Degenerate nonlocal Fokker–Planck equations

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Motivation: fully nonlinear mean field game

$$\begin{cases} -\partial_t u = F(\mathcal{L}u) + \mathfrak{f}(m) & \text{on } [0,T]\times\mathbb{R}^d, \\ u(T) = \mathfrak{g}\big(m(T)\big) & \text{on } \mathbb{R}^d, \\ \partial_t m = \mathcal{L}^*\big(F'(\mathcal{L}u)\, m\big) & \text{on } [0,T]\times\mathbb{R}^d, \\ m(0) = m_0 & \text{on } \mathbb{R}^d. \end{cases}$$

- Players control **the time rate** θ of a Lévy process ($\mathcal{L}$) – how fast they move along a (random) path of the process.
- The game's outcome is determined by the distribution m of the players. Their movement — towards maximizing their own individual chances u — changes that distribution, leading to more movement, *ad nauseam.*
- m is described by a Fokker–Planck equation (when u is fixed)
- We need at least existence and stability, and ideally also uniqueness for m with arbitrary fixed u to solve the game (u, m).

Lévy operators

- Lévy $\Leftrightarrow$ maximum principle
- Lévy–Khintchin–Courrège formula: $\mathcal{L} =$ local + nonlocal

$$\mathcal{L}\phi(x) = c \cdot \nabla\phi(x) + \operatorname{tr}\left(aa^T D^2\phi(x)\right) + \int_{\mathbb{R}^d} \Big(\phi(x+z) - \phi(x) - \mathbb{1}_{B_1}(z)\, z \cdot \nabla\phi(x)\Big)\, \nu(dz).$$

$$\int_{\mathbb{R}^d} \min(1, |x|^2)\, \nu(dy) < \infty, \quad \nu(\{0\}) = 0$$

- Examples:

local: $\quad \Delta, \quad \dfrac{d}{dx}, \quad c \cdot \nabla$

nonlocal: $\quad (-\Delta)^s u = \mathcal{F}^{-1}\big(|\xi|^{2s}\, \widehat{u}(\xi)\big), \ s \in (0,1)$

nonlocal, bounded: $\quad \dfrac{u(x+h) - u(x)}{h}, \quad \dfrac{u(x-h) + u(x+h) - 2u(x)}{h^2}$

- generator of a Lévy process (iid stationary increments) – a "diffusion".

Fokker–Planck–Kolmogorov

$$\begin{cases} \partial_t m = \mathcal{L}^*(bm) & \text{on } [0,T]\times\mathbb{R}^d, \\ m(0) = m_0 & \text{on } \mathbb{R}^d. \end{cases} \qquad \text{(FPK)}$$

$b = F'(\mathcal{L}u)$

- $b \in C\big([0,T]\times\mathbb{R}^d\big)$ and $b \geq 0$
- Natural space to look for solutions: $m \in C\big([0,T],\mathcal{P}(\mathbb{R}^d)\big)$:

$$m(t)[\phi(t)] = m_0[\phi(0)] + \int_0^t m(\tau)\big[\partial_t\phi(\tau) + b(\tau)(\mathcal{L}\phi)(\tau)\big]\,d\tau.$$

- Existence – set $\mathcal{M}$ of solutions is convex, compact and non-empty
- Stability – if $b_n \to b$ locally uniformly, then $\mathcal{M}_n \to \mathcal{M}$ as closed sets ("$K - \limsup$")
- Uniqueness – by the Holmgren method

- Even if m_0 is compactly supported, the action of an arbitrary Lévy measure ν may produce a solution – a probabilistic measure $m(t)$ – with unbounded (integer) moments. This is a purely nonlocal phenomenon.
- Recall Prokhorov: pre-compactness of probability measures $\Leftrightarrow$ tightness.
- Also: tightness of $\Pi \subset \mathcal{P}(\mathbb{R}^d) \Leftrightarrow \exists\, V : \mathbb{R}^d :\to [0,\infty)$ such that $\lim_{|x|\to\infty} V(x) = \infty$ and $\forall\, m \in \Pi\ m[V] < 1$.

Definition

A real function $V \in C^2(\mathbb{R}^d)$ is a Lyapunov function if $V(x) = V_0\big(\sqrt{1+|x|^2}\big)$ for some subadditive, non-decreasing function $V_0 : [0,\infty) \to [0,\infty)$ such that $\|V_0'\|_\infty, \|V_0''\|_\infty \leq 1$, and $\lim_{x\to\infty} V_0(x) = \infty$.

Lemma

A set $\Pi \subset \mathcal{P}(\mathbb{R}^d)$ is tight/pre-compact if and only if there exists a Lyapunov function V such that $m[V] \leq 1$ for every $m \in \Pi$.

Remark

We can also apply this lemma to tails $\nu \mathbb{1}_{B_1^c(0)}$ of Lévy measures (they're bounded).

Theorem

Assume $\mathcal{L}$ is a Lévy operator, $m_0 \in \mathcal{P}(\mathbb{R}^d)$ and $0 \leq b < \infty$ is continuous. The set $\mathcal{M}$ of solutions is convex, compact, and non-empty. Moreover, there are constants c_1, c_2 such that for every $m \in \mathcal{M}$,

$$\sup_{t\in[0,T]} m(t)[V] \leq c_1, \qquad \sup_{0<|t-s|\leq T} \frac{\|m(t)-m(s)\|_0}{\sqrt{|t-s|}} \leq c_2,$$

where V is a Lyapunov function such that $m_0[V], \|\mathcal{L}V\|_\infty < \infty$.

Proof.

- Step 1: Convexity by linearity. Bounds by Lyapunov. Compactness by AA.
- Step 2: Construct a sequence of approximations, where $\mathcal{L}^\varepsilon$ are bounded Lévy operators, uniform w.r.t. the Lyapunov function, and m_0^ε, b^ε are regularizations (use Banach FPT). These solution are in $L^1(\mathbb{R}^d)$
- Step 3: Show that the solutions are nonnegative and the mass is conserved.
- Step 4: Compactness of approximations in $\mathcal{P}(\mathbb{R}^d)$ by Lyapunov+AA. Each limit is a solution.

Lemma

Assume $\mathcal{L}$ is a Lévy operator, $m_0 \in \mathcal{P}(\mathbb{R}^d)$ and $\{b_n, b\}_{n\in\mathbb{N}}$ are non-negative, continuous and uniformly bounded. Let $\{\mathcal{M}_n, \mathcal{M}\}$ be the corresponding sets of solutions with m_0 as initial conditions. If $m_n \in \mathcal{M}_n$ for every $n \in \mathbb{N}$ and $b_n(t) \to b(t)$ uniformly on compact sets in $\mathbb{R}^d$ for every $t \in [0,\infty)$, then there exists a subsequence $\{m_{n_k}\}$ and $m \in \mathcal{M}$ such that $m_{n_k} \to m$ in $C\big([0,T], \mathcal{P}(\mathbb{R}^d)\big)$.

Remark

The long text above means: if $b_n \to b$ (locally uniformly in x) and all other things are equal, then $\mathcal{M}_n \to \mathcal{M}$ (in the sense of Kuratowski/Hausdorff).

Remark

In case $\mathcal{M}_n = \{m_n\}$ (i.e. uniqueness), we have $m_n \to m$ in $C\big([0,T], \mathcal{P}(\mathbb{R}^d)\big)$.

Uniqueness – Holmgren method

Prove existence for the dual problem (backward in time)

$$\begin{cases} \partial_t w = -b\,\mathcal{L}w, & \text{on } [0,\tau]\times\mathbb{R}^d \\ w(\tau) = \psi \in C_c^\infty(\mathbb{R}^d) & \text{on } \mathbb{R}^d \end{cases}$$

and then

$$(m_1 - m_2)[\psi] = (m_1 - m_2)[w(0)] + \int_0^t (m_1 - m_2)[\partial_t w + b\,\mathcal{L}w]\,ds = 0.$$

Uniqueness — comments

- We don't expect uniquness to always hold. Consider

$$\partial_t m(t,x) = \partial_x \Big(b(t,x)\, m(t,x) \Big) \qquad \text{in } [0,T] \times \mathbb{R}.$$

  If b is continuous, but not Lipschitz-continuous, we may not have uniqueness

- But if $b \geq \kappa > 0$ is uniformly continuous and Hölder in x for each t, then

$$\partial_t m(t,x) = \Delta \Big(b(t,x)\, m(t,x) \Big)$$

  has unique solutions [Lunardi + Holmgren method (next slide)]

- Also, if $b \geq \kappa > 0$ is continuous and Hölder in x for each t, then

$$\partial_t m(t,x) = -(-\Delta)^s \Big(b(t,x)\, m(t,x) \Big)$$

  has unique solutions (the operator can be slightly more general) [Mikulevičius and Pragarauskas + Holmgren]

- What can we say if either $b = 0$ somewhere or the operator is degenerate (in the sense of the lack of regularization properties of the semigroup)?

Uniqueness — degenerate case

Let $2\sigma \in (0,1)$ and for a constant $K \geq 0$ and every $p \in (2\sigma, 1]$, $r \in (0,1)$,

$$\mathcal{L}\phi(x) = \int_{\mathbb{R}^d} \left(\phi(x+z) - \phi(x)\right) \nu(dz), \qquad \int_{B_1} \left(1 \wedge \frac{|z|^p}{r^p}\right) \nu(dz) \leq \frac{K}{p-2\sigma} r^{-2\sigma}$$

- We employ visosity solutions techniques (even though, it is hardly a fully *nonlinear* problem).
- After long computations we obtain existence of sufficiently regular w.
- We hit a restriction: $b \in B([0,\infty), \mathcal{C}_b^{\beta}(\mathbb{R}^d))$ for $\beta > 2\sigma + \frac{2\sigma}{1-2\sigma}$; but that means $1 > 2\sigma + \frac{2\sigma}{1-2\sigma}$ and so $\sigma < \frac{3-\sqrt{5}}{4} \approx \frac{1}{5}$.
- If the measure is symmetric near $\{0\}$, then $\beta > 2\sigma + \frac{2\sigma}{1-\sigma}$, $\sigma < \frac{5-\sqrt{17}}{4} \approx \frac{2}{9}$.

Happy Birthday, Nenad!

Sto lat, sto lat
Niech żyje, żyje nam
Sto lat, sto lat
Niech żyje, żyje nam
Jeszcze raz, jeszcze raz
Niech żyje, żyje nam
Niech żyje nam

Niech mu gwiazdka pomyślności
Nigdy nie zagaśnie
Nigdy nie zagaśnie
A kto z nami nie wypije
Niech go piorun trzaśnie
A kto z nami nie wypije
Niech go piorun trzaśnie