One-scale H-measures

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BMS - summer school
Berlin, 26^{th} August, 2014

Joint work with Nenad Antonić and Martin Lazar

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If the defect measure is not trivial we need another objects to determine all the properties of the sequence:

- H-measures
- semiclassical measures
- ...

$\Omega \subseteq \mathbf{R}^d$ open.

Theorem

If $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_H \in \mathcal{M}_{\mathrm{b}}(\Omega\times \mathrm{S}^{d-1};\mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ *such that for any* $\varphi_1,\varphi_2 \in \mathrm{C}_0(\Omega)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$

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Measure $\boldsymbol{\mu}_H$ *we call the H-measure corresponding to the (sub)sequence* (u_n).

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[T1] Luc Tartar: *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proceedings of the Royal Society of Edinburgh,* **115A** (1990) 193–230.

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$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{sc}^{(\omega_n)}=\mathbf{0} \quad \& \quad (\mathsf{u}_n)\ \textit{is}\ (\omega_n)-\textit{oscillatory}\,.$$

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Definition

(u_n) is (ω_n)-oscillatory if
$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \quad \lim_{R\to\infty} \limsup_n \int_{|\boldsymbol{\xi}| \geqslant \frac{R}{\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = 0\,.$

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[PG] Patrick Gérard: *Mesures semi-classiques et ondes de Bloch, Sem. EDP 1990–91 (exp. 16)*, (1991)

Example: Oscillations - one characteristic length

$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,,\ n \to \infty$$

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$$\begin{aligned} \mu_H &= \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} \\ \mu_{sc}^{(\omega_n)} &= \lambda \boxtimes \begin{cases} \delta_0 & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty\rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases} \end{aligned}$$

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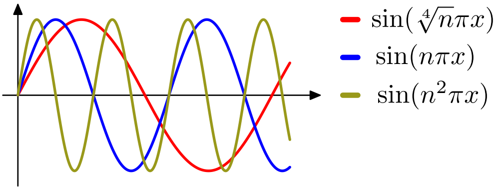
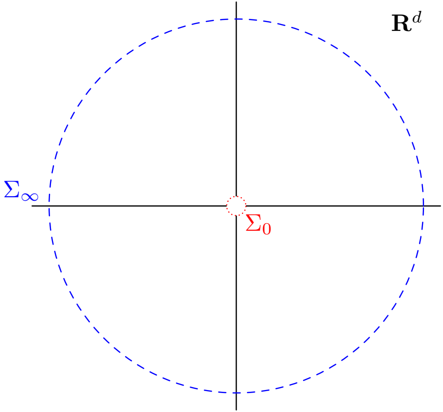
$n = 2$



Compatification of $\mathbf{R}^d \setminus \{0\}$



$$\Sigma_0 := \{0^{\boldsymbol{\xi}_0} \ : \ \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}$$
$$\Sigma_\infty := \{\infty^{\boldsymbol{\xi}_0} \ : \ \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}$$
$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Corollary

a) $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

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[T2] Luc Tartar: *The general theory of homogenization: A personalized introduction, Springer* (2009)

[T3] Luc Tartar: *Multi-scale H-measures, Discrete and Continuous Dynamical Systems, S* **8** (2015) 77–90.

Some properties

Theorem

$\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$, $\psi \in \mathcal{S}(\mathbf{R}^d)$, $\tilde{\psi} \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \searrow 0$

$$\begin{aligned} a) \quad & \left\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \right\rangle &=& \left\langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \right\rangle, \\ b) \quad & \left\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \tilde{\psi} \circ \boldsymbol{\pi} \right\rangle &=& \left\langle \boldsymbol{\mu}_H, \varphi_1\bar{\varphi}_2 \boxtimes \tilde{\psi} \right\rangle, \end{aligned}$$

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\end{array}$$

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

Theorem

$$\begin{array}{lccc}
a) & \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^* = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} & & \\
b) & \mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 & \iff & \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0} \\
c) & \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{(\omega_n)}(\Omega \times \Sigma_\infty) = 0 & \iff & (\mathsf{u}_n) \textit{ is } (\omega_n) - \textit{oscillatory}
\end{array}$$

Example revisited

$$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}},$$

$$\mu_H = \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$$

$$\mu_{sc}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{\mathsf{0}} & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty \rangle \\ 0 & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

$$\mu_{\mathrm{K}_{0,\infty}}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{\mathsf{0}^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \omega_n = c \in \langle 0, \infty \rangle \\ \delta_{\infty^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\alpha \omega_n = \infty \end{cases}$$

Motivation (Localisation principle for H-measures)

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$ and

$$\mathbf{P}\mathsf{u}_n := \sum_{|\boldsymbol{\alpha}|=m} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) \longrightarrow 0 \text{ in } \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)\,.$$

Then we have

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}_H^\top = \mathbf{0}\,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$ is the principle simbol of $\mathbf{P}$.

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Then we have

$$\operatorname{supp} \boldsymbol{\mu}_H \subseteq \{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{S}^{d-1} \ : \ \det \mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = 0\}\,,$$

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Idea: If $d = 1$ and p is nowhere zero (e.g. elliptic operator of the second order), we know $\mu_H = 0$, and that implies $u_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

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Applications:

- compactness by compensation
- small amplitude homogenisation
- velocity averaging
- averaged control

...

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$$\mathbf{P}_n \mathsf{u}_n := \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $\varepsilon_n \searrow 0$, $\varepsilon_n > 0$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

Then we have

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\mu}_{sc}^{\top} = \mathbf{0} \,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

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where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

Motivation (localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\mathbf{P}_n \mathsf{u}_n := \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $\varepsilon_n \searrow 0$, $\varepsilon_n > 0$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

Then we have

$$\operatorname{supp} \boldsymbol{\mu}_{sc} \subseteq \{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathbf{R}^d \ : \ \det \mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = 0\} \,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

Localisation principle

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $l \in 0..m$
- $\varepsilon_n \searrow 0$, $\varepsilon_n > 0$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

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Lemma

a) $(\mathrm{C}(\varepsilon_n))$ *is equivalent to*

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)\,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \quad \Longrightarrow \quad (\varepsilon_n^{k-l} \mathsf{f}_n)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$.

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem (Tartar (2009))

Under previous assumptions and $l=1$, 1-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) corresponding to (u_n) satisfies

$$\operatorname{supp}\,(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem

Under previous assumptions, 1-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) corresponding to (u_n) satisfies

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

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Localisation principle - final generalisation

Theorem

$\varepsilon_n > 0$ bounded $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ and

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n\,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega;\mathrm{M}_r(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ uniformly on compact sets, and $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ satisfies $(\mathrm{C}(\varepsilon_n))$.
Then for $\omega_n \searrow 0$ such that $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in [0,\infty]$, corresponding 1-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ω_n) satisfies

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0,\infty\rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

Theorem (cont.)

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *we can take*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

Theorem (cont.)

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *we can take*

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As a corollary from the previous theorem we can derive localisation principles for H-measures and semiclassica measures.

Summary

- H-measures do not catch frequency
- In some cases, semiclassical measures do not catch direction
- 1-scale H-measures are generalisation of H-measures and semiclassical measures and do not have above anomalies

- Localisation principle for 1-scale H-measures is obtained
- Localisation principles for H-measures and semiclassical measures via localisation principle for 1-scale H-measures