Generalised solutions to linear and non-linear Schrödinger-type equations

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Introduction

We consider two time-dependent Schrödinger-type equations in three spatial dimensions, that is,

$$i\partial_t u \;=\; -\Delta u + (w * |u|^2)u \tag{1}$$

and

$$i\partial_t v \;=\; -\Delta_\alpha v + (w * |v|^2)v\,, \tag{2}$$

for given $\alpha \in \mathbb{R}\cup\{\infty\}$ and given real-valued measurable potential w, in the unknowns $u \equiv u(t,x)$ and $v \equiv v(t,x)$, with $t\in\mathbb{R}$ and $x \in \mathbb{R}^3$, where the convolution in the cubic Hartree non-linearity is with respect to the x-variable.
The linear action $u \mapsto \Delta_\alpha u$ in (2) refers to the singular perturbation of the Laplacian with a point interaction supported at the origin. By this one means the self-adjoint extension, with respect to $L^2(\mathbb{R}^3)$, of the densely defined symmetric operator $(-\Delta)\upharpoonright C^\infty_c(\mathbb{R}^3\setminus\{0\})$.
Next, as approximants to (1) and (2), we consider, for each $\varepsilon\in(0,1]$, the time-dependent Schrödinger equation

$$i\partial_t u_\varepsilon \;=\; -\Delta u_\varepsilon + V_\varepsilon u_\varepsilon + (w*|u_\varepsilon|^2)u_\varepsilon\,, \tag{3}$$

in the unknown $u_\varepsilon \equiv u_\varepsilon(t,x)$, with $(t,x)\in\mathbb{R}\times\mathbb{R}^3$, where V_ε is a real-valued potential effectively supported around the origin at a spatial scale ε, and is meant to represent a singular, delta-like profile of some magnitude and centered at $x=0$.
The general law by which V_ε is assumed to shrink around the origin is

$$V_\varepsilon(x) \;:=\; \frac{\eta(\varepsilon)}{\varepsilon^\sigma}V\Big(\frac{x}{\varepsilon}\Big)\,, \tag{4}$$

for a given measurable function $V:\mathbb{R}^3\to\mathbb{R}$, a given smooth function $\eta:[0,1]\to[0,+\infty)$ such that $\eta(0)=\eta(1)=1$, and a given $\sigma\geqslant 0$.

Background

The scaling law (4) covers distinguished and fundamentally different regimes. A meaningful one is with $\sigma=3$, in which case $V_\varepsilon \to (\int_{\mathbb{R}^3}V)\delta(x)$ in $\mathcal{D}'(\mathbb{R}^3)$ as $\varepsilon\downarrow 0$, provided that $\int_{\mathbb{R}^3}V$ is (non-zero, and) finite.
In particular the case where $V\in C^\infty_c(\mathbb{R}^3)$ or $V\in\mathcal{S}(\mathbb{R}^3)$, $V\geqslant 0$ (and not identically zero), $\eta\equiv 1$, and $\sigma=3$ is the standard '*Colombeau regime*', the net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ then representing a generalised solution to the formal equation

$$i\partial_t u \;=\; -\Delta u + a_V\delta u + (w*|u|^2)u\,, \tag{5}$$

where $a_V := \int_{\mathbb{R}^3}V(x)\,dx$.
The following well-posedness result holds under standard working assumptions on the potentials V and w.

Theorem 1 *When*

$$w \;\in\; L^\infty(\mathbb{R}^3,\mathbb{R})\cap W^{1,3}(\mathbb{R}^3,\mathbb{R})\,,\qquad w \text{ is an even function,} \tag{6}$$

and

$$V \in L^p(\mathbb{R}^3,\mathbb{R}) + L^\infty(\mathbb{R}^3,\mathbb{R})\,,\qquad p>\frac{3}{2}\,, \tag{7}$$

all three equations (1)-(3) are globally well-posed in $L^2(\mathbb{R}^3)$. In particular, for given initial datum in $L^2(\mathbb{R}^3)$ at $t=0$ they all admit a unique strong L^2-solution preserving the L^2-norm at all times.

Here $W^{s,p}$ is the standard Sobolev space notation.
Next we denote by $\mathcal{R}$ the Rollnik class over $\mathbb{R}^3$, consisting of the functions f for which the Rollnik norm

$$\|f\|_{\mathcal{R}} \;:=\; \Big(\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\overline{f(x)}\,f(y)}{|x-y|^2}\,dx\,dy\Big)^{\frac{1}{2}} \tag{8}$$

is finite.

Main result, Colombeau algebras

For potentials $V\in\mathcal{R}+L^\infty(\mathbb{R}^3,\mathbb{R})$ the Schrödinger operator $-\Delta+V$ is self-adjointly realised in $L^2(\mathbb{R}^3)$ as a form sum with quadratic form $H^1(\mathbb{R}^3)$.
When

$$V \text{ is real-valued}\,,\qquad V\in\mathcal{R}\,, \tag{9}$$

the Birman-Schwinger operator $u(-\Delta)^{-1}v$, where $v(x) := \sqrt{|V(x)|}$ and $u(x) := \sqrt{|V(x)|}\,\mathrm{sign}(V(x))$, is compact in $L^2(\mathbb{R}^3)$ (it is actually a Hilbert-Schmidt operator).

Theorem 2 *The following are given: $\mathsf{T}>0$, w satisfying (6), V satisfying (9) and $V\in L^1(\mathbb{R}^3,\mathbb{R})$, and additionally (7) when $w\not\equiv \mathbf{0}$, $\eta:[0,1]\to[0,+\infty)$ such that it is smooth and $\eta(0)=\eta(1)=1$, and $a\in L^2(\mathbb{R}^3)$. For fixed $\sigma>0$ and $\varepsilon\in(0,1]$, define V_ε as in (4). Correspondingly, let u_ε be the unique solution in $C(\mathbb{R},L^2(\mathbb{R}^3))$ to the Cauchy problem associated with (3) with initial datum a.*
Let $\sigma\in(2,3]$ assume that V, if not identically zero, is non-negative. Then,

$$\lim_{\varepsilon\downarrow 0}\|u_\varepsilon - u\|_{L^\infty([0,\mathsf{T}],L^2(\mathbb{R}^3))} \;=\; 0\,, \tag{10}$$

where u the unique solution in $C(\mathbb{R},L^2(\mathbb{R}^3))$ to the Cauchy problem associated with (1).

The end-point regime $\sigma=3$ is relevant in the framework of the Colombeau generalised solution theory for non-linear Schrödinger equations.
We recall the main properties of the Colombeau algebra of generalised functions. The Colombeau algebra is an associative differential algebra, it displays properties that overcome the problem of multiplication of distributions, thus making it well suited as a solution framework for non-linear partial differential equations.
We present here in particular the H^2-based Colombeau algebra.
Given $T>0$, one denotes by

$$\mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^3)\,,\qquad \text{respectively,}\qquad \mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$$

the vector space of nets $(u_\varepsilon)_{\varepsilon\in(0,1]}$ (with natural component-wise linear structure), of functions

$$u_\varepsilon \;\in\; C([0,T),H^2(\mathbb{R}^3))\cap C^1([0,T),L^2(\mathbb{R}^3))$$

satisfying the following growth properties as $\varepsilon\downarrow 0$: for a $\mathcal{E}_{C^1,H^2}$-net there is $N\in\mathbb{N}$ such that

$$\max\Big\{\sup_{t\in[0,T)}\|u_\varepsilon(t)\|_{H^2}\,,\ \sup_{t\in[0,T)}\|\partial_t u_\varepsilon(t)\|_2\Big\} \overset{\varepsilon\downarrow 0}{=} O(\varepsilon^{-N})\,, \tag{11}$$

and for a $\mathcal{N}_{C^1,H^2}$-net the following holds true $\forall M\in\mathbb{N}$:

$$\max\Big\{\sup_{t\in[0,T)}\|u_\varepsilon(t)\|_{H^2}\,,\ \sup_{t\in[0,T)}\|\partial_t u_\varepsilon(t)\|_2\Big\} \overset{\varepsilon\downarrow 0}{=} O(\varepsilon^{M})\,. \tag{12}$$

Elements of $\mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$ and of $\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$ are called, respectively, *moderate nets* and *negligible nets*. $\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$ is a (multiplicative) bilateral ideal in $\mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$, and this allows to introduce the quotient space

$$\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^3) \;:=\; \mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^3)\,/\,\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^3) \tag{13}$$

which is a Colombeau-type vector space.
Each $H^2(\mathbb{R}^d)$ with $d\in\{1,2,3\}$ is a multiplicative algebra, so also is $\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$. The Sobolev spaces $H^s(\mathbb{R}^3)$, for $s\in\mathbb{R}$, as well as the space $\mathcal{E}'(\mathbb{R}^3)$ of distributions with compact support, are canonically identi-

Compatibility of solutions

Here

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^3}\rho\Big(\frac{x}{\varepsilon}\Big)\quad\text{for some}\quad \rho\in\mathcal{S}(\mathbb{R}^3)\quad\text{such that}\quad \rho>0\quad \int_{I\!R^3}\rho(x)dx = 1. \tag{14}$$

Next, let us review the notion of *generalised Colombeau solution* to the Cauchy problem

$$\begin{cases} i\partial_t u \;=\; -\Delta_x u + \delta\, u + (w*|u|^2)u\,,\\ u(0,\cdot) \;=\; a\in H^2(\mathbb{R}^3).\end{cases} \tag{15}$$

One says that (15) admits a (local in time) solution $u\in\mathcal{G}_{C^1,H^2}([0,T)\times I\!R^3)$, for some $T>0$, if there are three nets: $(u_\varepsilon)_{\varepsilon\in(0,1]}\in\mathcal{E}_{C^1,H^2}([0,T)\times I\!R^3)$, $(n_\varepsilon)_{\varepsilon\in(0,1]}\in\mathcal{N}_{H^2}(I\!R^3)$ and $(M_\varepsilon)_{\varepsilon\in(0,1]}\in C([0,T),L^2(\mathbb{R}^3))$ with $\|M_\varepsilon\|_{L^\infty([0,T),L^2(\mathbb{R}^3))}\overset{\varepsilon\downarrow 0}{=} O(\varepsilon^M)$, $\forall M\in\mathbb{N}$
satisfying

$$\begin{cases} i\partial_t u_\varepsilon - \big(-\Delta_x u_\varepsilon + \delta_\varepsilon\, u_\varepsilon + w*|u_\varepsilon|^2u_\varepsilon\big) \;=\; M_\varepsilon\,,\\ u(0,\cdot) \;=\; a_\varepsilon + n_\varepsilon\,,\end{cases} \tag{16}$$

where $a_\varepsilon := a*\rho_\varepsilon$ $\delta_\varepsilon := \delta*\rho_\varepsilon$, and ρ_ε is given by (14).
Notice that the classical Cauchy problem

$$\begin{cases} i\partial_t u \;=\; -\Delta_x u + (w*|u|^2)u\,,\\ u(0,\cdot) \;=\; a\in H^2(\mathbb{R}^3)\end{cases} \tag{17}$$

is globally well posed in $H^2(\mathbb{R}^3)$, say, with solution $u_{\mathrm{cl}}\in C(\mathbb{R},H^2(\mathbb{R}^3))$.
One says that the Colombeau generalised solution $u_{\mathrm{Col}} = [(u_\varepsilon)_{\varepsilon\in(0,1]}]\in\mathcal{G}_{C^1,H^2}([0,T)\times I\!R^3)$ to the Cauchy problem (15), thus with the u_ε's satisfying (16), is *compatible* with the classical solution $u_{\mathrm{cl}}\in C([0,T),H^2(\mathbb{R}^3))$ to the Cauchy problem (17), precisely when

$$\lim_{\varepsilon\downarrow 0}\|u_{\mathrm{cl}} - u_\varepsilon\|_{L^\infty([0,T),L^2(\mathbb{R}^3))} \;=\; 0\,. \tag{18}$$

Theorem 3 *Let $a\in L^2(\mathbb{R}^3)$ and let w be a function satisfying (6). The Colombeau generalised solution to the Cauchy problem for the singular non-linear Schrödinger equation*

$$i\partial_t u \;=\; -\Delta u + \delta u + (w*|u|^2)u \tag{19}$$

with initial datum a is compatible with the H^2-solution to the Cauchy problem for the classical non-linear Schrödinger equation

$$i\partial_t u \;=\; -\Delta u + (w*|u|^2)u \tag{20}$$

with the same initial datum.

Remark 4 *In [1] existence and uniqueness of a H^2-based Colombeau solution of* (19) *was established with different conditions on the convolution potential w (for even-symmetric $w\in W^{2,p}(\mathbb{R}^3,\mathbb{R})$, $p\in(2,\infty]$). Compatibility was also examined, but for odd-symmetric $a\in H^2(\mathbb{R}^3)$. Besides, the embedding of the δ-coefficient appearing in* (19) *was performed by means of* compactly supported *mollifiers, whereas our Theorem 3 requires the mollifiers to be* strictly positive *almost everywhere.*

References

1. Dugandžija N., Vojnović I., *Singular solution of the Hartree equation with a delta potential*, submitted for