Estimates on the mild solution of semilinear Cauchy problems and some notes on damped wave equations

Marko Erceg

Department of Mathematics, Faculty of Science
University of Zagreb
Joint work with Krešimir Burazin

Chemnitz, 4^{th} July, 2014

WeConMApp

Semilinear abstract Cauchy problem

- Assumptions
- Main thereom
- Examples

Generalised damped wave equation

$(X, \|\cdot\|)$ Banach space, $T > 0$,

(ACP)
$$\begin{cases} \mathsf{u}'(t) + A\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) & \text{in } \langle 0, T\rangle \\ \mathsf{u}(0) = \mathsf{g} \end{cases}$$

$(X, \|\cdot\|)$ Banach space, $T > 0$,

$$\text{(ACP)} \qquad \begin{cases} \mathsf{u}'(t) + A\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) & \text{in } \langle 0, T\rangle \\ \mathsf{u}(0) = \mathsf{g} \end{cases}$$

- $A : D(A) \subseteq X \longrightarrow X$ generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X, and $M \geqslant 1$, $\omega \geqslant 0$ such that

$$(\forall\, t \geqslant 0) \qquad \|T(t)\|_{\mathcal{L}(X)} \leqslant Me^{\omega t}\,,$$

- $\mathsf{g} \in X$,
- $\mathsf{f} : [0, T] \times X \longrightarrow X$ *Borel measurable* and *locally Lipschitz* in u: $(\exists\, \Psi \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbf{R}))(\forall\, r > 0)(\forall\, \mathsf{w}, \mathsf{z} \in \mathrm{B}_X[0, r])$

$$\|\mathsf{f}(t, \mathsf{z}) - \mathsf{f}(t, \mathsf{w})\| \leqslant \Psi(r)\|\mathsf{z} - \mathsf{w}\| \qquad (\text{a.e. } t \in [0, T])\,,$$

- $\mathsf{u} : [0, T\rangle \longrightarrow X$ is the unknown.

$(X, \|\cdot\|)$ Banach space, $T > 0$,

(ACP) $$\begin{cases} \mathsf{u}'(t) + A\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) & \text{in } \langle 0, T \rangle \\ \mathsf{u}(0) = \mathsf{g} \end{cases}$$

Semilinear abstract Cauchy problem

$(X, \|\cdot\|)$ Banach space, $T > 0$,

(ACP)
$$\begin{cases} \mathsf{u}'(t) + A\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) & \text{in } \langle 0, T\rangle \\ \mathsf{u}(0) = \mathsf{g} \end{cases}$$

$\mathsf{u} \in \mathrm{C}([0, S\rangle; X)$ is called a *mild solution* of (ACP) on $[0, S\rangle$ if

(CS)
$$\mathsf{u}(t) = T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s, \mathsf{u}(s))ds\,, \quad t \in [0, S\rangle\,.$$

Semilinear abstract Cauchy problem

$(X, \|\cdot\|)$ Banach space, $T > 0$,

(ACP) $$\begin{cases} \mathsf{u}'(t) + A\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) & \text{in } \langle 0, T\rangle \\ \mathsf{u}(0) = \mathsf{g} \end{cases}$$

$\mathsf{u} \in \mathrm{C}([0, S\rangle; X)$ is called a *mild solution* of (ACP) on $[0, S\rangle$ if

(CS) $$\mathsf{u}(t) = T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s, \mathsf{u}(s))ds\,, \quad t \in [0, S\rangle\,.$$

Standard theory [Pazy (1983), Cazenave-Haraux (1998)]: existence and uniqueness of the mild solution

$(X, \|\cdot\|)$ Banach space, $T > 0$,

(ACP) $$\begin{cases} \mathsf{u}'(t) + A\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) & \text{in } \langle 0, T\rangle \\ \mathsf{u}(0) = \mathsf{g} \end{cases}$$

$\mathsf{u} \in \mathrm{C}([0, S\rangle; X)$ is called a *mild solution* of (ACP) on $[0, S\rangle$ if

(CS) $$\mathsf{u}(t) = T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s, \mathsf{u}(s))ds\,, \quad t \in [0, S\rangle\,.$$

Standard theory [Pazy (1983), Cazenave-Haraux (1998)]: existence and uniqueness of the mild solution

Different apporach (based on [Tartar (2008), Burazin (2008)]): estimate on the mild solution and its time of existence

Local bound and ODE

Theorem

The function $\Phi(t,u) := \sup_{\|\mathsf{w}\| \leqslant u} \|\mathsf{f}(t,\mathsf{w})\|$, $t \in [0,T]$, $u \in \mathbf{R}_0^+$ *is (the smallest) local bound for* f*:*

$$(\forall\, r > 0)(\forall\, \mathsf{w} \in \mathrm{B}_X[\mathsf{0}, r]) \quad \|\mathsf{f}(t,\mathsf{w})\| \leqslant \Phi(t,r) \quad (\text{a.e. } (t,\mathsf{w}) \in [0,T] \times X)\,,$$

and has the following properties:

- $\Phi \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T] \times \mathbf{R}_0^+)\,;$
- $\Phi \geqslant 0$ *and* $\Phi(t,\cdot)$ *is nondecresing,* $t \in [0,T]\,;$
- Φ *is locally Lipschitz in* u*.*

Theorem

The function $\Phi(t,u) := \sup_{\|\mathsf{w}\|\leqslant u} \|\mathsf{f}(t,\mathsf{w})\|$, $t\in[0,T]$, $u\in\mathbf{R}_0^+$ *is (the smallest) local bound for* f*:*

$$(\forall\, r>0)(\forall\, \mathsf{w}\in \mathrm{B}_X[0,r])\quad \|\mathsf{f}(t,\mathsf{w})\|\leqslant \Phi(t,r) \qquad (\text{a.e. } (t,\mathsf{w})\in[0,T]\times X)\,,$$

and has the following properties:

- $\Phi\in \mathrm{L}^\infty_{\mathrm{loc}}([0,T]\times\mathbf{R}_0^+)$;
- $\Phi\geqslant 0$ *and* $\Phi(t,\cdot)$ *is nondecresing,* $t\in[0,T]$;
- Φ *is locally Lipschitz in* u.

The properties of the function Φ guarantee that the Cauchy problem

(ODE-Φ)
$$\begin{cases} v'(t) = e^{-\omega t}\Phi(t, Me^{\omega t}v(t)) \\ v(0) = \|\mathsf{g}\| \end{cases},$$

has the unique maximal solutions $v\in \mathrm{W}^{1,\infty}_{\mathrm{loc}}([0,S\rangle)$, for some $S>0$ (v is Lipschitz continuous on every $[a,b]\subseteq[0,S\rangle$).

Main theorem

Recall

(CS) $$\mathsf{u}(t) = T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s,\mathsf{u}(s))ds\,, \quad t \in [0,S\rangle\,.$$

Recall

(CS) $$\mathsf{u}(t) = T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s,\mathsf{u}(s))ds\,, \quad t \in [0,S\rangle\,.$$

Theorem

Let the previous assumptions hold, and assume that $v \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}([0,S\rangle)$ is the maximal solution of (ODE-Φ) for some $S \in \langle 0,T]$. Then there exists the unique mild solution on $[0,S\rangle$, $\mathsf{u} \in \mathrm{C}([0,S\rangle;X)$, of the problem (ACP). Additionally, u *satisfies the estimate*

(*E*) $$\|\mathsf{u}(t)\| \leqslant Me^{\omega t}v(t) \quad t \in [0,S\rangle\,.$$

Main theorem

Recall

$$\text{(CS)} \qquad \mathsf{u}(t) = T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s,\mathsf{u}(s))ds\,, \quad t \in [0,S\rangle\,.$$

Theorem

Let the previous assumptions hold, and assume that $v \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}([0,S\rangle)$ is the maximal solution of (ODE-Φ) for some $S \in \langle 0,T]$. Then there exists the unique mild solution on $[0,S\rangle$, $\mathsf{u} \in \mathrm{C}([0,S\rangle;X)$, of the problem (ACP). Additionally, u satisfies the estimate

$$(E) \qquad \|\mathsf{u}(t)\| \leqslant Me^{\omega t}v(t) \quad t \in [0,S\rangle\,.$$

Sketch of the proof. Uniqueness: $\mathsf{u}_1, \mathsf{u}_2 \in \mathrm{C}([0,S\rangle;X)$ two mild solutions,

$$\begin{aligned}\|\mathsf{u}_1(t)-\mathsf{u}_2(t)\| &\leqslant \int_0^t \|T(t-s)\|_{\mathcal{L}(X)}\|\mathsf{f}(s,\mathsf{u}_1(s))-\mathsf{f}(s,\mathsf{u}_2(s))\|\,ds \\ &\leqslant Me^{\omega t}\int_0^t \|\mathsf{f}(s,\mathsf{u}_1(s))-\mathsf{f}(s,\mathsf{u}_2(s))\|\,ds \\ &\leqslant Me^{\omega S}\|\Psi\|_{\mathrm{L}^\infty(0,S)}\int_0^t \|\mathsf{u}_1(s)-\mathsf{u}_2(s)\|\,ds\,.\end{aligned}$$

Sketch of the proof: Existence

Picards's iterations:

Sketch of the proof: Existence

Picards's iterations: $\mathsf{u}_0 \in \mathrm{C}([0,S];X)$ such that $\|\mathsf{u}_0(t)\| \leqslant Me^{\omega t}v(t)$, $t \in [0,S]$,

$$(\mathsf{CS}_n) \qquad \mathsf{u}_n(t) := T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s,\mathsf{u}_{n-1}(s))\,ds\,, \quad t \in [0,S\rangle\,.$$

Sketch of the proof: Existence

Picards's iterations: $\mathsf{u}_0 \in \mathrm{C}([0,S];X)$ such that $\|\mathsf{u}_0(t)\| \leqslant Me^{\omega t}v(t)$, $t \in [0,S]$,

$$(\mathsf{CS}_n) \qquad \mathsf{u}_n(t) := T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s,\mathsf{u}_{n-1}(s))\,ds\,, \quad t \in [0,S\rangle\,.$$

We have

$$\begin{aligned}
\|\mathsf{u}_1(t)\| &\leqslant \|T(t)\mathsf{g}\| + \int_0^t \|T(t-s)\mathsf{f}(s,\mathsf{u}_0(s))\|\,ds \\
&\leqslant Me^{\omega t}\|\mathsf{g}\| + Me^{\omega t}\int_0^t e^{-\omega s}\|\mathsf{f}(s,\mathsf{u}_0(s))\|\,ds \\
&\leqslant Me^{\omega t}\left(\|\mathsf{g}\| + \int_0^t e^{-\omega s}\Phi(s, Me^{\omega s}v(s))\,ds\right) \\
&\leqslant Me^{\omega t}v(t)\,,
\end{aligned}$$

and inductively we have for every $n \in \mathbf{N}$ the estimate $\|\mathsf{u}_n(t)\| \leqslant Me^{\omega t}v(t)$, $t \in [0,S\rangle$.

Sketch of the proof: Existence

Picards's iterations: $\mathsf{u}_0 \in \mathrm{C}([0,S];X)$ such that $\|\mathsf{u}_0(t)\| \leqslant M e^{\omega t} v(t)$, $t \in [0,S]$,

$$(\mathsf{CS}_n) \qquad \mathsf{u}_n(t) := T(t)\mathsf{g} + \int_0^t T(t-s)\mathsf{f}(s, \mathsf{u}_{n-1}(s))\, ds\,, \quad t \in [0,S\rangle\,.$$

We have

$$\begin{aligned}
\|\mathsf{u}_1(t)\| &\leqslant \|T(t)\mathsf{g}\| + \int_0^t \|T(t-s)\mathsf{f}(s,\mathsf{u}_0(s))\|\, ds \\
&\leqslant M e^{\omega t}\|\mathsf{g}\| + M e^{\omega t}\int_0^t e^{-\omega s}\|\mathsf{f}(s,\mathsf{u}_0(s))\|\, ds \\
&\leqslant M e^{\omega t}\left(\|\mathsf{g}\| + \int_0^t e^{-\omega s}\Phi(s, M e^{\omega s} v(s))\, ds\right) \\
&\leqslant M e^{\omega t} v(t)\,,
\end{aligned}$$

and inductively we have for every $n \in \mathbf{N}$ the estimate $\|\mathsf{u}_n(t)\| \leqslant M e^{\omega t} v(t)$, $t \in [0,S\rangle$.
After passing to the limit as $n \to \infty$ in (CS_n), we get the result.

Remarks

- Instead of a function defined on the whole $[0,T] \times X$, we can consider a function $\mathsf{f} : [0,T] \times \mathrm{B}_X(0,b) \longrightarrow X$, for some $b > 0$. v cannot blow-up, but it can quench when v approaches b.
- The mild solution of (ACP) exists at least as long as the solution v of (ODE-Φ).
- The best possible estimate of type (E) will be given by the smallest possible local bound for f, i.e. the function Φ.
- The estimate (E) is not optimal in general!
- The main theorem can be stated also for non-autonomous (evolution) abstract systems

$$\text{(eACP)} \qquad \begin{cases} \mathsf{u}'(t) + A(t)\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) \\ \mathsf{u}(0) = \mathsf{g} \end{cases} .$$

$\Omega \subset \mathbf{R}^d$ open, bounded with a Lipschitz boundary; $T, b, p > 0$,

$$\text{(nlHE)} \qquad \begin{cases} \partial_t u(t,\mathbf{x}) - \triangle u(t,\mathbf{x}) = \dfrac{\gamma(\mathbf{x})}{(b - u(t,\mathbf{x}))^p} & \text{in } \langle 0, T\rangle \times \Omega \\ u_{|_{\partial\Omega}} = 0 & \\ u(0,\cdot) = u_0 & \end{cases},$$

$\gamma, u_0 \in \mathrm{C}_0(\Omega)$ and $u : [0, T\rangle \times \Omega \longrightarrow \mathbf{R}$.

Nonlinear heat equation

$\Omega \subset \mathbf{R}^d$ open, bounded with a Lipschitz boundary; $T, b, p > 0$,

$$\text{(nlHE)} \qquad \begin{cases} \partial_t u(t,\mathbf{x}) - \triangle u(t,\mathbf{x}) = \dfrac{\gamma(\mathbf{x})}{(b - u(t,\mathbf{x}))^p} & \text{in } \langle 0, T\rangle \times \Omega \\ u_{|_{\partial\Omega}} = 0 & \\ u(0,\cdot) = u_0 & \end{cases},$$

$\gamma, u_0 \in \mathrm{C}_0(\Omega)$ and $u : [0, T\rangle \times \Omega \longrightarrow \mathbf{R}$.

- $X := \mathrm{C}_0(\Omega)$, $\|\cdot\| := \|\cdot\|_{\mathrm{L}^\infty(\Omega)}$
- $\mathsf{u}(t) := u(t,\cdot)$, $\mathsf{u}_0 := u_0(\cdot)$, $\boldsymbol{\gamma} := \gamma(\cdot)$
- $A := -\triangle$, $D(A) = \{v \in \mathrm{H}^1_0(\Omega) \cap X \ : \ \triangle v \in X\} \leq X$, is an infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geqslant 0}$

$\Omega \subset \mathbf{R}^d$ open, bounded with a Lipschitz boundary; $T, b, p > 0$,

$$\text{(nlHE)} \qquad \begin{cases} \partial_t u(t,\mathbf{x}) - \triangle u(t,\mathbf{x}) = \dfrac{\gamma(\mathbf{x})}{(b-u(t,\mathbf{x}))^p} & \text{in } \langle 0,T\rangle \times \Omega \\ u_{|_{\partial\Omega}} = 0 & \\ u(0,\cdot) = u_0 & \end{cases},$$

$\gamma, u_0 \in \mathrm{C}_0(\Omega)$ and $u : [0,T\rangle \times \Omega \longrightarrow \mathbf{R}$.

- $X := \mathrm{C}_0(\Omega)$, $\|\cdot\| := \|\cdot\|_{\mathrm{L}^\infty(\Omega)}$
- $\mathsf{u}(t) := u(t,\cdot)$, $\mathsf{u}_0 := u_0(\cdot)$, $\boldsymbol{\gamma} := \gamma(\cdot)$
- $A := -\triangle$, $D(A) = \{v \in \mathrm{H}^1_0(\Omega) \cap X \;:\; \triangle v \in X\} \leq X$, is an infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t\geqslant 0}$

$$\text{(ODE-}\Phi\text{)} \qquad \begin{cases} v'(t) = \Phi(t, v(t)) = \dfrac{\|\boldsymbol{\gamma}\|}{(b-v(t))^p} \\ v(0) = \|\mathsf{u}_0\| \end{cases}$$

$\Omega \subset \mathbf{R}^d$ open, bounded with a Lipschitz boundary; $T, b, p > 0$,

$$
\text{(nlHE)} \qquad \begin{cases} \partial_t u(t,\mathbf{x}) - \triangle u(t,\mathbf{x}) = \dfrac{\gamma(\mathbf{x})}{(b - u(t,\mathbf{x}))^p} & \text{in } \langle 0, T\rangle \times \Omega \\ u_{|\partial\Omega} = 0 & \\ u(0,\cdot) = u_0 & \end{cases},
$$

$\gamma, u_0 \in \mathrm{C}_0(\Omega)$ and $u : [0,T\rangle \times \Omega \longrightarrow \mathbf{R}$.

- $X := \mathrm{C}_0(\Omega)$, $\|\cdot\| := \|\cdot\|_{\mathrm{L}^\infty(\Omega)}$
- $\mathsf{u}(t) := u(t,\cdot)$, $\mathsf{u}_0 := u_0(\cdot)$, $\boldsymbol{\gamma} := \gamma(\cdot)$
- $A := -\triangle$, $D(A) = \{v \in \mathrm{H}^1_0(\Omega) \cap X \;:\; \triangle v \in X\} \leq X$, is an infinitesimal generator of a C_0-semigroup of contractions $(T(t))_{t \geqslant 0}$

$$
\text{(ODE-}\Phi\text{)} \qquad \begin{cases} v'(t) = \Phi(t, v(t)) = \dfrac{\|\boldsymbol{\gamma}\|}{(b - v(t))^p} \\ v(0) = \|\mathsf{u}_0\| \end{cases}
$$

$$
\Longrightarrow \quad v(t) = b - \Big((b - \|\mathsf{u}_0\|)^{p+1} - (p+1)\|\boldsymbol{\gamma}\| t \Big)^{\frac{1}{p+1}},
$$

exists until time $T_1 = \frac{(b - \|\mathsf{u}_0\|)^{p+1}}{(p+1)\|\boldsymbol{\gamma}\|}$ when it quenches

$$\left\{\begin{array}{rl} \partial_t u(t,\mathbf{x}) - i\triangle u(t,\mathbf{x}) = -\gamma(t)u(t,\mathbf{x}) - g(t)|u(t,\mathbf{x})|^2 u(t,\mathbf{x}) & \text{in } \langle 0,T\rangle \times \mathbf{R}^d \\ u(0,\cdot) = u_0 & \end{array}\right. ,$$

$\gamma, g \in \mathrm{C}([0,T];\mathbf{C})$, $u_0 \in \mathrm{L}^2(\mathbf{R}^d;\mathbf{C})$.

$$\begin{cases} \partial_t u(t,\mathbf{x}) - i\triangle u(t,\mathbf{x}) = -\gamma(t)u(t,\mathbf{x}) - g(t)|u(t,\mathbf{x})|^2 u(t,\mathbf{x}) & \text{in } \langle 0,T\rangle \times \mathbf{R}^d \\ u(0,\cdot) = u_0 \end{cases},$$

$\gamma, g \in \mathrm{C}([0,T];\mathbf{C})$, $u_0 \in \mathrm{L}^2(\mathbf{R}^d;\mathbf{C})$.

- $\mathsf{u}(t) := u(t,\cdot)$, $\mathsf{u}_0 := u_0(\cdot)$
- $A := -i\triangle$ is an infinitesimal generator of a C_0-semigroup of unitary operators $(T(t))_{t\geqslant 0}$ on $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C})$ with the domain $D(A) = \mathrm{H}^2(\mathbf{R}^d;\mathbf{C})$

$$\begin{cases} \partial_t u(t,\mathbf{x}) - i\triangle u(t,\mathbf{x}) = -\gamma(t)u(t,\mathbf{x}) - g(t)|u(t,\mathbf{x})|^2 u(t,\mathbf{x}) & \text{in } \langle 0,T\rangle \times \mathbf{R}^d \\ u(0,\cdot) = u_0 & \end{cases},$$

$\gamma, g \in \mathrm{C}([0,T];\mathbf{C})$, $u_0 \in \mathrm{L}^2(\mathbf{R}^d;\mathbf{C})$.

- $\mathsf{u}(t) := u(t,\cdot)$, $\mathsf{u}_0 := u_0(\cdot)$
- $A := -i\triangle$ is an infinitesimal generator of a C_0-semigroup of unitary operators $(T(t))_{t\geqslant 0}$ on $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C})$ with the domain $D(A) = \mathrm{H}^2(\mathbf{R}^d;\mathbf{C})$

Problem: the right hand side is not locally Lipshitz in u on $\mathrm{L}^2(\mathbf{R}^d)$!

$$\begin{cases} \partial_t u(t,\mathbf{x}) - i\triangle u(t,\mathbf{x}) = -\gamma(t)u(t,\mathbf{x}) - g(t)|u(t,\mathbf{x})|^2 u(t,\mathbf{x}) & \text{in } \langle 0,T\rangle \times \mathbf{R}^d \\ u(0,\cdot) = u_0 \end{cases},$$

$\gamma, g \in \mathrm{C}([0,T];\mathbf{C})$, $u_0 \in \mathrm{L}^2(\mathbf{R}^d;\mathbf{C})$.

- $\mathsf{u}(t) := u(t,\cdot)$, $\mathsf{u}_0 := u_0(\cdot)$
- $A := -i\triangle$ is an infinitesimal generator of a C_0-semigroup of unitary operators $(T(t))_{t\geqslant 0}$ on $\mathrm{L}^2(\mathbf{R}^d;\mathbf{C})$ with the domain $D(A) = \mathrm{H}^2(\mathbf{R}^d;\mathbf{C})$

Problem: the right hand side is not locally Lipshitz in u on $\mathrm{L}^2(\mathbf{R}^d)$!

- $X := D(A) = \mathrm{H}^2(\mathbf{R}^d;\mathbf{C})$
- $A_{|_X} : D(A_{|_X}) \subseteq X \longrightarrow X$, $A_{|_X}\mathsf{u} := A\mathsf{u}$ on the domain

$$D(A_{|_X}) := \{\mathsf{u} \in D(A) \cap X \ : \ A\mathsf{u} \in X\} \leqslant X\,.$$

is an infinitesimal generator of a C_0-semigroup of unitary operators $(T(t)_{|_X})_{t\geqslant 0}$ on X.

Sobolev emmbeding theorem implies that $\mathrm{w} \mapsto |\mathrm{w}|^2\mathrm{w}$ is locally Lipschitz in X. Under a stronger assumption on the inital data, $\mathrm{u}_0 \in X = \mathrm{H}^2(\mathbf{R}^d; \mathbf{C})$, we can apply the main theorem:

Sobolev emmbeding theorem implies that $\mathsf{w} \mapsto |\mathsf{w}|^2\mathsf{w}$ is locally Lipschitz in X. Under a stronger assumption on the inital data, $\mathsf{u}_0 \in X = \mathrm{H}^2(\mathbf{R}^d; \mathbf{C})$, we can apply the main theorem:

(ODE-Φ)
$$\begin{cases} v'(t) = \Phi(t, v(t)) = |\gamma(t)|v(t) + |g(t)|v(t)^3 \\ v(0) = \|\mathsf{u}_0\| \end{cases}$$

$$\implies \quad v(t) = \left(e^{-2\int_0^t |\gamma(\tau)|\, d\tau} \left(-2 \int_0^t |g(s)| e^{2\int_0^s |\gamma(\tau)|\, d\tau}\, ds + \|\mathsf{u}_0\|^{-2} \right) \right)^{-\frac{1}{2}},$$

that has a blow-up in finite time T_1.

Sobolev emmbeding theorem implies that w $\mapsto |\mathsf{w}|^2\mathsf{w}$ is locally Lipschitz in X. Under a stronger assumption on the inital data, $\mathsf{u}_0 \in X = \mathrm{H}^2(\mathbf{R}^d;\mathbf{C})$, we can apply the main theorem:

$$\text{(ODE-}\Phi\text{)} \qquad \begin{cases} v'(t) = \Phi(t, v(t)) = |\gamma(t)|v(t) + |g(t)|v(t)^3 \\ v(0) = \|\mathsf{u}_0\| \end{cases}$$

$$\implies \quad v(t) = \left(e^{-2\int_0^t |\gamma(\tau)|\,d\tau} \left(-2\int_0^t |g(s)| e^{2\int_0^s |\gamma(\tau)|\,d\tau}\,ds + \|\mathsf{u}_0\|^{-2} \right) \right)^{-\frac{1}{2}},$$

that has a blow-up in finite time T_1.
Finally, for $\mathsf{u}_0 \in \mathrm{H}^2(\mathbf{R}^d;\mathbf{C})$ we have the existence of the unique mild solution $\mathsf{u} \in \mathrm{C}([0, T_1\rangle; \mathrm{H}^2(\mathbf{R}^d;\mathbf{C}))$.

Generalised damped wave equation in 1D

In [K. Veselić, (2006)] this problem has been observed:

$$\rho(x)u_{tt} + \gamma(x)u_t - (d(x)u_{tx})_x - (k(x)u_x)_x = 0 \quad \text{in } \langle 0, \infty\rangle \times \langle a, b\rangle\,,$$

ρ, γ, d, k non-negative and "smooth enough" and $u : \langle 0, \infty\rangle \times \langle a, b\rangle \longrightarrow \mathbf{C}$.

Generalised damped wave equation in 1D

In [K. Veselić, (2006)] this problem has been observed:

$$\rho(x)u_{tt} + \gamma(x)u_t - (d(x)u_{tx})_x - (k(x)u_x)_x = 0 \quad \text{in } \langle 0, \infty \rangle \times \langle a, b \rangle\,,$$

ρ, γ, d, k non-negative and "smooth enough" and $u : \langle 0, \infty \rangle \times \langle a, b \rangle \longrightarrow \mathbf{C}$.
Additionally

- $\rho, \gamma \in \mathrm{L}^{\infty}(\langle a, b \rangle)$
- $k(x) > k_0 > 0$
- $\sup_{x \in \langle a,b \rangle} \frac{d(x)}{k(x)} < \infty$ and $d(b) = 0$

Generalised damped wave equation in 1D

In [K. Veselić, (2006)] this problem has been observed:

$$\rho(x)u_{tt} + \gamma(x)u_t - (d(x)u_{tx})_x - (k(x)u_x)_x = 0 \quad \text{in } \langle 0,\infty\rangle \times \langle a,b\rangle \,,$$

ρ, γ, d, k non-negative and "smooth enough" and $u : \langle 0,\infty\rangle \times \langle a,b\rangle \longrightarrow \mathbf{C}$.
Additionally

- $\rho, \gamma \in \mathrm{L}^{\infty}(\langle a,b\rangle)$
- $k(x) > k_0 > 0$
- $\sup_{x\in\langle a,b\rangle} \frac{d(x)}{k(x)} < \infty$ and $d(b) = 0$

Boundary conditions: $u(t,a) = 0$, $u_x(t,b) + \zeta u_t(t,b) = 0$ $(\zeta \geqslant 0)$

In [K. Veselić, (2006)] this problem has been observed:

$$\rho(x)u_{tt} + \gamma(x)u_t - (d(x)u_{tx})_x - (k(x)u_x)_x = 0 \quad \text{in } \langle 0, \infty\rangle \times \langle a, b\rangle\,,$$

ρ, γ, d, k non-negative and "smooth enough" and $u : \langle 0, \infty\rangle \times \langle a, b\rangle \longrightarrow \mathbf{C}$.
Additionally

- $\rho, \gamma \in \mathrm{L}^{\infty}(\langle a, b\rangle)$
- $k(x) > k_0 > 0$
- $\sup_{x\in\langle a,b\rangle} \frac{d(x)}{k(x)} < \infty$ and $d(b) = 0$

Boundary conditions: $u(t, a) = 0$, $u_x(t, b) + \zeta u_t(t, b) = 0$ $(\zeta \geqslant 0)$

Assume $u \in Y_0 := \{w \in \mathrm{C}^2(\langle a, b\rangle) \cap \mathrm{C}([a, b]) \;:\; w(a) = 0\}$. After multiplying equation by $v \in Y_0$, and using partial integration we get

$$\mu(u_{tt}, v) + \theta(u_t, v) + \kappa(u, v) = 0\,,$$

where

$$\mu(u, v) = \int_a^b \rho u\bar{v}dx\,,$$

$$\theta(u, v) = \int_a^b (\gamma u\bar{v} + du_x\bar{v}_x)dx + \zeta k(b)u(b)\bar{v}(b)\,,$$

$$\kappa(u, v) = \int_a^b ku_x\bar{v}_x dx\,.$$

$$\mu(u,v) = \int_a^b \rho u\bar{v}dx\,,$$

$$\theta(u,v) = \int_a^b (\gamma u\bar{v} + du_x\bar{v}_x)dx + \zeta k(b)u(b)\bar{v}(b)\,,$$

$$\kappa(u,v) = \int_a^b k u_x\bar{v}_x dx\,.$$

$$\mu(u,v) = \int_a^b \rho u\bar{v}dx\,,$$

$$\theta(u,v) = \int_a^b (\gamma u\bar{v} + du_x\bar{v}_x)dx + \zeta k(b)u(b)\bar{v}(b)\,,$$

$$\kappa(u,v) = \int_a^b k u_x\bar{v}_x dx\,.$$

- μ, θ and κ are symmetric
- μ, θ are positive and κ is strictly positive
- μ, θ are κ-bounded

$$\mu(u,v) = \int_a^b \rho u\bar{v}dx\,,$$

$$\theta(u,v) = \int_a^b (\gamma u\bar{v} + du_x\bar{v}_x)dx + \zeta k(b)u(b)\bar{v}(b)\,,$$

$$\kappa(u,v) = \int_a^b ku_x\bar{v}_x dx\,.$$

- μ, θ and κ are symmetric
- μ, θ are positive and κ is strictly positive
- μ, θ are κ-bounded

$\langle\, u \mid v\,\rangle_\kappa := \kappa(u,v)$ is a scalar product on Y_0, and $\|u\|_\kappa := \sqrt{\kappa(u,u)}$ is a norm.
$(Y, \langle\cdot \mid \cdot\rangle_\kappa)$ completion of Y_0 ... Hilbert space

$$\mu(u,v) = \int_a^b \rho u\bar{v} dx\,,$$
$$\theta(u,v) = \int_a^b (\gamma u\bar{v} + du_x\bar{v}_x)dx + \zeta k(b)u(b)\bar{v}(b)\,,$$
$$\kappa(u,v) = \int_a^b k u_x \bar{v}_x dx\,.$$

- μ, θ and κ are symmetric
- μ, θ are positive and κ is strictly positive
- μ, θ are κ-bounded

$\langle\, u \mid v \,\rangle_\kappa := \kappa(u,v)$ is a scalar product on Y_0, and $\|u\|_\kappa := \sqrt{\kappa(u,u)}$ is a norm.
$(Y, \langle\,\cdot \mid \cdot\,\rangle_\kappa)$ completion of Y_0 ... Hilbert space
Let us extend μ, θ to Y and denote by M, C bounded, selfadjoint and positive operators on Y such that

$$\mu(u,v) = \langle\, Mu \mid v \,\rangle_\kappa\,, \quad \theta(u,v) := \langle\, Cu \mid v \,\rangle_\kappa$$

$$\mu(u,v) = \int_a^b \rho u \bar{v} dx\,,$$

$$\theta(u,v) = \int_a^b (\gamma u \bar{v} + d u_x \bar{v}_x) dx + \zeta k(b) u(b) \bar{v}(b)\,,$$

$$\kappa(u,v) = \int_a^b k u_x \bar{v}_x dx\,.$$

- μ, θ and κ are symmetric
- μ, θ are positive and κ is strictly positive
- μ, θ are κ-bounded

$\langle\, u \mid v \,\rangle_\kappa := \kappa(u,v)$ is a scalar product on Y_0, and $\|u\|_\kappa := \sqrt{\kappa(u,u)}$ is a norm.
$(Y, \langle \cdot \mid \cdot \rangle_\kappa)$ completion of Y_0 ... Hilbert space
Let us extend μ, θ to Y and denote by M, C bounded, selfadjoint and positive operators on Y such that

$$\mu(u,v) = \langle\, Mu \mid v \,\rangle_\kappa\,, \quad \theta(u,v) := \langle\, Cu \mid v \,\rangle_\kappa$$

Our variational formulation reads: find $u \in Y$ such that

$$(\forall\, v \in Y) \quad \langle\, M u_{tt} + C u_t + u \mid v \,\rangle_\kappa = 0\,.$$

$$\mu(u,v) = \int_a^b \rho u\bar{v}dx\,,$$

$$\theta(u,v) = \int_a^b (\gamma u\bar{v} + du_x\bar{v}_x)dx + \zeta k(b)u(b)\bar{v}(b)\,,$$

$$\kappa(u,v) = \int_a^b ku_x\bar{v}_x dx\,.$$

- μ, θ and κ are symmetric
- μ, θ are positive and κ is strictly positive
- μ, θ are κ-bounded

$\langle\, u \mid v \,\rangle_\kappa := \kappa(u,v)$ is a scalar product on Y_0, and $\|u\|_\kappa := \sqrt{\kappa(u,u)}$ is a norm.
$(Y, \langle \cdot \mid \cdot \rangle_\kappa)$ completion of Y_0 ... Hilbert space
Let us extend μ, θ to Y and denote by M, C bounded, selfadjoint and positive operators on Y such that

$$\mu(u,v) = \langle\, Mu \mid v \,\rangle_\kappa\,, \quad \theta(u,v) := \langle\, Cu \mid v \,\rangle_\kappa$$

Our variational formulation reads: find $u \in Y$ such that

$$(\forall\, v \in Y) \quad \langle\, Mu_{tt} + Cu_t + u \mid v \,\rangle_\kappa = 0\,.$$

$$\iff \quad Mu_{tt} + Cu_t + u = 0$$

Abstract setting

$\mathsf{u} : [0, \infty\rangle \longrightarrow Y$, $\mathsf{u}(t) := u(t, \cdot)$
$'$ is derivative in Hilbert space Y

Abstract setting

$\mathsf{u} : [0, \infty\rangle \longrightarrow Y$, $\mathsf{u}(t) := u(t, \cdot)$
$'$ is derivative in Hilbert space Y

$$M\mathsf{u}'' + C\mathsf{u}' + \mathsf{u} = 0$$

Abstract setting

$\mathsf{u} : [0, \infty\rangle \longrightarrow Y$, $\mathsf{u}(t) := u(t, \cdot)$
$'$ is derivative in Hilbert space Y

$$M^{\frac{1}{2}}(M^{\frac{1}{2}}\mathsf{u}')' + C\mathsf{u}' + \mathsf{u} = 0$$

Abstract setting

$\mathsf{u} : [0, \infty\rangle \longrightarrow Y$, $\mathsf{u}(t) := u(t, \cdot)$
$'$ is derivative in Hilbert space Y

$$M^{\frac{1}{2}}(M^{\frac{1}{2}}\mathsf{u}')' + C\mathsf{u}' + \mathsf{u} = 0$$
$$\mathsf{u}(0) = \mathsf{u}_0$$
$$M^{\frac{1}{2}}\mathsf{u}'(0) = \mathsf{u}_1$$

Abstract setting

$\mathsf{u} : [0, \infty\rangle \longrightarrow Y$, $\mathsf{u}(t) := u(t, \cdot)$
$'$ is derivative in Hilbert space Y

$$
\begin{aligned}
M^{\frac{1}{2}}(M^{\frac{1}{2}}\mathsf{u}')' + C\mathsf{u}' + \mathsf{u} = 0 \\
\mathsf{u}(0) = \mathsf{u}_0 \\
M^{\frac{1}{2}}\mathsf{u}'(0) = \mathsf{u}_1
\end{aligned}
$$

$\mathsf{y}_1 := \mathsf{u}$, $\mathsf{y}_2 := M^{\frac{1}{2}}\mathsf{u}'$, $\mathsf{y} := [\mathsf{y}_1 \mathsf{y}_2]^\top$, $\mathsf{y}_0 := [\mathsf{u}_0 \mathsf{u}_1]^\top$

Abstract setting

$\mathsf{u} : [0,\infty\rangle \longrightarrow Y$, $\mathsf{u}(t) := u(t,\cdot)$
$'$ is derivative in Hilbert space Y

$$\begin{aligned} M^{\frac{1}{2}}(M^{\frac{1}{2}}\mathsf{u}')' + C\mathsf{u}' + \mathsf{u} &= 0 \\ \mathsf{u}(0) &= \mathsf{u}_0 \\ M^{\frac{1}{2}}\mathsf{u}'(0) &= \mathsf{u}_1 \end{aligned}$$

$\mathsf{y}_1 := \mathsf{u}$, $\mathsf{y}_2 := M^{\frac{1}{2}}\mathsf{u}'$, $\mathsf{y} := [\mathsf{y}_1\mathsf{y}_2]^\top$, $\mathsf{y}_0 := [\mathsf{u}_0\mathsf{u}_1]^\top$

$$\begin{cases} \mathcal{A}_+\mathsf{y}' = \mathsf{y} \\ \mathsf{y}(0) = \mathsf{y}_0 \end{cases},$$

where

$$\mathcal{A}_+ := \begin{bmatrix} -C & -M^{\frac{1}{2}} \\ M^{\frac{1}{2}} & 0 \end{bmatrix}.$$

Abstract setting

$\mathsf{u} : [0, \infty\rangle \longrightarrow Y$, $\mathsf{u}(t) := u(t, \cdot)$
$'$ is derivative in Hilbert space Y

$$\begin{aligned} M^{\frac{1}{2}}(M^{\frac{1}{2}}\mathsf{u}')' + C\mathsf{u}' + \mathsf{u} = 0 \\ \mathsf{u}(0) = \mathsf{u}_0 \\ M^{\frac{1}{2}}\mathsf{u}'(0) = \mathsf{u}_1 \end{aligned}$$

$\mathsf{y}_1 := \mathsf{u}$, $\mathsf{y}_2 := M^{\frac{1}{2}}\mathsf{u}'$, $\mathsf{y} := [\mathsf{y}_1 \mathsf{y}_2]^\top$, $\mathsf{y}_0 := [\mathsf{u}_0 \mathsf{u}_1]^\top$

$$\begin{cases} \mathcal{A}_+ \mathsf{y}' = \mathsf{y} \\ \mathsf{y}(0) = \mathsf{y}_0 \end{cases},$$

where

$$\mathcal{A}_+ := \begin{bmatrix} -C & -M^{\frac{1}{2}} \\ M^{\frac{1}{2}} & 0 \end{bmatrix}.$$

Problem: $\mathcal{A}_+^{-1}$ does not exist in general

$$\mathcal{A}_+ := \begin{bmatrix} -C & -M^{\frac{1}{2}} \\ M^{\frac{1}{2}} & 0 \end{bmatrix}.$$

- $\mathcal{D}(\mathcal{A}_+) = \mathcal{D}(\mathcal{A}_+^*) = Y \oplus Y$
- $N(\mathcal{A}_+) = N(\mathcal{A}_+^*) = (N(C) \cap N(M)) \oplus N(M)$
- $\mathcal{A}_+$ is maximal dissipative

$$\mathcal{A}_+ := \begin{bmatrix} -C & -M^{\frac{1}{2}} \\ M^{\frac{1}{2}} & 0 \end{bmatrix}.$$

- $\mathcal{D}(\mathcal{A}_+) = \mathcal{D}(\mathcal{A}_+^*) = Y \oplus Y$
- $N(\mathcal{A}_+) = N(\mathcal{A}_+^*) = (N(C) \cap N(M)) \oplus N(M)$
- $\mathcal{A}_+$ is maximal dissipative

Hence, for $X := (N(C) \cap N(M))^\perp \oplus N(M)^\perp$

$$\mathcal{A}_{+|_X} : X \longrightarrow R(\mathcal{A}_+)$$

is maximal dissipative and invertible which implies that

$$\mathcal{A} := (\mathcal{A}_{+|_X})^{-1} : R(\mathcal{A}_+) \subseteq X \longrightarrow X$$

is maximal dissipative, therefore generates a C_0-semigroup of contractions.

Final result

$$\begin{cases} \mathcal{A}_{+}\mathsf{y}' = \mathsf{y} \\ \mathsf{y}(0) = \mathsf{y}_0 \end{cases}$$

Final result

$$\begin{cases} \mathrm{y}' = \mathcal{A}\mathrm{y} \\ \mathrm{y}(0) = \mathrm{y}_0 \end{cases}$$

Final result

$$\begin{cases} \mathsf{y}' = \mathcal{A}\mathsf{y} \\ \mathsf{y}(0) = \mathsf{y}_0 \end{cases}$$

If $\mathsf{y}_0 \in R(\mathcal{A}_+)$ there exists the unique classical solution
$\mathsf{y} \in \mathrm{C}([0,\infty\rangle; X) \cap \mathrm{C}(\langle 0,\infty\rangle; R(\mathcal{A}_+)) \cap \mathrm{C}^2(\langle 0,\infty\rangle; X)$ of the system above.

Final result

$$\begin{cases} \mathsf{y}' = \mathcal{A}\mathsf{y} \\ \mathsf{y}(0) = \mathsf{y}_0 \end{cases}$$

If $\mathsf{y}_0 \in R(\mathcal{A}_+)$ there exists the unique classical solution $\mathsf{y} \in \mathrm{C}([0,\infty\rangle; X) \cap \mathrm{C}(\langle 0,\infty\rangle; R(\mathcal{A}_+)) \cap \mathrm{C}^2(\langle 0,\infty\rangle; X)$ of the system above.

Finally, $\mathsf{y}_1 \in \mathrm{C}([0,\infty\rangle; (N(C)\cap N(M))^\perp) \cap \mathrm{C}(\langle 0,\infty\rangle; R(C) + R(M^{\frac{1}{2}})) \cap \mathrm{C}^1(\langle 0,\infty\rangle; (N(C)\cap N(M))^\perp)$ satisfies

i) $M^{\frac{1}{2}}\mathsf{y}_1' \in \mathrm{C}([0,\infty\rangle; N(M)^\perp) \cap \mathrm{C}(\langle 0,\infty\rangle; R(M^{\frac{1}{2}})) \cap \mathrm{C}^1(\langle 0,\infty\rangle; N(M)^\perp)$

ii)

$$\begin{cases} M^{\frac{1}{2}}(M^{\frac{1}{2}}\mathsf{y}_1')' + C\mathsf{y}_1' + \mathsf{y}_1 = 0 \\ \mathsf{y}_1(0) = \mathsf{u}_0 \\ M^{\frac{1}{2}}\mathsf{y}_1'(0) = \mathsf{u}_1 \end{cases}$$

Good luck DEUTSCHLAND

FIFA WORLD CUP
Brasil