On extensions of bilinear functionals

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July, 2014

This work has been supported in part by Croatian Science Foundation under the project 9780.

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Standard situation

Assume that we have a bounded bilinear mapping

$$B : C(X) \times C(Y) \to \mathbf{R},$$

where X and Y are open subsets of Euclidean spaces. The question is whether we can extend it continuously to a measure $\mu : C(X \times Y) \to \mathbf{R}$, i.e. does it exist the continuous functional μ such that for any $f \in C(X)$ and $g \in C(Y)$ it holds

$$B(f, g) = \langle \mu, f \otimes g \rangle$$

The answer is NO in general situation. Actually, according to the Schwartz kernel theorem, we can merely find a distribution $\mu \in \mathcal{D}'(X \times Y)$ such that

$$B(f, g) = \langle \mu, f \otimes g \rangle, \quad f \in C(X), g \in C(Y).$$

Example

Consider the following mapping

$$B : (BV \cap C)(-1,1) \times (BV \cap C)(-1,1) \to \mathbf{R}$$

defined by

$$(f, g) \mapsto \int_{-1}^{1} \left(\text{p.v.} \int_{-1}^{1} \frac{f(x)g(y)}{x-y} dx \right) dy. \tag{1}$$

It is a bounded bilinear functional.

However, if we replace $f(x)g(y)$ by $\mathrm{sgn}_\sigma(x-y)$, where sgn_σ is a regularization of the sign function, we see that

$$\int_{-1}^{1}\left(\mathrm{p.v.}\int_{-1}^{1}\frac{\mathrm{sgn}_\sigma(x-y)}{x-y}dx\right)dy \to \infty$$

as $\sigma \to 0$ implying that

$$C(X\times Y)\cap BV(X\times Y)\ni\varphi\mapsto\int_{0}^{1}\left(\mathrm{p.v.}\int_{0}^{1}\frac{\varphi(x,y)}{x-y}dx\right)dy.$$

is not bounded.

Main result

Let B be a bilinear form on $L^p(\mathbf{R}^d) \otimes E$, where E is a separable Banach space and $p \in \langle 1, \infty\rangle$. Then, it can be extended as a continuous functional on $L^p(\mathbf{R}^d; E)$ if and only if there exists a nonnegative function $b \in L^{p'}(\mathbf{R}^d)$ such that for every $\psi \in E$ and almost every $(\mathbf{x})$, it holds

$$|\tilde{B}\psi(\mathbf{x})| \leq b(\mathbf{x})\|\psi\|_E. \tag{2}$$

where $\tilde{B}$ is a linear bounded operator $E \to L^{p'}(\mathbf{R}^d)$ defined by $\langle \tilde{B}\psi, \phi\rangle = B(\phi, \psi)$.

Proof

Let us assume that (2) holds. In order to prove that B can be extended as a linear functional on $\mathrm{L}^p(\mathbf{R}^d; E)$, it is enough to obtain an appropriate bound on the following dense subspace of $\mathrm{L}^p(\mathbf{R}^d; E)$:

$$\left\{ \sum_{j=1}^{N} \psi_j \chi_j(\mathbf{x}) : \ \psi_j \in E, N \in \mathbf{N} \right\}, \tag{3}$$

where χ_i are characteristic functions associated to mutually disjoint, finite measure sets.

For an arbitrary function $g = \sum_{i=1}^{N} \psi_i \chi_i$ from (3), the bound follows easily once we notice that

$$\left|B\Big(\sum_{j=1}^{N} \psi_j \chi_j\Big)\right| := \Big|\sum_{j=1}^{N} B(\chi_j, \psi_j)\Big| = \Big|\int_{\mathbf{R}^d} \sum_{j=1}^{N} \tilde{B}\psi(\mathbf{x})\chi_i(\mathbf{x}) d\mathbf{x}\Big|$$

$$\leq \int_{\mathbf{R}^d} b(\mathbf{x}) \sum_{j=1}^{N} \chi_j(\mathbf{x}) \|\psi_j\|_E d\mathbf{x} \leq \|b\|_{L^{p'}(\mathbf{R}^d)} \|g\|_{L^p(\mathbf{R}^d;E)}.$$

Converse

In order to prove the converse, take a countable dense set of functions from the unit ball of E, and denote them by ψ_j, $j \in \mathbf{N}$. Assume that the functions $\psi_{-j} := -\psi_j$ are also in E. For each function $\tilde{B}\psi_j \in \mathrm{L}^{p'}(\mathbf{R}^d)$ denote by D_j the corresponding set of Lebesgue points, and their intersection by $D = \cap_j D_j$.

For any $\mathbf{x} \in D$ and $k \in \mathbf{N}$ denote

$$b_k(\mathbf{x}) = \max_{|j| \leq k} \Re(\tilde{B}\psi_j)(\mathbf{x}) = \sum_{j=1}^{k} \Re(\tilde{B}\psi_j)(\mathbf{x})\chi_j^k(\mathbf{x})$$

where $\chi_{j_0}^k$ is the characteristic function of set $X_{j_0}^k$ of all points $\mathbf{x} \in D$ for which the above maximum is achieved for $j = j_0$. Furthermore, we can assume that for each k the sets X_j^k are mutually disjoint.

The sequence (b_k) is clearly monotonic sequence of positive functions, bounded in $\mathrm{L}^{p'}(\mathbf{R}^d)$, whose limit (in the same space) we denote by $b^{\Re}$. Indeed, choose $\varphi \in L^p(\mathbf{R}^d)$, $g = \sum_{j=1}^{k} \varphi(\mathbf{x})\chi_j^k(\mathbf{x})\psi_j \in L^p(\mathbf{R}^d; E)$, and consider:

$$\int_{\mathbf{R}^d} b_k(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \Re\Big(\int_{\mathbf{R}^d} \tilde{B} \sum_{j=1}^{k} \psi_j \chi_j^k(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\Big)$$

$$= \Re\Big(\sum_{j=1}^{k} B(\chi_j^k \varphi, \psi_j)\Big) = \Re\big(B(g)\big) \leq C\|g\|_{L^p(\mathbf{R}^d;E)} = C\|\varphi\|_{L^p(\mathbf{R}^d)},$$

where C is the norm of B on $(L^p(\mathbf{R}^d; E))'$. Since $\varphi \in L^p(\mathbf{R}^d)$ is arbitrary, we get that (b_k) is bounded in $L^{p'}(\mathbf{R}^d)$.

As D is a set of full measure, for every ψ_j we have

$$|\Re(\tilde{B}\psi_j)(\mathbf{x})| \leq b^{\Re}(\mathbf{x}), \qquad (\text{a.e. } \mathbf{x} \in \mathbf{R}^d).$$

We are able to obtain a similar bound for the imaginary part of $\tilde{B}\psi_j$. In other words, there exists $b^{\Im} \in \mathrm{L}^{p'}(\mathbf{R}^d)$ such that

$$|\Im(\tilde{B}\psi_j)(\mathbf{x})| \leq b^{\Im}(\mathbf{x}), \qquad (\text{a.e. } \mathbf{x} \in \mathbf{R}^d).$$

The assertion now follows since (2) holds for $b = b^{\Re} + b^{\Im}$ on the dense set of functions ψ_j, $j \in \mathbf{N}$.

The End

Thank you for listening.