Singular solutions concept for systems of conservation laws.

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Joint work with: Henrik Kalisch

University of Bergen, Norway

WeConMApp

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General system

Consider a general 2x2 system of conservation laws

$$\begin{cases} \partial_t u + \partial_x f(u, v) = 0 \\ \partial_t v + \partial_x g(u, v) = 0 \end{cases} \tag{1}$$

with the Riemann initial data

$$u|_{t=0} = U_0(x) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0 \end{cases}, \quad v|_{t=0} = V_0(x) = \begin{cases} V_L, & x < 0 \\ V_R, & x > 0. \end{cases} \tag{2}$$

If the system is genuinely nonlinear and strictly hyperbolic, then the latter Riemann problem has a unique solution consisting of rarefaction waves and compressive shock waves (Lax admissible waves) if the states (U_L, V_L) and (U_R, V_R) are *close to each other*.

Wave front tracking blows up

A. Bressan et al. *Lack of BV bounds for approximate solutions to the p-system with large data*, J.of Differential Equation, (2014)

Glimm scheme blows up

C. Tsikkou, *Hyperbolic conservation laws with large initial data. Is the Cauchy problem well-posed?*, Quart. Appl. Math. (2010)

No solution for certain Riemann problems
B. Hayes and P. LeFloch, *Measure solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity (1996).

Existence of solutions to Riemann problems is resolved by expanding the space of possible solutions by δ-distributions.

C. Korchinski, *Solution of a Riemann problem for a* 2×2 *system of conservation laws possessing no classical weak solution* PhD Thesis Adelphi University, 1977

B. L. Keyfitz, H. C. Krantzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, J. Differential Equations 118 (1995) 420-451.
Hugoniot locus is compact $\implies$ no weak solutions

What does it mean to have δ-distribution as a part of the solution?

- Define various viscous and other approximations
- These approximations converge to singular functions

A relaxation of the vanishing viscosity

Definition

Leg $f_\varepsilon \in \mathcal{D}'(\mathbb{R})$. We say that $f_\varepsilon = o_{\mathcal{D}'}(1)$ if for any $\phi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle f_\varepsilon, \phi \rangle = o(1) \text{ as } \varepsilon \to 0.$$

Definition

The families (u_ε) and (v_ε) are weak asymptotic solutions of (1) if

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \text{ in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$$

and

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon, v_\varepsilon) = o_{\mathcal{D}'}(1) \\ \partial_t v_\varepsilon + \partial_x g(u_\varepsilon, v_\varepsilon) = o_{\mathcal{D}'}(1). \end{cases} \tag{3}$$

for every $t \in \mathbb{R}^+$.

The work that has been the motivation for our research is

B. Hayes and P. LeFloch, *Measure solutions to a strictly hyperbolic system of conservation laws,* Nonlinearity 9 (1996), 1547–1563.

where the following system was considered (the Brio system)

$$\begin{aligned} \partial_t u + \partial_x\left(\tfrac{u^2+v^2}{2}\right) &= 0, \\ \partial_t v + \partial_x(v(u-1)) &= 0. \end{aligned} \tag{4}$$

The system is strictly hyperbolic; it is genuinely nonlinear at $\{(u,v): \ u \in \mathbb{R}, \ v > 0\}$ and $\{(u,v): \ u \in \mathbb{R}, \ v < 0\}$, but not on the whole of $\mathbb{R}^2$.

No weak solution for certain Riemann data.

For the Riemann initial data (2) such that $v_1 < 0 < v_2$, the system does not admit Lax admissible solutions. Moreover, for certain combination of the Riemann initial data, no weak solutions exist.

We want to construct a weak asymptotic solution to the Brio system for any combination of the Riemann initial data. The weak asymptotic solution converges toward the δ-shock solution.

Let $\rho \in C_c^\infty(\mathbb{R})$ be an even non-negative, smooth, compactly supported function such that

$$\mathrm{supp}\rho \subset (-1,1), \quad \int_{\mathbb{R}} \rho(z)dz = 1, \quad \rho \geq 0.$$

We take:

$$\begin{aligned} R_\varepsilon(x,t) &= \tfrac{i}{\varepsilon}\rho((x-ct-2\varepsilon)/\varepsilon) - \tfrac{i}{\varepsilon}\rho((x-ct+2\varepsilon)/\varepsilon), \\ \delta_\varepsilon(x,t) &= \tfrac{1}{\varepsilon}\rho((x-ct-4\varepsilon)/\varepsilon) + \tfrac{1}{\varepsilon}\rho((x-ct+4\varepsilon)/\varepsilon). \end{aligned} \tag{5}$$

Next, define smooth functions U_ε and V_ε such that

$$U_\varepsilon(x,t) = \begin{cases} u_1, & x < ct - 20\varepsilon, \\ c+1, & ct - 10\varepsilon < x < ct + 10\varepsilon, \\ u_2, & x > ct + 20\varepsilon, \end{cases}$$

$$V_\varepsilon(x,t) = \begin{cases} v_1, & x < ct - 20\varepsilon, \\ 0, & ct - 10\varepsilon < x < ct + 10\varepsilon, \\ v_2, & x > ct + 20\varepsilon. \end{cases}$$

Notice that

$$R_\varepsilon \rightharpoonup 0, \quad U_\varepsilon R_\varepsilon \rightharpoonup 0, \quad \text{and} \quad U_\varepsilon \delta_\varepsilon \rightharpoonup 2(c+1)\delta(x-ct). \tag{6}$$

Moreover, we have

$$V_\varepsilon R_\varepsilon \equiv 0, \quad V_\varepsilon \delta_\varepsilon \equiv 0, \quad \text{and} \quad \delta_\varepsilon R_\varepsilon \equiv 0. \tag{7}$$

Now, the ansatz

$$
\begin{aligned}
u_\varepsilon(x,t) &= U_\varepsilon(x,t), \\
v_\varepsilon(x,t) &= V_\varepsilon(x,t) + \alpha(t)(\delta_\varepsilon(x,t) + R_\varepsilon(x,t)).
\end{aligned}
\tag{8}
$$

represents the weak asymptotic solution to (4), (2).

a) If $u_1 \neq u_2$, $c = \frac{u_1^2+v_1^2-u_2^2-v_2^2}{2(u_1-u_2)}$ and,

$$\alpha(t) = \frac{1}{2}\left(c(v_2 - v_1) + (v_1(u_1 - 1) - v_2(u_2 - 1))\right) t,$$

then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system, such that the families (u_ε) and (v_ε) have distributional limits

$$\begin{aligned} u(x,t) &= U_0(x-ct), \\ v(x,t) &= V_0(x-ct) + \alpha(t)\delta(x-ct). \end{aligned}$$

b) If $v_1 \neq v_2$, $c = \frac{v_1(u_1-1)-v_2(u_2-1)}{v_1-v_2}$ and ,

$$\alpha(t) = \left(c(u_2 - u_1) + \frac{u_1^2 + v_1^2 - u_2^2 - v_2^2}{2}\right) t,$$

then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system, such that the families (u_ε) and (v_ε) have distributional limits

$$\begin{aligned} u(x,t) &= U_0(x-ct) + \alpha(t)\delta(x-ct), \\ v(x,t) &= V_0(x-ct). \end{aligned}$$

Solution concept

Question:
Can we find the sense in which the limiting distributions satisfy (4)?

More generally, if we have the equation

$$\partial_t u + \partial_x F(u) = 0. \tag{9}$$

what is the sense of the equality sign?

- If $u \in C^1(\mathbb{R}^2)$ it is clear since all the operations in the previous expression are well defined;
- If $u \in L^1_{loc}(\mathbb{R}^2)$, it can still be a solution to (9) in a weaker sense (weak solution concept). It is defined so that the differentiation of the function u is avoided;
- If u contains δ, we need to apply even weaker concept in which we shall avoid nonlinear operations on δ;

A generalization of the classical weak solution concept on δ-shock solution concept for system (1) was provided by Danilov and Shelkovich (2005) but for systems which are linear with respect to one of the variables. There are no obstacles for extending the definition on an arbitrary system of form (1) (Kalisch and Mitrovic (2012)).

Suppose that γ is the Lipschitz continuous arc and let x^0 be the initial point of the arc γ. Let (u, v) be a pair of distributions represented in the form

$$\begin{aligned} u(x,t) &= U(x,t) + \alpha(t)\delta(\gamma) \\ v(x,t) &= V(x,t) + \beta(t)\delta(\gamma), \end{aligned} \tag{10}$$

and where $U, V \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$. Finally, the expression $\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}$ denotes the tangential derivative of a function φ on the graph γ, and $\int_\gamma$ connotes the line integral over the arc γ.

Definition

The pair of distributions (10) is called a generalized δ-shock solution of (1) with the initial data $U_0(x) + \alpha_k(x_0, 0)\delta(x - x^0)$ and $V_0(x) + \beta(x_0, 0)\delta(x - x^0)$ if it satisfies

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (U\partial_t\varphi + f(U, V)\partial_x\varphi)\ dxdt \tag{11}$$

$$+\int_\gamma \alpha(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\ dx + \alpha(x^0,0)\varphi(x^0,0) = 0,$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} (V\partial_t\varphi + g(U, V)\partial_x\varphi)\ dxdt \tag{12}$$

$$+\int_\gamma \beta(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}} + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\ dx + \beta(x^0,0)\varphi(x^0,0) = 0,$$

for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

Remark that in the case when $\gamma = \{x = ct\}$, the term involving tangential derivatives reduces to

$$\int_\gamma \alpha(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}} = -\int_{\mathbb{R}^+} \alpha'(t)\varphi(ct,t)dt = -\langle\delta(x-ct), \alpha'(t)\varphi(x,t)\rangle$$

This operator, we shall call the tangential derivative of the measure $\delta(x - ct)$.

General system

a) If $U_L \neq U_R$ then the pair of distributions

$$\begin{aligned} u(x,t) &= U_0(x-ct), \\ v(x,t) &= V_0(x-ct) + \alpha(t)\delta(x-ct), \end{aligned}$$

where

$$c = \frac{[f(U,V)]}{[U]} = \frac{f(U_R,V_R) - f(U_L,V_L)}{U_R - U_L}, \text{ and}$$

$\alpha(t) = (c[V] - [g(U,V)])t = (c(U_L - U_R) - (g(U_R,V_R) - g(U_L,V_L)))t$, represents the δ-shock wave solution of (1), (2).

b) If $V_L \neq V_R$ then the pair of distributions

$$\begin{aligned} u(x,t) &= U_0(x-ct) + \alpha(t)\delta(x-ct), \\ v(x,t) &= V_0(x-ct), \end{aligned}$$

where

$c = \frac{[g(U,V)]}{[V]} = \frac{g(U_R,V_R) - g(U_L,V_L)}{V_R - V_L}$, and $\alpha(t) = (c[U] - [f(U,V)])t$

represents the δ-shock solution of (1), (2).

Uniqueness issues

As usual when passing to weaker solution concept, problem of uniqueness arises and additional demands must be imposed on the solution.

Non-uniqueness due to:

- shock speed;
- multiple δ;

Usual additional demand for δ-shock solution is the overcompressivity condition:

$$\lambda_i(u_2, v_2) \leq c \leq \lambda_i(u_1, v_1), \quad i = 1, 2,$$

where λ_i are characteristic speeds corresponding to the system, and c is speed of the δ-shock connecting a left state $L = (u_1, v_1)$ and a right state $R = (u_2, v_2)$.

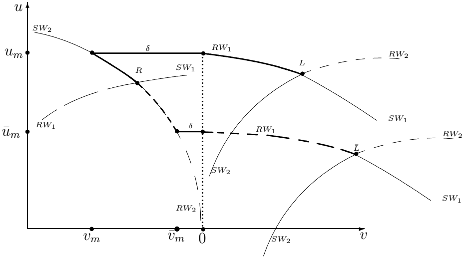
In the case of the Brio system, we are able to obtain only compressivity conditions, i.e.

Definition
A δ-shock solution of (4), connecting a left state $L = (u_1, v_1)$ and a right state $R = (u_2, v_2)$ is *i*-admissible if

$$\lambda_i(u_2, v_2) \le c \le \lambda_i(u_1, v_1), \tag{13}$$

for $i = 1$ or $i = 2$.

Theorem
Given any Rieman initial data (2) such that $v_2 < 0 < v_1$, there exists a δ-shock solution of (4) which consists of a combination of the classical Lax admissible simple waves (shock or rarefaction) and compressive 1-admissible δ waves.



u
SW_2
u_m
δ
RW_1
L
RW_2
R
SW_1
RW_1
SW_1
$\bar{u}_m$
δ
RW_1
$\bar{L}$
RW_2
SW_2
SW_1
RW_2
v_m
$\bar{v}_m$
0
SW_2
v

Cauchy problem with BV-initial data

Definition

We denote by $\mathcal{G}_e$ the space of all linear combinations over the field of affine functions $\mathbb{R}^+ \ni t \mapsto at + b$, $a, b \in \mathbb{R}$, of the Radon measures $\mathcal{M}(\mathbb{R} \times \mathbb{R}^+)$ of the form $\delta(x - x_0 - c(t - t_0))\kappa_{(t_0,T_0)}(t)$ for some constants $c \in \mathbb{R}$ and $T_0 > t_0 \geq 0$, and the characteristic function $\kappa_{(t_0,T_0)}$ of the interval (t_0, T_0).
By $\mathcal{G}$ we denote the closure of $\mathcal{G}_e$ with respect to the weak topology in $\mathcal{M}(\mathbb{R} \times \mathbb{R}^+)$.

Let us now define the tangential derivative of an element from $\mathcal{G}$.

Definition

Let $m \in \mathcal{G}$ such that for the family

$$m_\varepsilon = \sum_{i \in I_\varepsilon} \alpha_{i\varepsilon}(t)\delta(x - x_{i\varepsilon} - c_{i\varepsilon}(t - t_{i\varepsilon})\kappa_{(t_{i\varepsilon}, T_{i\varepsilon})}(t) \tag{14}$$

it holds for every $\varphi \in C_c(\mathbb{R} \times \mathbb{R}^d)$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} \sum_{i \in I_\varepsilon} \alpha_{i\varepsilon}(t)\varphi(x_{i\varepsilon} + c_{i\varepsilon}(t - t_{i\varepsilon}), t)\kappa_{(t_{i\varepsilon}, T_{i\varepsilon})}(t)dt = \int_{\mathbb{R}^+} \langle m, \varphi(\cdot, t)\rangle dt. \tag{15}$$

We say that the functional $m_l = \partial_l m$ is the generalized tangential derivative of $m \in \mathcal{G}$ if along some subsequence it holds

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^+} \sum_{i \in I_\varepsilon} \alpha'_{i\varepsilon}(t)\varphi(x_{i\varepsilon} + c_{i\varepsilon}(t - t_{i\varepsilon}), t)\kappa_{(t_{i\varepsilon}, T_{i\varepsilon})}(t)dt = \int_{\mathbb{R}^+} \langle m_l, \varphi(\cdot, t)\rangle dt. \tag{16}$$

Definition

We say that the pair of distributions

$$u(x,t) = U(x,t) + m_u(x,t),$$
$$v(x,t) = V(x,t) + m_v(x,t),$$

for a bounded functions U and V, and measures $m_u, m_v \in \mathcal{G}$ is a measure-type solution to (1) with the initial data $u|_{t=0} = u_0$, $v|_{t=0} = v_0$ if the following relations hold for any $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}^+)$

$$\int_{\mathbb{R}^+}\int (U\partial_t\varphi + f(U,V)\partial_x\varphi)\, dxdt + \int_{-\infty}^{\infty} U_0(x)\varphi(x,0)\, dx, \tag{17}$$
$$- \int_{\mathbb{R}^+} \langle \partial_l m_u, \varphi(\cdot,t)\rangle dt = 0,$$

$$\int_{\mathbb{R}^+}\int (V\partial_t\varphi + g(U,V)\partial_x\varphi)\ dxdt + \int_{-\infty}^{\infty} v^0(x)\varphi(x,0)\, dx \tag{18}$$
$$- \int_{\mathbb{R}^+} \langle \partial_l m_v, \varphi(\cdot,t)\rangle dt = 0$$

for some generalized tangential derivatives $\partial_l m_u$ and $\partial_l m_v$ of

Theorem

Assume that $u_0, v_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. *Then, system* (1) *augmented with the initial data* $u|_{t=0} = u_0$, $v|_{t=0} = v_0$ *admits a solution in the sense of Definition 5.*

Shallow water system

The system has the form:

$$\left.\begin{array}{rcl} \partial_t h + \partial_x (uh) & = & 0 \\ \partial_t u + \partial_x \left(h + \frac{u^2}{2} \right) & = & 0 \end{array}\right\} \qquad (19)$$

where h is the height of the water and u is its velocity.

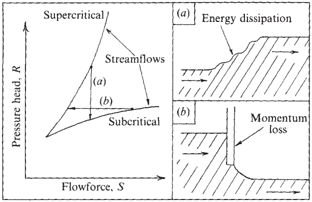


FIGURE 1. Flow transitions in the (*R*, *S*)-diagram: (*a*) the bore, (*b*) the sluice (or hydroplane).

Figure: Hydraulic jump and flow through sluice gate

The case a) – no energy conservation

The phenomenon occurring in the case a) is called a hydraulic jump. When liquid at high velocity discharges into a zone of lower velocity, a rather abrupt rise occurs in the liquid surface. The rapidly flowing liquid is abruptly slowed and increases in height, converting some of the flow's initial kinetic energy into an increase in potential energy, with some energy irreversibly lost through turbulence to heat.

The case b) – no momentum conservation

On the other hand, considering the flow over a weir as in case b), mass and energy are to be conserved. Momentum is lost because the sluice gate exerts a force F on the fluid (where the two touch). However energy is conserved since when the fluid touches the gate, its velocity is zero. Therefore no work is done ($dW/dt = Fu$ where W is the energy, and $u = 0$).

If the solutions of the system are smooth, they also satisfy corresponding conservation equation for horizontal momentum given by

$$\partial_t(uh) + \partial_x\left(hu^2 + \frac{1}{2}h^2\right) = 0, \tag{20}$$

and an energy equation which takes the form

$$\partial_t\left(\frac{1}{2}hu^2 + \frac{1}{2}h^2\right) + \partial_x\left(\frac{1}{2}hu^3 + uh^2\right) = 0. \tag{21}$$

However, in both cases a) and b), we do not have smooth solutions and we need to solve systems consisting of

Case a)

$$\begin{cases} \partial_t h + \partial_x (uh) & = 0 \\ \partial_t (uh) + \partial_x \left(hu^2 + \frac{1}{2} h^2 \right) & = 0 \end{cases} \tag{22}$$

Case b)

$$\begin{cases} \partial_t h + \partial_x (uh) & = 0 \\ \partial_t \left(\frac{1}{2} hu^2 + \frac{1}{2} h^2 \right) + \partial_x \left(\frac{1}{2} hu^3 + uh^2 \right) & = 0 \end{cases} \tag{23}$$

Conclusion

The above means that the physically proper solution to (19) is δ-type solution with the Rankine-Hugoniot deficit which corrects the mistake that is made by not using the physically proper models (22) or (23), but model (19).

Possible reasons for appearance of the δ-distributions:

- They are experimentally confirmed
- Too simplified model
- Wrong constitutive relations
- Mathematically ineligible operations during the model derivation