L^p version of the First commutation lemma

Marin Mišur

email: mmisur@math.hr
PMF-Matematički odsjek
Joint work with Nenad Antonić and Darko Mitrović.

21^{st} of June 2016.

What is the First commutation lemma?

- $\mathcal{A}_\psi u := (\psi \hat{u})^\vee$
- $M_b u := bu$

$$[\mathcal{A}_\psi, M_b] := \mathcal{A}_\psi M_b - M_b \mathcal{A}_\psi$$

Compactness on L^2 - Cordes' result[1]

Theorem

If bounded continuous functions b and ψ satisfy

$$\lim_{|\boldsymbol{\xi}|\to\infty} \sup_{|\mathbf{h}|\leq 1} \{|\psi(\boldsymbol{\xi}+\mathbf{h})-\psi(\boldsymbol{\xi})|\} = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{|\mathbf{h}|\leq 1} \{|b(\mathbf{x}+\mathbf{h})-b(\mathbf{x})|\} = 0 \, ,$$

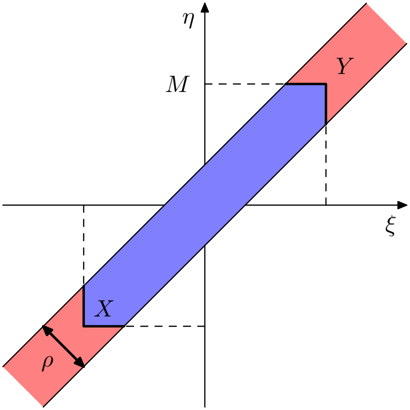
then the commutator $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$.

[1] H. O. Cordes, *On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators*, J. Funct. Anal. **18** (1975) 115–131.

Compactness on L^2 - Tartar's version

For given $M, \varrho \in \mathbf{R}^+$ we denote the set

$$Y(M,\varrho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geq M \& |\boldsymbol{\xi} - \boldsymbol{\eta}| \leq \varrho\} \ .$$



Compactness on L^2 - Tartar's version[2]

Lemma (general form of the First commutation lemma)

If $b \in \mathrm{C}_0(\mathbf{R}^d)$, while $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ satisfies the condition

$$(\forall\, \varrho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |\psi(\boldsymbol{\xi}) - \psi(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\mathrm{s.s.}\;\; (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \varrho)) \,, \quad (1)$$

then $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$.

Lemma

Let $\pi : {\mathbf{R}^d}_* \to \Sigma$ be a smooth projection to a smooth compact hypersurface Σ, such that $\|\nabla\pi(\boldsymbol{\xi})\| \to 0$ for $|\boldsymbol{\xi}| \to \infty$, and let $\psi \in \mathrm{C}(\Sigma)$. Then $\psi \circ \pi$ (ψ extended by homogeneity of order 0) satisfies (1).

[2] L. Tartar, *The general theory of homogenization: A personalized introduction*, Springer, 2009.

Where is it used?

- L. Tartar, *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations,* Proc. Roy. Soc. Edinburgh **115A** (1990) 193–230.[3]
- E. Ju. Panov, *Ultra-parabolic H-measures and compensated compactness*, Ann. Inst. H. Poincaré Anal. Non Linéaire C **28** (2011) 47–62.
- N. Antonić, M. Lazar, *Parabolic H-measures*, J. Funct. Anal. **265** (2013) 1190–1239.
- Z. Lin, *Instability of nonlinear dispersive solitary waves*, J. Funct. Anal. **255** (2008) 1191–1224.
- Z. Lin, *On Linear Instability of 2D Solitary Water Waves*, International Mathematics Research Notices **2009** (2009) 1247–1303.
- S. Richard, R. T. de Aldecoa, *New Formulae for the Wave Operators for a Rank One Interaction*, Integr. Equ. Oper. Theory **66** (2010) 283–292.

[3] P. Gérard, *Microlocal defect measures,* Comm. Partial Diff. Eq. **16** (1991) 1761–1794.

Boundedness on L^p - the Hörmander-Mihlin theorem

Theorem

Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ *have partial derivatives of order less than or equal to* $\kappa = [d/2] + 1$. *If for some* $k > 0$

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \quad |\boldsymbol{\alpha}| \le \kappa \Longrightarrow \int_{r/2 \le |\boldsymbol{\xi}| \le r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \le k^2 r^{d-2|\boldsymbol{\alpha}|},$$

then for any $p \in \langle 1, \infty \rangle$ *and the associated multiplier operator* $\mathcal{A}_\psi$ *there exists a constant* C_d *such that*

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \le C_d \max\{p, 1/(p-1)\}(k + \|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}).$$

What about the L^p variant of the First commutation lemma?

- one variant in the article by Cordes - complicated proof and higher regularity assumptions

- N. Antonić, D. Mitrović, *H-distributions: an extension of H-measures to an $\mathrm{L}^p - \mathrm{L}^q$ setting*, Abs. Appl. Analysis **2011** Article ID 901084 (2011) 12 pp.

A variant of Krasnoselskij's type of result[4]

Lemma

Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle \setminus \{2\}$. Then A is also compact on $\mathrm{L}^p(\mathbf{R}^d)$, for any p between 2 and r (i.e. such that $1/p = \theta/2 + (1-\theta)/r$, for some $\theta \in \langle 0, 1\rangle$).

Corollary

If $b \in \mathrm{C}_0(\mathbf{R}^d)$, while $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d)$ satisfies the conditions of the Hörmander-Mihlin theorem, then the commutator $[\mathcal{A}_\psi, M_b]$ is a compact operator on $\mathrm{L}^p(\mathbf{R}^d)$, for any $p \in \langle 1, \infty\rangle$.

---

[4] M. A. Krasnoselskij, *On a theorem of M. Riesz*, Dokl. Akad. Nauk SSSR **131** (1960) 246–248 (in russian); translated as Soviet Math. Dokl. **1** (1960) 229–231.

Theorem

Let $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ *be bounded and satisfy Hörmander's condition, while* $b \in \mathrm{C}_c(\mathbf{R}^d)$*. Then for any* $u_n \xrightarrow{*} 0$ *in* $\mathrm{L}^\infty(\mathbf{R}^d)$ *and* $p \in \langle 1, \infty\rangle$ *one has:*

$$(\forall \varphi, \phi \in \mathrm{C}_c^\infty(\mathbf{R}^d)) \qquad \phi C(\varphi u_n) \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^p(\mathbf{R}^d)\ .$$

Corollary

Let (u_n) *be a bounded, uniformly compactly supported sequence in* $\mathrm{L}^\infty(\mathbf{R}^d)$*, converging to 0 in the sense of distributions. Assume that* $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d \setminus \{0\})$ *satisfies Hörmander's condition and condition from the general form of the First commutation lemma.*
Then for any $b \in \mathrm{L}^s(\mathbf{R}^d)$*,* $s > 1$ *arbitrary, it holds*

$$\lim_{n\to\infty} \|b\mathcal{A}_\psi(u_n) - \mathcal{A}_\psi(bu_n)\|_{\mathrm{L}^r(\mathbf{R}^d)} = 0, \quad r \in \langle 1, s\rangle.$$

Further comments...

- $b \in \mathrm{VMO}$; variant of H-measures
- results of I.-L. Hwang and A. Stefanov