Jednoskalne H-mjere i inačice

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Seminar za diferencijalne jednadžbe i numeričku analizu
PMF - Matematički odsjek

Zagreb, 14. srpnja 2014.

WeConMApp

Promatramo sljedeću Cauchyjevu zadaću

$$\begin{cases} i\varepsilon_n \partial_t \mathsf{u}_n + \mathbf{P}_n \mathsf{u}_n = 0 \quad \text{u} \quad \mathbf{R}^+ \times \mathbf{R}^d \\ \mathsf{u}_n(0,\cdot) = \mathsf{u}_n^0 \end{cases},$$

pri čemu

- $\varepsilon_n > 0$, $\varepsilon_n \to 0$
- u_n^0 omeđen u $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$ (najčešće titrajući niz)
- $\mathbf{P}_n$ (pseudo)diferencijalni operator

Uvod

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Pretpostavimo da je $\mathsf{u}_n : \mathbf{R}_0^+ \longrightarrow \mathbf{C}^r$,,dovoljno dobro" rješenje gornje zadaće.

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Definirajmo gustoću energije za fiksan t: $e_n^t(\mathbf{x}) := |\mathsf{u}_n(t,\mathbf{x})|^2$.
Ukoliko je $\mathsf{u}(t,\cdot)$ jednoliko omeđeno u $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, e_n^t je omeđeno u $\mathrm{L}^1(\mathbf{R}^d)$

$$(\forall\, t \in \mathbf{R}_0^+)(\exists\, e^t \in \mathcal{M}_{\mathrm{b}}) \quad e_n^t \xrightarrow{*} e^t$$

Posebno, e^0 možemo odrediti iz početnih uvjeta.

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Cilj: pronaći e_0^t bez određivanja niza rješenja u_n

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Nije nam dovoljno znanje e^0!

Schrödingerova jednadžba

Gibanje jednog elektrona pod utjecajem potencijala V u $\mathbf{R}^d$ dano je s

$$\begin{cases} i\varepsilon_n \partial_t u_n + \frac{\varepsilon_n^2}{2} \triangle u_n - V u_n = 0 \quad \text{u} \quad \mathbf{R} \times \mathbf{R}^d \\ u_n(0, \cdot) = u_n^0 \in \mathrm{H}^2(\mathbf{R}^d) \end{cases}.$$

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Za fiksan $n \in \mathbf{N}$ i $V \in \mathrm{L}^\infty(\mathbf{R}^d)$ operator $P_n := \frac{\varepsilon_n^2}{2} \triangle - V$ je generator unitarne grupe na $\mathrm{L}^2(\mathbf{R}^d)$ pa postoji $u_n \in \mathrm{C}(\mathbf{R}; \mathrm{H}^2(\mathbf{R}^d)) \cap \mathrm{C}^1(\mathbf{R}; \mathrm{L}^2(\mathbf{R}^d))$, te

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iz čega slijedi omeđenost za svako vrijeme t gustoće vjerojatnosti nalaženja čestice u danom položaju e_n^t u $\mathrm{L}^1(\mathbf{R}^d)$, tj. postoji $e^t \in \mathcal{M}_\mathrm{b}$.

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Fizikalna interpretacija: matematička formulacija za prijelaz s kvantne na klasičnu mehaniku [Wigner (1932)]

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Određivanje e^t:

- WKBJ metoda: pojavljuje se eikonalna jednadžba koja ima singularitet u konačnom vremenu

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$\Omega \subseteq \mathbf{R}^d$ otvoren.

Teorem

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[T1] Luc Tartar: *H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proceedings of the Royal Society of Edinburgh*, **115A** (1990) 193–230.

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$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \quad \lim_{R\to\infty} \limsup_n \int_{|\boldsymbol{\xi}| \geqslant \frac{R}{\varepsilon_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = 0\,.$

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[PG] Patrick Gérard: *Mesures semi-classiques et ondes de Bloch, Sem. EDP 1990–91 (exp. 16)*, (1991)

(u_n) iz $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, $\varepsilon_n \to 0$,

$$\mathbf{W}_n(\mathbf{x},\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\xi}} \mathsf{u}_n\left(\mathbf{x}+\frac{\varepsilon_n \mathbf{y}}{2}\right) \otimes \mathsf{u}_n\left(\mathbf{x}-\frac{\varepsilon_n \mathbf{y}}{2}\right) d\mathbf{y}$$

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Teorem

Ako je (u_n) niz u prostoru $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, takav da $\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0$ (slabo), onda postoji podniz $(\mathsf{u}_{n'})$ takav da

$$\mathbf{W}_{n'} \xrightarrow{\mathcal{S}'} \boldsymbol{\mu}_{sc}\ ,$$

pri čemu je $\boldsymbol{\mu}_{sc}$ poluklasična mjera skale ε_n.

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[LP] Pierre Louis Lions, Thierry Paul: *Sur les measures de Wigner, Revista Mat. Iberoamericana* **9,** (1993) 553-618

Primjer 1: Titranje - jedan smjer

$\alpha > 0$, $\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,, \;\; n \to \infty$$

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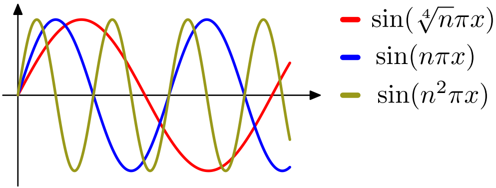
$n = 2$



Primjer 2: Titranje - dva smjera

$0 < \alpha < \beta$, $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{\mathsf{0}\}$,

$$u_n(\mathbf{x}) := e^{2\pi i n^{\alpha} \mathsf{k}\cdot\mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,, \;\; n \to \infty$$

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μ_H (μ_{sc}) je H-mjera (poluklasična mjera skale ε_n, $\varepsilon_n \to 0$) pridružena nizu $(u_n + v_n)$.

$$\mu_H = \lambda(\mathbf{x}) \boxtimes \left(\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\frac{\mathsf{s}}{|\mathsf{s}|}}\right)(\boldsymbol{\xi})$$

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$$\begin{cases} i\varepsilon_n \partial_t u_n + \frac{\varepsilon_n^2}{2} \triangle u_n - V u_n = 0 \quad \text{u} \quad \mathbf{R} \times \mathbf{R}^d \\ u_n(0,\cdot) = u_n^0 \in \mathrm{H}^2(\mathbf{R}^d) \end{cases}.$$

Neka je W_n^t Wignerova pretvorba niza rješenja $(u_n(t,\cdot))$ skale ε_n. Vrijedi

$$e_n^t(\mathbf{x}) = \int_{\mathbf{R}^d} W_n^t(\mathbf{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}$$

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DA, ako je za svaki t niz $(u_n(t,\cdot))$ (ε_n)-titrajući.
U našem slučaju će to biti ispunjeno ako je niz početnih uvjeta (u_n^0) (ε_n)-titrajući.

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Koristeći jednadžbu dobivamo da Wignerove pretvorbe W_n^t niza rješenja $(u_n(t, \cdot))$ zadovoljavaju Wignerovu jednadžbu:

$$\partial_t W_n^t + \boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} W_n^t + K_n^t *_{\boldsymbol{\xi}} W_n^t = 0 \,,$$

pri čemu je

$$K_n^t(\mathbf{x}, \boldsymbol{\xi}) = -\int_{\mathbf{R}^d} e^{-2\pi i \mathbf{z} \cdot \boldsymbol{\xi}} \frac{V(\mathbf{x} + \frac{\varepsilon_n \mathbf{z}}{2}) - V(\mathbf{x} - \frac{\varepsilon_n \mathbf{z}}{2})}{i\varepsilon_n} \,.$$

(klasična) Liouvilleova jednadžba

Formalnim prijelazom na limes dobivamo

$$\partial_t \mu_{sc}^t + \boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} \mu_{sc}^t - \nabla_{\mathbf{x}} V \cdot \nabla_{\boldsymbol{\xi}} \mu_{sc}^t = 0 \,,$$

uz početni uvjet $\mu_{sc}^0 = \nu_{sc}$, gdje je ν_{sc} poluklasična mjera skale ε_n pridružena nizu početnih uvjeta (u_n^0).

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Rješavanjem prethodne transportne jednadžbe dobivamo μ^t_{sc} pa iz $e^t(\mathbf{x}) = \int_{\mathbf{R}^d} \mu^t_{sc}(\mathbf{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}$ dobivamo traženi kvadratični izraz.

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Prethodna jednadžba odgovara jednadžbi (klasične) statističke fizike, pa možemo reći da *kvantna zadaća konvergira klasičnoj zadaći kada kvantni parametar ε_n iščezava.*

Problemi u provedbi metode

- Singularnost potencijala V
- Nejedinstvenost poluklasične mjere
- Postojanje i jedinstvenost rješenja transportne zadaće: sustavi i nelinearne zadaće

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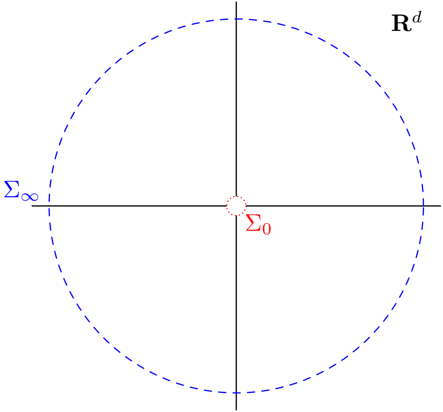
- Lokalizacijsko svojstvo mikrolokalnih defektnih funkcionala
- Objediniti dobra svojstva H-mjera i poluklasičnih mjera
- Poopćenje postojećih objekata na $\mathrm{L}^p - \mathrm{L}^{p'}$ prostore



$$\Sigma_0 := \{0^{\boldsymbol{\xi}_0} \;:\; \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}$$
$$\Sigma_\infty := \{\infty^{\boldsymbol{\xi}_0} \;:\; \boldsymbol{\xi}_0 \in \mathrm{S}^{d-1}\}$$
$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Korolar

a) $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \pi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *pri čemu je* $\pi(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

Teorem

Ako $\mathsf{u}_n \rightharpoonup 0$ *u* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, $\varepsilon_n \searrow 0$, *tada postoje podniz* $(\mathsf{u}_{n'})$ *i* $\boldsymbol{\mu}_{sc} \in \mathcal{M}_{\rm b}(\mathbf{R}^d \times \mathbf{R}^d; \mathrm{M}_r(\mathbf{C}))$ *takvi da za svake* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ *i* $\psi \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} (\widehat{\varphi_1 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \otimes (\widehat{\varphi_2 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \psi(\varepsilon_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}_{sc}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \,.$$

Mjeru $\boldsymbol{\mu}_{sc}$ *nazivamo poluklasičnom mjerom skale* ε_n *pridruženom (pod)nizu* (u_n).

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Mjeru $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *nazivamo* *jednoskalna H-mjera skale* ε_n *pridruženom (pod)nizu* (u_n).

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Mjeru $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *nazivamo jednoskalna H-mjera skale* ε_n *pridruženom (pod)nizu* (u_n).

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Ako $\mathsf{u}_n \rightharpoonup 0$ *u* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{C}^r)$, $\varepsilon_n \searrow 0$, *tada postoje podniz* $(\mathsf{u}_{n'})$ *i* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\mathbf{R}^d \times \mathrm{K}_{0,\infty}(\mathbf{R}^d); \mathrm{M}_r(\mathbf{C}))$ *takvi da za svake* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *i* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

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Distribuciju reda nula $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *nazivamo jednoskalna H-mjera skale* ε_n *pridruženom (pod)nizu* (u_n).

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Distribuciju reda nula $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *nazivamo jednoskalna H-mjera skale* ε_n *pridruženom (pod)nizu* (u_n)*.*

[T2] Luc Tartar: *The general theory of homogenization: A personalized introduction, Springer* (2009)

Teorem

$\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathcal{S}(\mathbf{R}^d)$, $\tilde{\psi} \in \mathrm{C}(\mathrm{S}^{d-1})$.

$$\begin{aligned} &a) \quad \langle \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \psi\rangle &&= \langle \boldsymbol{\mu}_{sc}, \varphi_1\bar{\varphi}_2 \boxtimes \psi\rangle\,, \\ &b) \quad \langle \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \tilde{\psi}\circ\pi\rangle &&= \langle \boldsymbol{\mu}_{H}, \varphi_1\bar{\varphi}_2 \boxtimes \tilde{\psi}\rangle\,, \end{aligned}$$

pri čemu je $\pi(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

Svojstva

Teorem

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Teorem

$$\begin{aligned} &a) \quad && \boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \\ &b) \quad && \mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 && \iff && \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0} \\ &c) \quad && \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 && \iff && (\mathsf{u}_n) \text{ je } (\varepsilon_n) - \text{titrajući} \end{aligned}$$

Primjer 1 - još jednom

$$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}},$$

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$$\mu_{\mathrm{K}_{0,\infty}} = \lambda(\mathbf{x}) \boxtimes \begin{cases} \delta_{0\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) & , \quad \lim_n n^\alpha \varepsilon_n = 0 \\ \delta_{c\mathsf{k}}(\boldsymbol{\xi}) & , \quad \lim_n n^\alpha \varepsilon_n = c \in \langle 0, \infty\rangle \\ \delta_{\infty\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) & , \quad \lim_n n^\alpha \varepsilon_n = \infty \end{cases}$$

Primjer 2 - još jednom

$u_n(\mathbf{x}) = e^{2\pi i n^\alpha \mathsf{k}\cdot\mathbf{x}}$, $v_n(\mathbf{x}) = e^{2\pi i n^\beta \mathsf{s}\cdot\mathbf{x}}$,
Pripadne mjere niza $(u_n + v_n)$:

$$\mu_H = \lambda(\mathbf{x}) \boxtimes \left(\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\frac{\mathsf{s}}{|\mathsf{s}|}}\right)(\boldsymbol{\xi})$$

$$\mu_{sc} = \lambda(\mathbf{x}) \boxtimes \begin{cases} 2\delta_0(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = 0 \\ (\delta_0 + \delta_{c\mathsf{s}})(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = c \in \langle 0, \infty\rangle \\ \delta_0(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = \infty \ \& \ \lim_n n^\alpha \varepsilon_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\alpha \varepsilon_n = c \in \langle 0, \infty\rangle \\ 0 & , \quad \lim_n n^\alpha \varepsilon_n = \infty \end{cases}$$

$$\mu_{\mathrm{K}_{0,\infty}} = \lambda(\mathbf{x}) \boxtimes \begin{cases} (\delta_{0\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{0\frac{\mathsf{s}}{|\mathsf{s}|}})(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = 0 \\ (\delta_{0\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{c\mathsf{s}})(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = c \in \langle 0, \infty\rangle \\ (\delta_{0\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\infty\frac{\mathsf{s}}{|\mathsf{s}|}})(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = \infty \ \& \ \lim_n n^\alpha \varepsilon_n = 0 \\ (\delta_{c\mathsf{k}} + \delta_{\infty\frac{\mathsf{s}}{|\mathsf{s}|}})(\boldsymbol{\xi}) & , \quad \lim_n n^\alpha \varepsilon_n = c \in \langle 0, \infty\rangle \\ (\delta_{\infty\frac{\mathsf{k}}{|\mathsf{k}|}} + \delta_{\infty\frac{\mathsf{s}}{|\mathsf{s}|}}) & , \quad \lim_n n^\alpha \varepsilon_n = \infty \end{cases}$$

Neka je $\Omega \subseteq \mathbf{R}^d$ otvoren, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ u $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ i

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{u } \Omega ,$$

pri čemu je $l \in 0..m$, $\varepsilon_n \to 0$, $\varepsilon_n > 0$, $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$, te $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ takav da

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

Lokalizacijsko svojstvo

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Teorem (Tartar (2009))

Uz prethodne pretpostavke i $l = 1$, jednoskalna H-mjera $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ skale ε_n pridružena (u_n) zadovoljava

$$\mathrm{supp}\,(\mathbf{p}\boldsymbol{\mu}^\top_{\mathrm{K}_{0,\infty}}) \subseteq \Omega \times \Sigma_0 \,,$$

gdje je

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{1 \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}| + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

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Teorem

Uz prethodne pretpostavke, jednoskalna H-mjera $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ skale ε_n pridružena (u_n) zadovoljava

$$\mathbf{p} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0} \,,$$

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Theorem

$\varepsilon_n > 0$ *omeđen* $\mathsf{u}_n \rightharpoonup 0$ *u* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *i*

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

pri čemu je $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *jednoliko na kompaktima, te* $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *zadovoljava* $(\mathrm{C}(\varepsilon_n))$.
Tada za $\omega_n \to 0$ *takav da* $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in [0, \infty]$, *pripadna jednoskalna H-mjera* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *skale* ω_n *zadovoljava*

$$\mathbf{p} \boldsymbol{\mu}^{\top}_{\mathrm{K}_{0,\infty}} = \mathbf{0} \,,$$

gdje je

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

Lokalizacijsko svojstvo - završno poopćenje

Teorem (nastavak)

Nadalje, ako postoji $\varepsilon_0 > 0$ *takav da* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *možemo uzeti*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

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Skica dokaza. Pretpostavimo da tvrdnja vrijedi za $\lim_n \frac{\omega_n}{\varepsilon_n} \in \langle 0, \infty \rangle$ i dokažimo rezultat za preostala dva slučaj.
U slučaju $\lim_n \frac{\omega_n}{\varepsilon_n} = \infty$ možemo zapisati jednadžbu u obliku

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \omega_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{B}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n ,$$

za $\mathbf{B}_n^{\boldsymbol{\alpha}} := \left(\frac{\varepsilon_n}{\omega_n}\right)^{|\boldsymbol{\alpha}|-l} \mathbf{A}_n^{\boldsymbol{\alpha}}$, i slično za slučaj $\lim_n \frac{\omega_n}{\varepsilon_n} = 0$ imamo

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \omega_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}} (\mathbf{B}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{g}_n ,$$

pri čemu su $\mathbf{B}_n^{\boldsymbol{\alpha}} := \left(\frac{\omega_n}{\varepsilon_n}\right)^{m-|\boldsymbol{\alpha}|} \mathbf{A}_n^{\boldsymbol{\alpha}}$, $\mathsf{g}_n := \left(\frac{\omega_n}{\varepsilon_n}\right)^{m-l} \mathsf{f}_n$.

- Korištenjem prethodnog teorema, te $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_H$ na $\Omega \times \mathrm{S}^{d-1}$, dobivamo poznato lokalizacijsko svojstvo H-mjera.

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Teorem

Uz pretpostavke prethodnog teorema imamo

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}_{sc}^{\top} = \mathbf{0}\,,$$

gdje je

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = \infty \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad \lim_n \frac{\omega_n}{\varepsilon_n} = 0 \end{cases}$$

Dokaz (samo slučaj $\lim_n \frac{\omega_n}{\varepsilon_n} = c \in \langle 0, \infty \rangle$)

$$\psi \in \mathcal{S}(\mathbf{R}^d) \quad \Longrightarrow \quad \boldsymbol{\xi} \mapsto (|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|)\psi(\boldsymbol{\xi}) \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$$

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$$\begin{aligned}
0 &= \Big\langle \overline{\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \Big(\frac{2\pi i}{c}\Big)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \varphi \boxtimes (|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m)\psi \Big\rangle \\
&= \Big\langle \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}, \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \Big(\frac{2\pi i}{c}\Big)^{|\boldsymbol{\alpha}|} \varphi \mathbf{A}^{\boldsymbol{\alpha}} \boxtimes \boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \Big\rangle \\
&= \Big\langle \boldsymbol{\mu}_{sc}, \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \Big(\frac{2\pi i}{c}\Big)^{|\boldsymbol{\alpha}|} \varphi \mathbf{A}^{\boldsymbol{\alpha}} \boxtimes \boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \Big\rangle = \Big\langle \overline{\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \Big(\frac{2\pi i}{c}\Big)^{|\boldsymbol{\alpha}|} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}} \boldsymbol{\mu}_{sc}, \varphi \boxtimes \psi \Big\rangle ,
\end{aligned}$$

pri čemu smo koristili $\boldsymbol{\xi}^{\boldsymbol{\alpha}} \psi \in \mathcal{S}(\mathbf{R}^d)$, te da se $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ i $\boldsymbol{\mu}_{sc}$ podudaraju na $\mathcal{S}(\mathbf{R}^d)$.

$\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^r)$, $p \in \langle 0, \infty \rangle$

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Uz kakve uvjete na ψ je $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^r) \longrightarrow \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^r)$ neprekinut?

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Teorem (Mihlin)

Neka je $\psi \in \mathrm{L}^\infty(\mathbf{R}^d \setminus \{0\})$ *takva da postoje* $\partial^{\boldsymbol\alpha}\psi$, $|\boldsymbol\alpha| \leqslant \kappa$, *pri čemu* $\kappa = [d/2] + 1$, *te postoji* $k > 0$ *za koju vrijedi*

$$(\forall\, \boldsymbol\alpha \in \mathbf{N}_0^d)\ |\boldsymbol\alpha| \leqslant \kappa \quad\Longrightarrow\quad |\partial^{\boldsymbol\alpha}\psi(\boldsymbol\xi)| \leqslant k|\boldsymbol\xi|^{-|\boldsymbol\alpha|}\,,$$

tada

$$(\exists\, C_d > 0)(\forall\, p \in \langle 0, \infty\rangle)\ \|\mathcal{A}_\psi\|_{\mathrm{L}^p\to\mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_{\mathrm{L}^\infty})\,.$$

Teorem

Ako $u_n \longrightarrow 0$ *u* $\mathrm{L}^p(\mathbf{R}^d)$ *i* $v_n \xrightarrow{*} 0$ *u* $\mathrm{L}^q(\mathbf{R}^d)$, $q \geqslant p'$, *tada postoje* $(u_{n'}), (v_{n'})$ *i* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ *reda ne većeg od* $\kappa = [d/2] + 1$ *u* $\boldsymbol{\xi}$, *takvi da* $(\forall\, \varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d))(\forall\, \psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1}))$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v_{n'})}\, d\mathbf{x} = \langle \mu, \varphi_1 \bar\varphi_2 \boxtimes \psi \rangle \,.$$

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Ključnu ulogu u dokazu ima sljedeća komutacijska lema.

$$C := [\mathcal{A}_\psi, M_b] = \mathcal{A}_\psi M_b - M_b \mathcal{A}_\psi \ , \ b \in \mathrm{C}_0(\mathbf{R}^d)$$

Lema

(v_n) *omeđen u* $\mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^r(\mathbf{R}^d)$, $r \in \langle 2, \infty]$, *i* $v_n \longrightarrow 0$ *u smislu distribucija, tada* $Cv_n \to 0$ *u* $\mathrm{L}^q(\mathbf{R}^d)$, $q \in [2, r] \setminus \{\infty\}$.

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[AM] Nenad Antonić, Darko Mitrović: *H-distributions - an extension of the H-measures to an* $\mathrm{L}^p - \mathrm{L}^q$ *setting, Abstr. Appl. Anal.* (2011)

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Provjeravanje Mihlinovog uvjeta za funkcije iz $\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ je vrlo tehičko i zahtjeva precizno opisivanje prostora $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

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Lokalizacijsko svojstvo...
Očekujemo da će ovaj objekt biti prikladniji za nelinearne zadaće.

Jednoskalne H-distribucije

Teorem

Ako $u_n \longrightarrow 0$ *u* $\mathrm{L}^p(\mathbf{R}^d)$, $v_n \xrightarrow{*} 0$ *u* $\mathrm{L}^q(\mathbf{R}^d)$, $q \geqslant p'$, *i* $\varepsilon_n \to 0$, *tada postoje* $(u_{n'}), (v_{n'})$ *i* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathbf{R}^d)$ *reda ne većeg od* $\kappa = [d/2] + 1$ *u* $\boldsymbol{\xi}$, *takvi da* $(\forall\, \varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d))(\forall\, \psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)))$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})\overline{\mathcal{A}_{\bar\psi_{n'}}(\varphi_2 v_{n'})}\, d\mathbf{x} = \langle \mu, \varphi_1 \bar\varphi_2 \boxtimes \psi \rangle\,,$$

pri čemu je $\psi_n(\boldsymbol{\xi}) := \psi(\varepsilon_n \boldsymbol{\xi})$.

Provjeravanje Mihlinovog uvjeta za funkcije iz $\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ je vrlo tehičko i zahtjeva precizno opisivanje prostora $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Lokalizacijsko svojstvo...
Očekujemo da će ovaj objekt biti prikladniji za nelinearne zadaće.

Veza s Wignerovom pretvorbom?

Primjer 3: Titranje - dvije skale

$0 < \alpha < \beta$, $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i (n^\alpha \mathsf{s} + n^\beta \mathsf{k}) \cdot \mathbf{x}} \xrightharpoonup{\mathrm{L}^2_{\mathrm{loc}}} 0\,, \ n \to \infty$$

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$$\mu_{\mathrm{K}_{0,\infty}} = \lambda(\mathbf{x}) \boxtimes \begin{cases} \delta_{0\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = 0 \\ \delta_{c\mathsf{k}}(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = c \in \langle 0, \infty \rangle \\ \delta_{\infty\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}) & , \quad \lim_n n^\beta \varepsilon_n = \infty \end{cases}$$

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Sporiju skalu n^α, odnosno pripadni smjer titranja s, u niti jednom slučaju ne uspijemo dohvatiti.

$\Longrightarrow$ potrebni novi objekti i/ili metode

[T3] LUC TARTAR: *Multi-scale H-measures, Discrete and Continuous Dynamical Systems - Series S* (2015)

Višeskalni problemi

U [T3] Tartar je uveo višeskalne objekte, ali za sad još uvijek bez nekih konkretnih rezultata.

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Ideja: počevši od paraboličkih H-mjera, konstruirati paraboličke jednoskalne H-mjere. Objekt s dvije skale u omjeru 1:2.

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$$\lim_{n'} \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 \mathsf{u}_{n'}}(\tau, \boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\tau, \boldsymbol{\xi}) \psi(\varepsilon_{n'}^2 \tau, \varepsilon_{n'} \boldsymbol{\xi}) \, d\tau d\boldsymbol{\xi} = \langle \boldsymbol{\nu}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \, .$$

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