_c^{l,0}(X \times Y)$:

$$\langle K, \Psi \rangle = \int_Y F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) \, d\mathbf{y} \ ,$$

which is obviously linear in Ψ, as $F_{\mathbf{y}}$ is.

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However, we can check that K is continuous at zero (modifications for $l = \infty$):

$$(\forall\, H \in \mathcal{K}(X))(\forall\, L \in \mathcal{K}(Y))(\exists\, C > 0)(\forall\, \Psi \in \mathrm{C}_c^{l,0}(X \times Y))$$
$$\operatorname{supp}\Psi \subseteq H \times L \quad \Longrightarrow \quad |\langle K, \Psi\rangle| \leqslant C \|\Psi\|_{\mathrm{C}^{l,0}_{K\times L}(X\times Y)}\,.$$

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The continuity of $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$, for Ψ supported in $H \times L$ and the fact that the support of $A\Psi(\cdot, \mathbf{y})$ is contained in L gives us the estimate

$$\left|\int_Y F_{\mathbf{y}}(\Psi(\cdot,\mathbf{y}))\, d\mathbf{y}\right| \leqslant (\mathrm{vol} L) C\|\Psi\|_{\mathrm{C}^{l,0}_{K\times L}(X\times Y)} \,,$$

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as needed.

Finally, it is easy to check that for $\varphi \in \mathrm{C}_c^\infty(X)$ and $\psi \in \mathrm{C}_c^\infty(Y)$, we have:

$$\langle K, \varphi \boxtimes \psi\rangle = \int_Y F_{\mathbf{y}}(\varphi \boxtimes \psi(\mathbf{y}))d\mathbf{y} = \int_Y F_{\mathbf{y}}(\varphi)\psi(\mathbf{y})d\mathbf{y}$$
$$= \int_Y (A\varphi)(\mathbf{y})\psi(\mathbf{y})d\mathbf{y} = \langle A\varphi, \psi\rangle\ .$$

(b) existence in general: reduction to charts

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It is sufficient to show existence of distributions $K_{\alpha\beta}$ on $U_\alpha \times V_\beta$, which satisfy

$$\langle A\varphi, \psi \rangle = \langle K_{\alpha\beta}, \varphi \boxtimes \psi \rangle \ , \quad \varphi \in \mathrm{C}_c^\infty(U_\alpha), \psi \in \mathrm{C}_c^\infty(V_\beta) \ .$$

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Indeed, the uniqueness of $K \in \mathcal{D}'(X \times Y)$ then follows from the fact that two distributions $K_{\alpha\beta}$ and $K_{\gamma\delta}$ will coincide on open sets $(U_\alpha \cap U\gamma) \times (V_\beta \cap V_\delta)$ of $X \times Y$, while the existence of K will be a result of the localisation theorem [Dieudonne, 17.4.2].

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Furthermore, if we assume that U_α and V_β lie within domains of some charts of X and Y, in the light of results of [Gösser, Kunzinger & al., Chapter 3.1.4], we can identify the distributions localised to these chart domains with distributions on open subsets of $\mathbf{R}^d$. Thus, without loss of generality, we assume that U and V are relatively compact open subsets of $\mathbf{R}^d$.

(b) existence in general: the structure theorem

Consider $\tilde{A} : \mathrm{C}^l_c(U) \to \mathcal{D}'_m(V)$ defined by: for $\varphi \in \mathrm{C}^l_c(U)$ and $\psi \in \mathrm{C}^m_c(V)$

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Take a relatively compact open neighbourhood W of $\mathrm{Cl}\, V$ in Y and pick a smooth cut-off function ρ being one on $\mathrm{Cl}\, V$ and supported in W.

For $\varphi \in \mathrm{C}_c^l(U)$, $\rho\tilde{A}\varphi \in \mathcal{D}'_m(W)$ and has a compact support. Next we use the Structure theorem for distributions: from its proof [Friedlander & Joshi, Theorem 5.4.1], we can write

$$\rho\tilde{A}\varphi = \left(\partial_1^{m+2} \ldots \partial_d^{m+2}\right) \Big(E_{m+2} * (\rho\tilde{A}\varphi) \Big) \ ,$$

where E_{m+2} is the fundamental solution of $\partial_1^{m+2} \ldots \partial_d^{m+2}$ (derivatives in $\mathbf{y}$), i.e. it satisfies the equation $\left(\partial_1^{m+2} \ldots \partial_d^{m+2}\right) E_{m+2} = \delta_0$ (explicit formula for E_{m+2} in loc.cit.), and $E_{m+2} * (\rho\tilde{A}\varphi)$ is a continuous function.

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Denoting by $\widetilde{E}_{m+2}*$ the transpose of $E_{m+2}*$, for $\varphi \in \mathrm{C}^l_c(U)$ and $\psi \in \mathrm{C}^m_c(W)$

$$\Big\langle E_{m+2} * (\rho\tilde{A}\varphi), \psi \Big\rangle = \Big\langle \tilde{A}\varphi, \rho\widetilde{E}_{m+2} * \psi \Big\rangle \ ,$$

concluding that $\varphi \mapsto E_{m+2} * (\rho\tilde{A}\varphi)$ is continuous from $\mathrm{C}^l_c(U)$ to $\mathcal{D}'_m(W)$.

(b) existence in general: reduction to special case

Now we can find $R \in \mathcal{D}'_{l,0}(U \times W)$ such that for all $\varphi \in \mathrm{C}_c^\infty(U)$ and $\psi \in \mathrm{C}_c^\infty(W)$ it holds

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$$\left\langle E_{m+2} * (\rho\tilde{A}\varphi), \psi \right\rangle = \langle R, \varphi \boxtimes \psi \rangle \ .$$

Taking $\varphi \in \mathrm{C}_c^\infty(U)$ and $\psi \in \mathrm{C}_c^\infty(V)$, we have

$$\begin{aligned}
\left\langle R, \varphi \boxtimes \left(\partial_1^{m+2} \dots \partial_d^{m+2}\right) \psi \right\rangle &= \left\langle E_{m+2} * (\rho\tilde{A}\varphi), \left(\partial_1^{m+2} \dots \partial_d^{m+2}\right) \psi \right\rangle \\
&= (-1)^{d(m+2)} \left\langle \left(\partial_1^{m+2} \dots \partial_d^{m+2}\right) \left(E_{m+2} * (\rho\tilde{A}\varphi)\right), \psi \right\rangle \\
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&= (-1)^{d(m+2)} \left\langle \tilde{A}\varphi, \rho\psi \right\rangle \\
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\end{aligned}$$

which gives $\langle A\varphi, \psi \rangle = (-1)^{d(m+2)} \left\langle \left(\partial_1^{m+2} \dots \partial_d^{m+2}\right) R, \varphi \boxtimes \psi \right\rangle$, where the derivatives are taken with respect to the variable $\mathbf{y}$. Since R was an element of $\mathcal{D}'_{l,0}(U \times W)$, we conclude that $A \in \mathcal{D}'_{l,d(m+2)}(U \times V)$.

Remarks

Note that in part (b) we did not get $K \in \mathcal{D}'_{l,m}(X \times Y)$, as one would expect. The order with respect to $\mathbf{x}$ variable remained the same, but the order with respect to $\mathbf{y}$ increased from m to $d(m+2)$. Interchanging the roles of X and Y, the same proof gives $K \in \mathcal{D}'_{d(l+2),m}(X \times Y)$, where order with respect to $\mathbf{y}$ remained the same, but order with respect to the $\mathbf{x}$ variable increased from l to $d(l+2)$. Since uniqueness of $K \in \mathcal{D}'(X \times Y)$ has already been determined, we conclude that $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y) \cap \mathcal{D}'_{d(l+2),m}(X \times Y)$. It might be interesting to see some additional properties of that intersection.

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If one used a more constructive proof of the Schwartz kernel theorem, for example [Simanca, Theorem 1.3.4], one would end up increasing the order with respect to both variables $\mathbf{x}$ and $\mathbf{y}$. This occurs naturally, because one needs to secure the integrability of the function which is used to define the kernel function.

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One interesting approach to the kernel theorem is given in [Trèves, Chapter 51]. This approach is based on deep results of functional analysis on tensor products of nuclear spaces of Alexander Grothendieck. This approach might result in further improvements of the preceeding theorem. This is a subject of our current ongoing research.

Consequence for H-distributions

By the previous theorem the H-distribution μ mentioned at the beginning belongs to the space $\mathcal{D}'_{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than $d(\kappa+2)$ in $\boldsymbol{\xi}$.

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$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leqslant C \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} ,$$

where C does not depend on φ and ψ.

Now we just need to apply the Schwartz kernel theorem given above to conclude that μ is a continuous linear functional on $\mathrm{C}_c^{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$.

Thank you for your attention.