Expressing limits of non-quadratic terms via H-measures

Martin Lazar
University of Dubrovnik

March 4, 2015
Zagreb
Outline

Introduction

Exploring non-quadratic terms

Computation of higher order correction terms in small amplitude homogenisation
H-measures

- introduced around 1990. by L. Tartar and P. Gérard
- Radon measures associated to bounded $L^2(\mathbb{R}^d)$ sequences

$$\mu \sim (u_n)$$

- express limit of $\int u_n^2$
- a microlocal defect tool
  - include dual variable ($\xi$), in addition to the physical one ($x$),
  - measure deflection of weak from strong $L^2$ convergence.

$(u_n)$ - bounded in $L^2(\mathbb{R}^d)$, $u_n \to 0$.

$$\mu = 0 \iff u_n \to 0 \text{ in } L^2_{\text{loc}}(\mathbb{R}^d)$$
H-measures

**Theorem 1.** (Existence)\(^{a}\)

Let \(u_n \rightharpoonup 0\) in \(L^2(\mathbb{R}^d)\). There exists a subsequence \((u_{n'})\) and a non-negative Radon measure \(\mu_H\) on \(\mathbb{R}^d \times S^{d-1}\) such that for all \(\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d)\), \(\psi \in C(S^{d-1})\):

\[
\lim_{n'} \int_{\mathbb{R}^d} A_\psi(\varphi_1 u_n)(x)(\varphi_2 u_n)(x) \, dx = \langle \mu_H, \varphi \boxtimes \psi \rangle
\]

\[
= \int_{\mathbb{R}^d \times S^{d-1}} \varphi \psi(\xi) \, d\mu(x, \xi).
\]

where: \(A_\psi\) is the (Fourier) multiplier operator \(F(A_\psi u)(\xi) = \psi(\frac{\xi}{|\xi|})\hat{u}(\xi)\), \(\varphi = \varphi_1 \bar{\varphi}_2\).

Measure \(\mu_H\) we call H-measure associated to the (sub)sequence \((u_n)\).


The theorem is also valid:

- for \(u_n\) of class \(L^2_{loc}\), but the associated H-measure does not need to be finite, test functions \(\varphi \in C_c(\mathbb{R}^d)\),
- for vector functions \(u_n \in L^2(\mathbb{R}^d; C^r)\), the H-measure is a positive semi-definite matrix Radon measure.
Applications:

- **Compensated compactness** \(^1\)
  - if \(u_n, v_n \to 0\), does \(u_n v_n \to 0\)?

- **Homogenisation** \(^2\)
  - defect measures describe limits of quadratic terms.

- **Velocity averaging** \(^3\)
  - under which conditions \(\int_{\mathbb{R}^y} u_n(x, y) \rho(y) dy \to 0\) in \(L^2\)?

- **(Averaged) control theory** \(^4\)
  - under which conditions can we control the averaged quantity \(\int_{\mathbb{R}^y} u_n(x, y) \rho(y) dy\)?

\[\ldots\]

---


H-measures – restricted to quadratic terms of $L^2$ sequences.

**H-distributions:**

- a generalisation of the concept to the $L^p, p \geq 1$ framework.
- explore products of a form

$$\int u_n v_n, u_n \in L^p, v_n \in L^{p'}.$$ 

The aim of the paper:

- to deal with higher order terms

$$\int u_n^p, u_n \in L^p.$$ 

More precisely

$$\lim_{n} \int_{\mathbb{R}^d} A_{\psi_1} (\varphi_1 u_n)(x) A_{\psi_2} (\varphi_2 u_n)(x) \cdots A_{\psi_p} (\varphi_p u_n)(x) dx = ?$$

---

Split the integrand into two parts

\[
\int_{\mathbb{R}^d} A_{\psi_1}(\varphi_1 u_n) \ldots A_{\psi_{p/2}}(\varphi_{p/2} u_n) \underbrace{A_{\psi_{p/2+1}}(\varphi_{p/2+1} u_n) \ldots A_{\psi_p}(\varphi_p u_n)}_{v_n} d\mathbf{x} = \int_{\mathbb{R}^d} (\varphi v)(\mathbf{x}) w(\mathbf{x}) d\mathbf{x},
\]

Theorem 2.

Let \( u_n \longrightarrow 0 \) in \( L^{p+\varepsilon}_{\text{loc}}(\mathbb{R}^d), p \in \mathbb{N}, \varepsilon > 0 \).

Then for any choice of test functions \( \varphi_i \in C_c(\mathbb{R}^d), \psi_i \in C^1(S^{d-1}), i = 1..p \) it holds

\[
\lim_n \int_{\mathbb{R}^d} A_{\psi_1}(\varphi_1 u_n)(\mathbf{x}) \cdot \ldots \cdot A_{\psi_p}(\varphi_p u_n)(\mathbf{x}) d\mathbf{x} = \langle \mu_{vw}, \varphi \boxtimes 1 \rangle
\]

where:

- \( \varphi = \prod_{i=1}^p \varphi_i \),
- \( \mu_{vw} \) – off-diagonal component of the matrix H-measure associated to \( (v_n - v, w_n - w) \).
Proof

The proof is based on:

- the Marcinkiewicz multiplier theorem
  \(A_\psi\) is a bounded operator on \(L^p(\mathbb{R}^d)\), for any \(p \in (1, \infty)\), \(\psi \in C^d(S^{d-1})\).

- the (First) commutation lemma
  Let:
  \[u_n \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), \quad p \in [2, \infty].\]
  \[C = A_\psi \varphi - \varphi A_\psi\] the commutator determined by \(\varphi \in C_0(\mathbb{R}^d), \quad \psi \in C^d(S^{d-1}).\)

Then:

\[Cu_n \rightharpoonup 0 \text{ in } L^q(\mathbb{R}^d), \quad q \in [2, p).\]

- \(u_n \in L^p \implies v_n, w_n \in L^2\)
\[
\lim_{n} \int_{\mathbb{R}^d} A_{\psi_1}(\varphi_1 u_n) \cdots A_{\psi_p}(\varphi_p u_n)
\]

- depends on \( \varphi = \prod_i \varphi_i \in C_0(\mathbb{R}^d) \);
- defines a continuous \((p + 1)\)-linear form \( B \) on \( C_0(\mathbb{R}^d) \times C^d(S^{d-1}) \times \cdots \times C^d(S^{d-1}) \);
- the special case \( p = 2 \)
  by Plancherel theorem it depends on \( \varphi = \prod_i \varphi_i \) and \( \psi = \prod_i \psi_i \) only,
  results in a distribution, and eventually an \((\text{H}-)\)measure on \( \mathbb{R}^d \times S^{d-1} \);
- in general, the form \( B \) is a measure with respect to \( x \) variable.
  for any test functions \( \psi_i, i = 1..p \),
  \[
  B(\cdot, \psi_1, \ldots, \psi_p) = \mu_{vw} + \nu \bar{w} \lambda(x).
  \]
The periodic setting

Let \((u_n)\) be a sequence of periodic functions

\[
u_n(x) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k e^{2\pi in k \cdot x} \to 0.
\]

The associated H-measure:

\[
\mu(x, \xi) = \sum_k |\hat{u}_k|^2 \delta_{k / |k|} (\xi) \lambda(x).
\]

Can we express \(\lim \int A_{\psi_1}(\varphi_1 u_n) \ldots A_{\psi_p}(\varphi_p u_n)\) explicitly?

**Remark**

\(A_{\psi} u_n\) is a periodic function:

\[
A_{\psi} u_n(x) = \sum_k \hat{u}_k \psi(k) e^{2\pi in k \cdot x}.
\]
Specially,

\[ v_n(x) = (A_{\psi_1} u_n)(A_{\psi_2} u_n)(x) = \sum_{j,k} \hat{u}_j \hat{u}_k \psi_1(j) \psi_2(k) e^{2\pi i n(j+k) \cdot x} \]

\[ \rightarrow \sum_k \hat{u}_k \hat{u}_{-k} \psi_1(k) \psi_2(-k). \]

Similarly for

\[ \overline{w_n}(x) = (A_{\psi_3} u_n)(A_{\psi_4} u_n)(x). \]

The measure \( \mu_{vw} \) determined by the sequences \((v_n - v)\) and \((w_n - w)\) reads

\[ \mu_{vw} = \sum_{j,k} \left( \sum_{l,m} \hat{u}_j \hat{u}_k \hat{u}_l \hat{u}_m \psi_1(j) \psi_2(k) \psi_3(l) \psi_4(m) \right) \delta_{\frac{j+k}{|j+k|}}(\xi) \lambda(x). \]

Taking into account the form of the limits \( v \) and \( w \):

\[ \lim_n \int_{\mathbb{R}^d} A_{\psi_1}(\varphi_1 u_n)(x) \cdot \ldots \cdot A_{\psi_4}(\varphi_4 u_n)(x) dx \]

\[ = \sum_{j,k,l,m} \hat{u}_j \hat{u}_k \hat{u}_l \hat{u}_m \psi_1(j) \psi_2(k) \psi_3(l) \psi_4(m) \int_{\mathbb{R}^d} \varphi(x) dx, \]

where \( \varphi = \prod_i \varphi_i. \)
Explicit formula for general $p$

**Theorem 3.**

Let $(u_n)$ be a bounded sequence of periodic functions in $L^\infty_{\text{loc}}(\mathbb{R}^d)$, $u_n \rightharpoonup 0$. For any $p \in \mathbb{N}$, $\varphi_i \in C_c(\mathbb{R}^d)$, $\psi_i \in C^d(S^{d-1})$, $i = 1..p$ it holds

$$
\lim_{n} \int_{\mathbb{R}^d} A_{\psi_1}(\varphi_1 u_n)(x) \cdot \ldots \cdot A_{\psi_p}(\varphi_p u_n)(x) dx = \sum_{\mathbb{k}_i \in \mathbb{Z}^d, \sum_i k_i=0} \left( \prod_{i=1}^{p} \hat{u}_{k_i} \psi_i(k_i) \right) \int_{\mathbb{R}^d} \varphi(x) dx,
$$

where $\varphi = \prod_{i=1}^{p} \varphi_i$.

The theorem:

- easily generalises to a case when each factor in the integrand above is associated to a different sequence $(u_n^i)_{n}$, $i = 1..p$, just by adjusting the Fourier coefficients on the right hand side;
- incorporates the expression for an $H$-measure associated to a sequence of periodic functions.
H-convergence

A sequence of elliptic problems:

\[
\begin{align*}
-\text{div} (A^n \nabla u^n) &= f \in H^{-1}(\Omega) \\
u^n &\in H^1_0(\Omega),
\end{align*}
\]

where \( \Omega \subseteq \mathbb{R}^d \) is an open, bounded domain. The coefficients \( A^n \) are taken from the set (with \( 0 < \alpha < \beta \))

\[
\mathcal{M}(\alpha, \beta; \Omega) := \{ A \in L^\infty(\Omega; M_d(\mathbb{R}^d)) : A(x)\xi \cdot \xi \geq \alpha |\xi|^2, A^{-1}(x)\xi \cdot \xi \geq \frac{1}{\beta} |\xi|^2 \},
\]

Then

\[
A^n \overset{H}{\rightarrow} A^\infty \in \mathcal{M}(\alpha, \beta; \Omega),
\]

i.e. for any choice of \( f \in H^{-1}(\Omega) \), solutions to (1) satisfy:

\[
u^n \rightharpoonup u^\infty \quad \text{in} \ H^1_0(\Omega)
\]

\[
A^n \nabla u^n \rightharpoonup A^\infty \nabla u^\infty \quad \text{in} \ L^2(\Omega),
\]

where \( u^\infty \) is the solution of (1) with \( \infty \) instead of \( n \).
Small amplitude homogenisation

The coefficients $A^n$ are perturbations of a constant:

$$A^n_\gamma(t, x) = A_0 + \gamma A^1_n(t, x) + \gamma^2 A^2_n(t, x) + \gamma^3 A^3_n(t, x) + o(\gamma^3),$$

where $A^n_i \to^* 0$ in $L^\infty(\Omega)$ for any $i \geq 1$.

Assuming $A_0 \in \mathcal{M}(\alpha, \beta; \Omega)$, we have (for small values of $\gamma$)

$$A^n_\gamma \xrightarrow{H} A^\infty_\gamma = A_0 + \gamma A^1_\infty(t, x) + \gamma^2 A^2_\infty(t, x) + \gamma^3 A^3_\infty(t, x) + o(\gamma^3),$$

where the limit $A^\infty_\gamma$ is measurable in $x$ and analytic in $\gamma$.

Existing results:

- $A^\infty_1 = 0$

- $A^\infty_2$ – the limit of a quadratic term in $A^1_n$, expressed via H-measure $\mu \sim A^1_n$.

Missing:

- Higher order correction terms, $A^\infty_3$, etc.
  - the limit of expressions involving higher order powers,
  - beyond the scope of H-measures.
Expansion in $\gamma$

Fix $u \in H^1_0(\Omega)$, denote by $u^n_\gamma$ the solution of

$$-\text{div} \ (A^n_\gamma \nabla u^n_\gamma) = -\text{div} \ (A^\infty_\gamma \nabla u) .$$  \hspace{1cm} (2)

Because of H-convergence,

$$u^n_\gamma \rightharpoonup u \quad \text{in} \ H^1_0(\Omega)$$

$$A^n_\gamma \nabla u^n_\gamma \rightharpoonup A^\infty_\gamma \nabla u^\infty \quad \text{in} \ L^2(\Omega) .$$  \hspace{1cm} (3)

By expansion of $u^n_\gamma$ in powers of $\gamma$:

$$u^n_\gamma = u^n_0 + \gamma u^n_1 + \gamma^2 u^n_2 + o(\gamma^2) ,$$

we get

$$u^n_0 \rightharpoonup u, \quad u^n_i \rightharpoonup 0, \quad i \geq 1 .$$

Putting the expansions for $A^n_\gamma, u^n_\gamma$ in (2) and (3) we try to find $A^n_\infty$.

Immediately:

$$\nabla u^n_0 = \nabla u, \quad A^n_1 = 0 .$$
(3) for $\gamma = 2$ gives:

$$A_0 \nabla u_2^n + A_1 \nabla u_1^n + A_2 \nabla u \longrightarrow \lim_{n} A_1 \nabla u_1^n = A_2^\infty \nabla u.$$ 

On the other side (2) for $\gamma = 1$ implies

$$-\operatorname{div} (A_0 \nabla u_1^n) = -\operatorname{div} (A_1 \nabla u).$$

The Fourier transform yields

$$\nabla u_1^n(x) = -A_\Psi (A_1^n \nabla u)(x),$$

where $A_\Psi$ is the multiplier operator with the symbol $\Psi(\xi) = \frac{\xi \otimes \xi}{A_0 \xi \cdot \xi}$. Thus

$$A_2^\infty \nabla u = -\lim_{n} A_1^n A_\Psi (A_1^n \nabla u)$$

yielding the (existing) expression for $A_2^\infty$:

$$\int_{\Omega} (A_2^\infty)_{ij}(x) \phi(x) dx = -\sum_{k,l} \left< \mu_{11}^{iklj}, \phi \frac{\xi_k \xi_l}{A_0 \xi \cdot \xi} \right>, \quad \text{with} \ \mu_{11} \ \text{standing for an H-measure (with four indices) associated to} \ A_1^n.$$
Higher order correction terms

Similarly

\[ \mathbf{A}_0 \nabla u_3^n + \mathbf{A}_1^n \nabla u_2^n + \mathbf{A}_2^n \nabla u_1^n + \mathbf{A}_3^n \nabla u \longrightarrow \lim_{n} (\mathbf{A}_1^n \nabla u_2^n + \mathbf{A}_2^n \nabla u_1^n) = \mathbf{A}_3^\infty \nabla u \]

and

\[ \nabla u_2^n (x) = -\mathbf{A}_\Psi (\mathbf{A}_2^n \nabla u + \mathbf{A}_1^n \nabla u_1^n - \mathbf{A}_2^\infty \nabla u) (x) \]

provide

\[ \mathbf{A}_3^\infty \nabla u = \lim_{n} \left( - \mathbf{A}_1^n \mathbf{A}_\Psi \mathbf{A}_2^n \nabla u - \mathbf{A}_2^n \mathbf{A}_\Psi \mathbf{A}_1^n \nabla u + \mathbf{A}_1^n \mathbf{A}_\Psi (\mathbf{A}_1^n \mathbf{A}_\Psi \mathbf{A}_1^n \nabla u) \right). \]

Finally:

\[ \int_{\Omega} (\mathbf{A}_3^\infty)^{ij} \varphi \, dx = -\left\langle 2 \Re \mu_{12}^{ij}, \varphi \frac{\xi \otimes \xi}{\mathbf{A}_0 \xi \cdot \xi} \right\rangle + \left\langle \tr \mu_{VW}^{ij}, \varphi \bigotimes 1 \right\rangle, \]

where \( \mu_{12} \sim (\mathbf{A}_1, \mathbf{A}_2) \)

\( \mu_{VW} \sim (\mathbf{V}_n, \mathbf{W}_n). \)

And so on: \( \mathbf{A}_4^\infty, \ldots \)
Periodic setting

Periodic coefficients

\[ A^*_n(nx) = A_i(nx) = \sum_{k \in \mathbb{Z}^d} \hat{A}_{i,k} e^{2\pi i n k \cdot x}, \quad i \in \mathbb{N} \]

We have explicit expressions for H-measures associated to (an arbitrary power of \( A^*_n \)).

Specially:

\[ A_2^\infty = - \sum_{k \in \mathbb{Z}^d} \frac{1}{A_0 k \cdot k} (\hat{A}_{1,k} k) \otimes \hat{A}_{1,-k} k. \]

and

\[ A_3^\infty = \sum_{k \in \mathbb{Z}^d} \frac{1}{A_0 k \cdot k} (\hat{A}_{1,k} k) \otimes \left( -2\hat{A}_{2,-k} k + \sum_{l,m \in \mathbb{Z}^d, k+l+m=0} \hat{A}_{1,l} k \cdot m \hat{A}_{1,m} m \right). \]
Similarly:

\[
A^\infty_4 = \sum_{k \in \mathbb{Z}^d} \frac{1}{A_0 k \cdot k} \left( -2(\hat{A}_{1,k}k) \otimes \hat{A}_{3,-k}k - (\hat{A}_{2,k}k) \otimes (\hat{A}_{2,-k}k) \right) \\
+ \sum_{1,m \in \mathbb{Z}^d} \frac{1}{A_0 m \cdot m} \left( \hat{A}_{1,1}k \cdot m (\hat{A}_{1,k}k) \otimes (\hat{A}_{2,m}m) \right) \\
+ \hat{A}_{2,1}k \cdot m (\hat{A}_{1,k}k) \otimes (\hat{A}_{1,m}m) + \hat{A}_{1,1}k \cdot m (\hat{A}_{2,k}k) \otimes (\hat{A}_{1,m}m) \\
- \sum_{j \in \mathbb{Z}^d} \frac{1}{A_0 (j+k) \cdot (j+k)} (\hat{A}_{1,1}j + \hat{A}_{1,1}m) \cdot (j+k)(\hat{A}_{1,k}k) \otimes (\hat{A}_{1,m}m) \right) 
\]
Conclusion

Presented:

▶ a method for expressing limits of non-quadratic terms by means of original H-measures,
▶ application to the small amplitude homogenisation problem for a stationary diffusion equation.

Perspectives:

▶ non-stationary diffusion problems \(^6\) by means of parabolic H-measures, \(^7\)
▶ coefficients \(A_i^n\) oscillating on different scales – multiscale H-measures, \(^8\)
▶ various situations requiring limits of higher order powers.

Thanks for your attention!

