One-scale H-distributions

Nenad Antonić and Marko Erceg

nenad@math.hr, maerceg@math.hr

Department of Mathematics, Faculty of Science, University of Zagreb

Introduction

Theorem. [6], [4] *If $u_n, v_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0$, then there exist subsequences $(u_{n'}), (v_{n'})$ and $\mu_H \in \mathcal{M}(\mathbf{R}^d \times \mathrm{S}^{d-1})$ such that $(\forall \varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d))(\forall \psi \in \mathrm{C}(\mathrm{S}^{d-1}))$*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{\varphi_2 v_{n'}}(\mathbf{x})\,d\mathbf{x} = \langle \mu_H, \varphi_1\bar\varphi_2\psi\rangle\,,$$

$\mathcal{A}_\psi(u) = \bar{\mathcal{F}}(\psi\hat u)$ being the Fourier multiplier with symbol $\psi\circ\boldsymbol{\pi}$, $\boldsymbol{\pi}(\boldsymbol{\xi}) := \boldsymbol{\xi}/|\boldsymbol{\xi}|$. ∎

Measure μ_H we call *the H-measure* associated to the (sub)sequences $(u_{n'}), (v_{n'})$.

Theorem. [3] *If $u_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} u$, $v_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} v$ and $\omega_n \to 0^+$, then there exist subequences $(u_{n'}), (v_{n'})$ and $\mu_{sc} \in \mathcal{M}(\mathbf{R}^d \times \mathbf{R}^d)$ such that $(\forall \varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d))(\forall \psi \in \mathcal{S}(\mathbf{R}^d))$*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})(\mathbf{x})\overline{\varphi_2 v_{n'}}(\mathbf{x})\,d\mathbf{x} = \langle \mu_{sc}, \varphi_1\bar\varphi_2\psi\rangle\,,$$

where $\mathcal{A}_{\psi_{n'}}$ is the F.m. with symbol $\psi_{n'} := \psi(\omega_{n'}\cdot)$. ∎

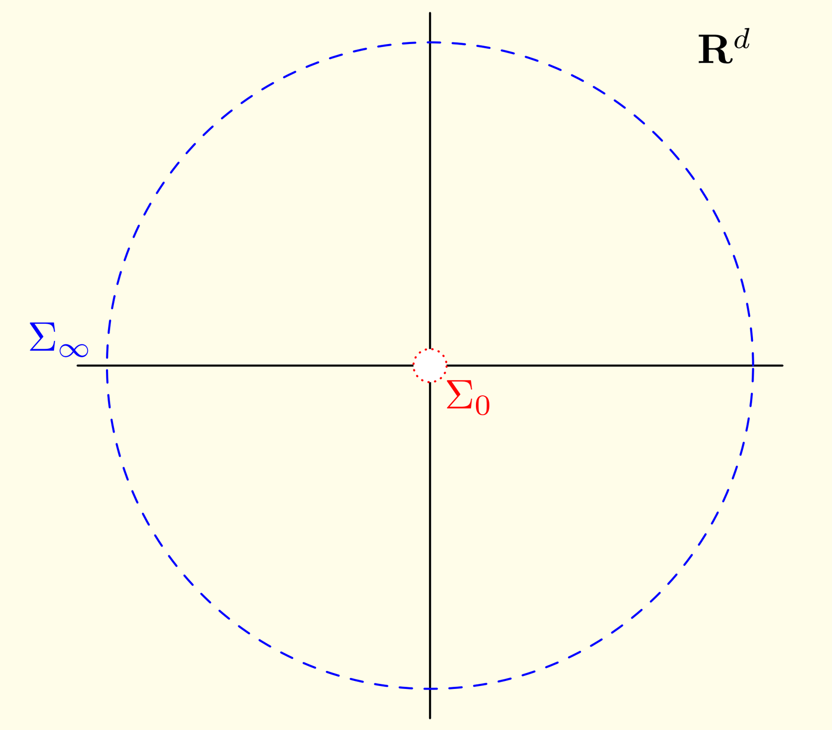
Measure μ_{sc} we call *the semiclassical measure with characteristic length* $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'}), (v_{n'})$.
Alternative, semiclassical measures can be defined via *the Wigner transform* [5]:

$$W_n(\mathbf{x},\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i\mathbf{y}\cdot\boldsymbol{\xi}} u_n\Big(\mathbf{x} + \frac{\omega_n\mathbf{y}}{2}\Big) v_n\Big(\mathbf{x} - \frac{\omega_n\mathbf{y}}{2}\Big)\,d\mathbf{y} \xrightarrow{\mathcal{S}'} \mu_{sc}\,.$$

In general we cannot reconstruct semiclassical measures from H-measures and vice versa (see [7, 1]). That can be considered as a motivation for introducing new object, *one-scale H-measures*.

1-scale H-measure

Tartar introduced new space of test functions ψ in the Fourier space [7] as continuous functions on a compactification $\mathrm{K}^d_{0,\infty}$ of $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{0\}$ by adding two spheres (around the origin, Σ_0, and in the infinity, Σ_∞):



$$\Sigma_0 := \{0^{\boldsymbol{\xi}} : \boldsymbol{\xi} \in \mathrm{S}^{d-1}\}\,,\ \Sigma_\infty := \{\infty^{\boldsymbol{\xi}} : \boldsymbol{\xi} \in \mathrm{S}^{d-1}\}$$
$$\mathrm{K}^d_{0,\infty} := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

In the sequel we need differential calculus on $\mathrm{K}^d_{0,\infty}$ so we choose

$$\mathrm{K}^d_{0,\infty} := \mathrm{Cl}\,\mathcal{J}(\mathbf{R}^d_*) = \mathrm{Cl}\,A(2^{-1/2}, 1)\,,$$

where $A(2^{-1/2}, 1) := \{\boldsymbol{\xi} \in \mathbf{R}^d : 2^{-1/2} < |\boldsymbol{\xi}| < 1\}$ and $\mathcal{J} : \mathbf{R}^d_* \longrightarrow A(2^{-1/2}, 1)$ is a smooth bijection given by

$$\mathcal{J}(\boldsymbol{\xi}) := \frac{\boldsymbol{\xi}}{\sqrt{|\boldsymbol{\xi}|^2 + \frac{|\boldsymbol{\xi}|^2}{(|\boldsymbol{\xi}|+1)^2}}}$$

(see the figure on the right).

For $\psi \in \mathrm{C}(\mathrm{K}^d_{0,\infty})$ by $\psi^* := \psi\circ\mathcal{J}$ we denote the pullback of ψ by $\mathcal{J}$.

Theorem. [7], [1] *If $u_n, v_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0$, $\omega_n \to 0^+$, then there exist subsequences $(u_{n'}), (v_{n'})$ and $\mu_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\mathbf{R}^d \times \mathrm{K}^d_{0,\infty})$ such that $(\forall \varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d))(\forall \psi \in \mathrm{C}(\mathrm{K}^d_{0,\infty}))$*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi^*_{n'}}(\varphi_1 u_{n'})(\mathbf{x})\overline{\varphi_2 v_{n'}}(\mathbf{x})\,d\mathbf{x} = \langle \mu_H, \varphi_1\bar\varphi_2\psi\rangle\,,$$

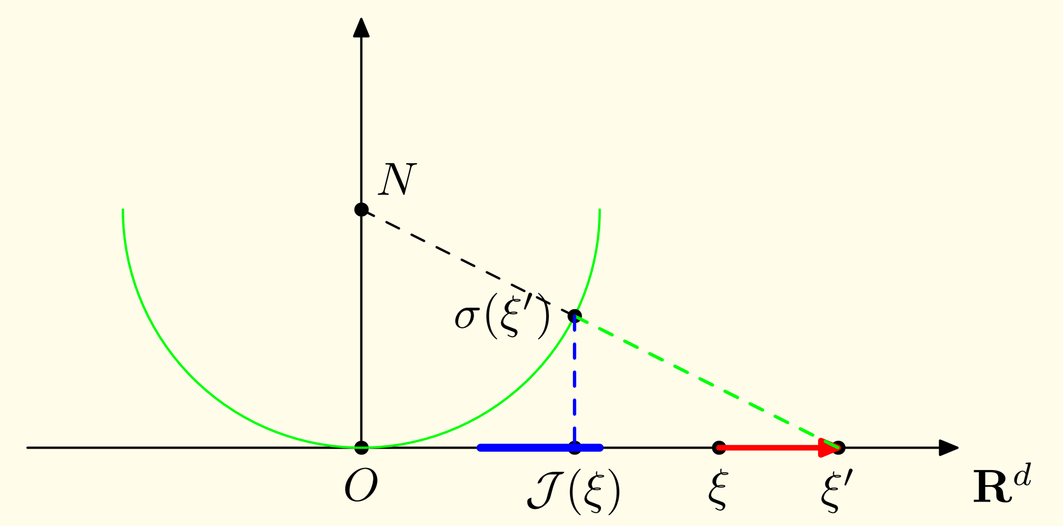
where $\mathcal{A}_{\psi^*_{n'}}$ is the F.m. with symbol $\psi^*_{n'} := \psi^*(\omega_{n'}\cdot)$. ∎

Measure $\mu_{\mathrm{K}_{0,\infty}}$ we call *the one-scale H-measure with characteristic length* $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'})$, $(v_{n'})$.
By the very definition of $\mathrm{K}^d_{0,\infty}$ we have

- $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathcal{J}^*(\mathrm{C}(\mathrm{K}^d_{0,\infty}))$,
- $\psi \in \mathrm{C}(\mathrm{S}^{d-1}), \psi\circ\boldsymbol{\pi} \in \mathcal{J}^*(\mathrm{C}(\mathrm{K}^d_{0,\infty}))$,

which implies that $\mu_{\mathrm{K}_{0,\infty}}$ is an *extension* of μ_H and μ_{sc}: for arbitrary $\psi \in \mathcal{S}(\mathbf{R}^d)$, $\tilde\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\varphi \in \mathrm{C}_c(\Omega)$ we have

- $\langle \mu_{\mathrm{K}_{0,\infty}}, \varphi\boxtimes\psi\rangle = \langle \mu_{sc}, \varphi\boxtimes\psi\rangle$
- $\langle \mu_{\mathrm{K}_{0,\infty}}, \varphi\boxtimes\tilde\psi\circ\boldsymbol{\pi}\rangle = \langle \mu_H, \varphi\boxtimes\tilde\psi\rangle$



Localisation principle for one-scale H-measures

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $l \in 0..m$, $\varepsilon_n \to 0^+$. If $u_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0$, consider sequence of equations

$$(1)\qquad \sum_{l\leqslant|\boldsymbol\alpha|\leqslant m} \varepsilon_n^{|\boldsymbol\alpha|-l}\partial_{\boldsymbol\alpha}(a^{\boldsymbol\alpha}_n u_n) = f_n \quad \text{in } \Omega\,,$$

where $a^{\boldsymbol\alpha}_n \in \mathrm{C}(\Omega)$ and for every $\boldsymbol\alpha$, $a^{\boldsymbol\alpha}_n \to a^{\boldsymbol\alpha}$ uniformly on compact sets, and $f_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega)$ such that for any $\varphi \in \mathrm{C}^\infty_c(\Omega)$

$$(2)\qquad \frac{\widehat{\varphi f_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol\xi|^s} \xrightarrow{\mathrm{L}^2} 0\,.$$

In [7] Tartar proved the localisation principle for one-scale H-measure, but that result does not provide any information on the structure of the measure on Σ_0. This disadvantage has been resolved and the result generalised in [1].

Theorem. [1] *Under the previous assumptions and (ε_n) not neccesary convergent, take $\omega_n \to 0^+$ such that $\lim_n \frac{\varepsilon_n}{\omega_n} = c \in [0,\infty]$. Then for $\mu_{\mathrm{K}_{0,\infty}}$ with characteristic length (ω_n) associated to (u_n) we have*

$$p\mu_{\mathrm{K}_{0,\infty}} = 0\,,$$

where

$$p(\mathbf{x},\boldsymbol\xi) := \begin{cases} \sum_{|\boldsymbol\alpha|=l} \frac{\boldsymbol\xi^{\boldsymbol\alpha}}{|\boldsymbol\xi|^l + |\boldsymbol\xi|^m} a^{\boldsymbol\alpha}(\mathbf{x}) & , \ \frac{\varepsilon_n}{\omega_n} \to 0 \\ \sum_{l\leqslant|\boldsymbol\alpha|\leqslant m} (2\pi i c)^{|\boldsymbol\alpha|} \frac{\boldsymbol\xi^{\boldsymbol\alpha}}{|\boldsymbol\xi|^l + |\boldsymbol\xi|^m} a^{\boldsymbol\alpha}(\mathbf{x}) & , \ \frac{\varepsilon_n}{\omega_n} \to c \in \mathbf{R}^+ \\ \sum_{|\boldsymbol\alpha|=m} \frac{\boldsymbol\xi^{\boldsymbol\alpha}}{|\boldsymbol\xi|^l + |\boldsymbol\xi|^m} a^{\boldsymbol\alpha}(\mathbf{x}) & , \ \frac{\varepsilon_n}{\omega_n} \to \infty \end{cases}.$$

Moreover, if there exists $\varepsilon_0 > 0$ such that $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, we can take

$$p(\mathbf{x},\boldsymbol\xi) := \sum_{|\boldsymbol\alpha|=m} \frac{\boldsymbol\xi^{\boldsymbol\alpha}}{|\boldsymbol\xi|^m} a^{\boldsymbol\alpha}(\mathbf{x})\,.$$ ∎

From the previous theorem we can obtain the known localisation principle for H-measures and the localisation principle for semiclassical measures under weaker assumptions on the convergence of the right hand side ((2) is a weaker assumption then the convergence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$) (cf. [1]).

H-distributions

Aim: Generalise the notion of H-measures to the L^p setting.
Problem: Find $M \subseteq \mathrm{C}(\mathrm{S}^{d-1})$ such that for every $\psi \in M$ we have $\mathcal{A}_\psi \in \mathcal{L}(\mathrm{L}^p; \mathrm{L}^p)$, $p \in \langle 1, \infty\rangle$.
Answer: The Mihlin multiplier theorem: Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d_*)$ satisfies, $\kappa := \lfloor d/2\rfloor + 1$,

$$(\forall |\boldsymbol\alpha| \leqslant \kappa)(\forall \boldsymbol\xi \in \mathbf{R}^d_*)(\exists A > 0)\quad |\partial^{\boldsymbol\alpha}\psi(\boldsymbol\xi)| \leqslant \frac{A}{|\boldsymbol\xi|^{|\boldsymbol\alpha|}}\,.$$

Theorem. [2] *If $u_n \xrightarrow{\mathrm{L}^p_{\mathrm{loc}}} 0$ and $v_n \xrightarrow{\mathrm{L}^{p'}_{\mathrm{loc}}} v$ then there exist $(u_{n'}), (v_{n'})$ and $\nu_H \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$, such that $(\forall \varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d))(\forall \psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1}))$*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\,d\mathbf{x} = \langle \nu_H, \varphi_1\bar\varphi_2\psi\rangle\,.$$ ∎

Distribution ν_H we call *the H-distribution* associated to the (sub)sequences $(u_{n'}), (v_{n'})$.
The localisation principle for H-distributions is also provided in [2]. Our aim is to introduce the generalisation of one-scale H-measures, *one-scale H-distributions*, which will play the same role with H-distributions as one-scale H-measures plays with H-measures in L^2 case.

Existence of one-scale H-distributions

For any $\psi \in \mathrm{C}^\kappa(\mathrm{K}^d_{0,\infty})$ we have

$$(\forall |\boldsymbol\alpha| \leqslant \kappa)(\forall \boldsymbol\xi \in \mathbf{R}^d_*)\quad |\partial^{\boldsymbol\alpha}\psi^*(\boldsymbol\xi)| \leqslant \frac{\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}^d_{0,\infty})}}{|\boldsymbol\xi|^{|\boldsymbol\alpha|}}\,,$$

hence $\mathcal{A}_{\psi^*_n}$ is a bounded operator on $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$.

Theorem. *If $u_n \xrightarrow{\mathrm{L}^p_{\mathrm{loc}}} 0$, $v_n \xrightarrow{\mathrm{L}^{p'}_{\mathrm{loc}}} v$ and $\omega_n \to 0^+$, then there exist $(u_{n'})$, $(v_{n'})$ and $\nu_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{K}^d_{0,\infty})$, such that $(\forall \varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d))(\forall \psi \in \mathrm{C}^\kappa(\mathrm{K}^d_{0,\infty}))$*

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi^*_{n'}}(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\,d\mathbf{x} = \langle \nu_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2\psi\rangle\,.$$ ∎

Distribution $\nu_{\mathrm{K}_{0,\infty}}$ we call *the one-scale H-distribution with characteristic length* (ω_n) associated to the (sub)sequences $(u_{n'}), (v_{n'})$. As in the L^2 case, we have that $\nu_{\mathrm{K}_{0,\infty}}$ is an *extension* of ν_H.
As soon as we get boundedness of $\mathcal{A}_{\psi^*_n}$, the most important part of the proof (like in [2, 6, 7]) is based on the variation of the First commutation lemma. Here we just use an easy generalisation of the L^2 result given in [1].
Let $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{K}^d_{0,\infty})$ and denote by B_φ a bounded operator on $\mathrm{L}^p(\mathbf{R}^d)$ such that $(B_\varphi u)(\mathbf{x}) := \varphi(\mathbf{x})u(\mathbf{x})$.

Lemma. *Let (v_n) be bounded in $\mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 2, \infty]$, and weakly convergent to zero in the sense of distributions. The sequence $(C_n v_n)$ strongly converges to zero in $\mathrm{L}^q(\mathbf{R}^d)$, for any $q \in [2, r] \setminus \{\infty\}$, where $C_n := [B_\varphi, \mathcal{A}_{\psi^*_n}]$.* ∎

Localisation principle. We study (1) in the L^p setting. The assumption on the right hand side (2) have to be rephrased using Fourier multipliers as for $f_n \in \mathrm{W}^{-m,p}(\Omega)$ we cannot satisfactorily describe the fraction in (2). The Mihlin theorem requires smooth symbols, therefore we define $K_n(\boldsymbol\xi) := (1 + |\boldsymbol\xi|^{2l} + \varepsilon_n^{2m-2l}|\boldsymbol\xi|^{2m})^{-1/2}$, and for $f_n \in \mathrm{W}^{-m,p}(\mathbf{R}^d)$ introduce:

$$(3)\qquad (\forall \varphi \in \mathrm{C}^\infty_c(\Omega))\quad \mathcal{A}_{K_n}(\varphi f_n) \xrightarrow{\mathrm{L}^p} 0\,.$$

This condition is well defined since for any $n \in \mathbf{N}$ we have $\mathcal{A}_{K_n}(\varphi f_n) \in \mathrm{L}^p(\mathbf{R}^d)$. It is easy to see that (3) and (2) are equivalent for $p = 2$.
Following the approach in [1], with some more additional technical details, we can obtain the localisation principle for one-scale H-distributions as the generalisation of the result presented for one-scale H-measures.

References

[1] Nenad Antonić, Marko Erceg, Martin Lazar: *Localisation principle for one-scale H-measures*, preprint.

[2] Nenad Antonić, Darko Mitrović: *H-distributions: an extension of H-measures to an $\mathrm{L}^p - \mathrm{L}^q$ setting*, *Abst. appl. analysis* (2011) 12 pages.

[3] Patrick Gérard: *Mesures semi-classiques et ondes de Bloch*, *Sem. EDP 1990–91 (exp. 16)*.

[4] Patrick Gérard: *Microlocal defect measures*, Comm. in PDEs **16** (1991) 1761–1794.

[5] Pierre-Louis Lions, Thierry Paul: *Sur les mesures de Wigner*, *Revista Mat. Iberoamericana* **9** (1993) 553–618.

[6] Luc Tartar: *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations*, *Proc. R. Soc. Edinburgh* **115A** (1990) 193–230.

[7] Luc Tartar: *The General Theory of Homogenization*, *Springer*, 2009.