# Entropy solutions of degenerate parabolic equations: existence and strong traces


Marko Erceg

Department of Mathematics, Faculty of Science, University of Zagreb


Applied Mathematics Seminar

UNC Greensboro, $21^{\text{st}}$ November 2022

Joint work with D. Mitrović


UIP-2017-05-7249 (MANDphy)

IP-2018-01-2449 (MiTPDE)

# Degenerate parabolic equation - introduction

$$\begin{aligned}\partial_t u(t,\mathbf{x}) + \mathsf{div}_{\mathbf{x}}\mathsf{f}(t,\mathbf{x},u(t,\mathbf{x})) &= D^2_{\mathbf{x}} \cdot A(u(t,\mathbf{x})) \\ &= \mathsf{div}_{\mathbf{x}}\big(A'(u(t,\mathbf{x}))\nabla_{\mathbf{x}} u(t,\mathbf{x})\big)\,,\end{aligned}$$

where $\mathsf{f} : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $A : \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ are given and $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is unknown.

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- LHS: convection effects ($\mathsf{f}$ flux, $\mathsf{f} = \mathsf{f}(u)$ homogeneous case);
- RHS: diffusion effects ($A'$ diffusion matrix – direction and intensity of the diffusion).

**Motivation for the equation:**

- flow in porous media (e.g. $\mathsf{f} = 0$ and $A(u) = u^m \mathrm{I}$ – porous media equation)
  - heterogeneous layers $\longrightarrow$ discontinuous flux and a lack of diffusion in some directions
- sedimentation-consolidation process

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- Existence (and uniqueness) of solution to the Cauchy problem;
- Existence of traces of solutions, i.e. give meaning to $u(0,\cdot)$.

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- Formulation, well-posedness, optimal control, etc., for initial-boundary problems.
- Characterising the limit of hyperbolic relaxation towards a scalar conservation law.

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- degeneracy of $A'$, i.e. $A' \geq 0$.

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We start with: $A = 0$

# First order quasilinear equations

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,, \end{cases}$$

where f : $\mathbb{R} \to \mathbb{R}^d$ (homogeneous) flux, $u : \mathbb{R}^{d+1}_+ \to \mathbb{R}$ unknown.

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**Weak solutions:** $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+)$ s.t. $\mathsf{f}(u) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^d)$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$

$$\int_{\mathbb{R}^{d+1}_+} u\varphi_t + \mathsf{f}(u) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x}dt + \int_{\mathbb{R}^d} u_0 \varphi(0, \cdot) \, d\mathbf{x} = 0 \,.$$

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Even for smooth f's non-uniqueness:

$d = 1$, $f(\lambda) = \frac{\lambda^2}{2}$ (Burgers equation), $u_0(x) = \begin{cases} 0\,, & x < 0 \\ 1\,, & x \geq 0 \end{cases}$.

Both functions are a weak solution:

$$u_1(t,x) = \begin{cases} 0\,, & x < t/2 \\ 1\,, & x \geq t/2 \end{cases} \,, \qquad u_2(x) = \begin{cases} 0\,, & x < 0 \\ x/t\,, & 0 \leq x < t \\ 1\,, & x \geq t \end{cases} \text{(rarefaction wave)}$$

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For the uniqueness we need to impose some conditions on discontinuities.

# Vanishing Viscosity 1/2

Consider only those weak solutions that can be reached as a limit $\varepsilon \to 0^+$ of the sequence of solutions $(u^\varepsilon)$:

$$\begin{cases} \partial_t u^\varepsilon + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u^\varepsilon) = \varepsilon \Delta u^\varepsilon \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u^\varepsilon|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

For $\eta \in \mathrm{C}^2(\mathbb{R})$ convex (i.e. $\eta'' \geq 0$) and $\varphi \in \mathrm{C}^2_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$, we multiply the equation by $-\eta'(u^\varepsilon)\varphi$ and integrate over $\mathbb{R}^{1+d}_+$:

$$-\int_{\mathbb{R}^{d+1}_+} \partial_t u^\varepsilon \eta'(u^\varepsilon)\varphi + \mathsf{f}'(u^\varepsilon) \cdot \nabla u^\varepsilon \eta'(u^\varepsilon)\varphi \, d\mathbf{x}dt = -\varepsilon \int_{\mathbb{R}^{d+1}_+} \Delta u^\varepsilon \eta'(u^\varepsilon)\varphi \, d\mathbf{x}dt$$

LHS:

$$-\int_{\mathbb{R}^{1+d}_+} \partial_t u^\varepsilon \eta'(u^\varepsilon)\varphi + \mathsf{f}'(u^\varepsilon)\cdot\nabla u^\varepsilon \eta'(u^\varepsilon)\varphi \, d\mathbf{x}dt$$

$$= \int_{\mathbb{R}^{1+d}_+} \eta(u^\varepsilon)\partial_t\varphi + \mathsf{f}^\eta(u^\varepsilon)\cdot\nabla\varphi \, d\mathbf{x}dt + \int_{\mathbb{R}^d} \eta(u_0)\varphi(0,\cdot)\, d\mathbf{x}\,,$$

where $\mathsf{f}^\eta(\lambda) = \int_0^\lambda \mathsf{f}'(s)\eta'(s)\, ds$, and we have used

$$\partial_t u^\varepsilon \eta'(u^\varepsilon) = \partial_t\big(\eta(u^\varepsilon)\big) \quad \text{and} \quad \mathsf{f}'(u^\varepsilon)\cdot\nabla u^\varepsilon\eta'(u^\varepsilon) = \mathrm{div}\big(\mathsf{f}^\eta(u^\varepsilon)\big)\,.$$

RHS:

$$-\varepsilon\int_{\mathbb{R}^{1+d}_+} \Delta u^\varepsilon \eta'(u^\varepsilon)\varphi\, d\mathbf{x}dt = \varepsilon\int_{\mathbb{R}^{1+d}_+} \eta'(u^\varepsilon)\nabla u^\varepsilon\cdot\nabla\varphi + \underbrace{|\nabla u^\varepsilon|^2\eta''(u^\varepsilon)\varphi}_{\geq 0}\, d\mathbf{x}dt$$

$$\geq -\varepsilon\int_{\mathbb{R}^{1+d}_+} \eta(u^\varepsilon)\Delta\varphi\, d\mathbf{x}dt$$

# Vanishing Viscosity 2/2

LHS:

$$-\int_{\mathbb{R}^{1+d}_+} \partial_t u^\varepsilon \eta'(u^\varepsilon)\varphi + \mathsf{f}'(u^\varepsilon)\cdot\nabla u^\varepsilon \eta'(u^\varepsilon)\varphi \, d\mathbf{x}dt$$

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$$\geq -\varepsilon\int_{\mathbb{R}^{1+d}_+} \eta(u^\varepsilon)\Delta\varphi\, d\mathbf{x}dt \overset{\varepsilon\to 0}{\longrightarrow} 0$$

# Entropy solutions

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

**Entropy solutions:** $u$ a weak solution and s.t. $\forall \eta \in \mathrm{C}(\mathbb{R})$ convex and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$,

$$\int_{\mathbb{R}^{d+1}_+} \eta(u)\varphi_t + \mathsf{f}^\eta(u) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} dt + \int_{\mathbb{R}^d} \eta(u_0)\varphi(0, \cdot) \, d\mathbf{x} \geq 0 \,,$$

here $\mathsf{f}^\eta(\lambda) = \int_0^\lambda \mathsf{f}' \eta' \, ds$ is an entropy-flux.

- $\eta$ is called (mathematical) entropy ($-\eta$ corresponds to physical entropy)
- The above inequality is due to the fact that the physical entropy has a tendency to increase in time, i.e. the mathematical entropy decreases in time

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**Entropy solutions:** (Kružkov) $u \in \mathrm{L}^\infty(\mathbb{R}^{d+1}_+)$ s.t. $\forall \lambda \in \mathbb{R}$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$,

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$$\lambda = \|u\|_{L^\infty} \implies -\int_{\mathbb{R}^{d+1}_+} u\varphi_t + \mathsf{f}(u) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} dt - \int_{\mathbb{R}^d} u_0 \varphi(0, \cdot) \, d\mathbf{x} \geq 0$$

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Kružkov (1970): existence and uniqueness of entropy solutions for smooth heterogeneous fluxes $\mathsf{f}$.

- Existence: vanishing viscosity method; Uniqueness: method of doubling variables (used for developing numerical schemes as well)

Panov (2010): existence of entropy solutions for non-smooth heterogeneous fluxes under non-degeneracy assumptions

- $u_n$ solution for the regularised flux $\mathsf{f}_n$, and apply a compactness result

# Strong traces ($A = 0$)

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$\Longleftrightarrow$

(a.e. $\lambda \in \mathbb{R}$) $\quad \partial_t |u - \lambda| + \mathsf{div}_{\mathbf{x}} \Big( \operatorname{sgn}(u - \lambda)(\mathsf{f}(u) - \mathsf{f}(\lambda)) \Big) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^{d+1}_+) \,,$

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Vasseur (2001): existence of strong traces for entropy solutions for smooth fluxes f and with a non-degeneracy condition

Panov (2005, 2007): existence of strong traces for entropy solutions (without non-degeneracy conditions)

Neves, Panov, Silva (2018): existence of strong traces for entropy solutions for heterogeneous fluxes f and with a non-degeneracy condition

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The result does not hold for weak solutions!

# Strong traces ($A = 0$) – comments

$$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d$$

**Idea of the proof:**

$u$ admits the strong trace $\iff$

$$u_m(t, \mathbf{x}, \mathbf{y}) := u\Big(\frac{t}{m}, \frac{\mathbf{x}}{m} + \mathbf{y}\Big) \text{ is precompact in } \mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+ \times \mathbb{R}^d)\,.$$

**Some applications:**

- The strong boundary condition in the sense of Bardos, LeRoux, Nédélec for rough initial $u_0$ and boundary $u_b$ data: $(\forall \lambda \in \mathbb{R})$

$$\big(\mathrm{sgn}(u - \lambda) + \mathrm{sgn}(\lambda - u_b)\big)(\mathsf{f}(u) - \mathsf{f}(\lambda)) \cdot \vec{\nu} \geq 0 \ \text{ on } \ \partial\Omega\,.$$

- Bürger, Frid, Karlsen (2007): The well-posedness of the initial-boundary problem with zero-flux boundary condition.
- Pfaff, Ulbrich (2015): The optimal control of initial-boundary value problems.

# Degenerate parabolic equation ($A' \geq 0$)

$$\text{(DP)} \quad \begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,, & \Big(\mathsf{div}_{\mathbf{x}}(A'(u)\nabla_{\mathbf{x}} u)\Big) \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d)\,, \end{cases}$$

where $\mathsf{f} : \mathbb{R} \to \mathbb{R}^d$, $A : \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, and $u : \mathbb{R}^{d+1}_+ \to \mathbb{R}$ unknown.

# Definition of solutions (kinetic formulation)

(DP) $$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

## Definition

$u \in \mathrm{L}^\infty(\mathbb{R}^{d+1}_+)$ is called a quasi-solution to ($\mathrm{DP}_1$) if for a.e. $\lambda \in \mathbb{R}$

$$\partial_t |u - \lambda| + \mathsf{div}_{\mathbf{x}} \Big( \mathsf{sgn}(u - \lambda) \left( \mathsf{f}(u) - \mathsf{f}(\lambda) \right) \Big) - D^2_{\mathbf{x}} \cdot [\mathsf{sgn}(u - \lambda)(A(u) - A(\lambda))] = -\gamma(t, \mathbf{x}, \lambda) \,,$$

holds in $\mathcal{D}'(\mathbb{R}^{d+1}_+)$, where $\gamma \in \mathrm{C}(\mathbb{R}_\lambda; \mathcal{M}(\mathbb{R}^{d+1}_+))$.

Vol'pert, Hudjaev (1969)

For $A = 0$ and $\gamma \geq 0$ coincides with the previous definition of entropy solutions.

# Definition of solutions (kinetic formulation)

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## Theorem

*If function $u$ is a bounded quasi-solution to (DP$_1$), then the function*

$$h(t, \mathbf{x}, \lambda) := \mathsf{sgn}(u(t, \mathbf{x}) - \lambda) = -\partial_\lambda |u(t, \mathbf{x}) - \lambda|$$

*is a weak solution to the following linear equation:*

$$\partial_t h + \mathsf{div}_{\mathbf{x}} \left( \mathsf{f}' \, h \right) - D^2_{\mathbf{x}} \cdot [A'(\lambda) h] = \partial_\lambda \gamma(t, \mathbf{x}, \lambda) \,.$$

Lions, Perthame, Tadmor (1994)

# Existence of entropy solutions to (DP)

(DP) $$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,, \\ u\big|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

- smooth fluxes
  - Carrillo (1999): $\mathrm{L}^\infty$ solutions
  - Chen, Perthame (2003): $\mathrm{L}^1$ solutions
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  - Tadmor, Tao (2007): improved regularity under a non-degeneracy condition
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  - Lazar, Mitrović (2012): the result for heterogeneous ultra-parabolic equations using a velocity averaging approach
  - Holden, Karlsen, Mitrović, Panov (2009): general but homogeneous $A$ (in E., Mišur, Mitrović (to appear in JLMS) a similar result via velocity averaging approach)

# Existence of entropy solutions to (DP)

(DP) $$\begin{cases} \partial_t u + \mathrm{div}_{\mathbf{x}} \mathsf{f}(\mathbf{x}, u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \ , \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \ . \end{cases}$$

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# A detour on H-measures 1/2

If we have $u_n \rightharpoonup 0$ in $L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open, what we can say about $|u_n|^2$?

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It is bounded in $L^1_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{M}(\Omega)$, so

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What if $(u_n)$ satisfies certain differential relations with variable coefficients:

$$P_n(\mathbf{x}, D)u_n = f_n\,?$$

$\nu$ "lives" only on the physical space $\Omega$ (depends only on $\mathbf{x}$), but to exploit the above we need a microlocal object.

H-measure $\mu$ to $(u_n)$:

$$\langle \mu, \varphi_1 \bar{\varphi}_2 \otimes \psi \rangle = \lim_n \int_{\mathbb{R}^d} \widehat{\varphi_1 u_n}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 u_n}(\boldsymbol{\xi})} \psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) d\boldsymbol{\xi}$$

If $P_n(\mathbf{x}, D)u_n = f_n$, under certain conditions, we have

$$p(\mathbf{x}, \boldsymbol{\xi})\mu = 0$$

(one can think that $p$ is the principle symbol of $P_n$).

If $p$ is nonzero on the support of $\mu$, we get

$$\mu = 0 \implies u_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0\,.$$

# A detour on H-measures 2/2

H-measure $\mu$ to $(u_n)$:

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If the equation is not of the same order in all variables, we study different variants.

# Velocity averaging

1. Regularisation of the flux
2. Kinetic fomulation
3. Localisation principle and non-degeneracy condition
4. Adaptive H-measures

# Velocity averaging

1. Regularisation of the flux

   Replace f by $f_n$, which defines sequence of solutions $(u_n)$.
   It is sufficient to get the strong convergence of $(u_n)$.
2. Kinetic fomulation
3. Localisation principle and non-degeneracy condition
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# Velocity averaging

1. Regularisation of the flux
2. Kinetic fomulation

$h_n(t, \mathbf{x}, \lambda) := \operatorname{sgn}(u_n(t, \mathbf{x}) - \lambda).$

$$\partial_t h_n + \mathsf{div}_{\mathbf{x}} \left( \mathsf{f}_n' \, h_n \right) - D^2_{\mathbf{x}} \cdot [A'(\lambda) h_n] = \partial_\lambda \gamma_n(t, \mathbf{x}, \lambda) \ .$$

$$2u_n(t, \mathbf{x}) - \alpha - \beta = \int_\alpha^\beta h_n(t, \mathbf{x}, \lambda) \, d\lambda$$

3. Localisation principle and non-degeneracy condition
4. Adaptive H-measures

# Velocity averaging

1. Regularisation of the flux
2. Kinetic fomulation
3. Localisation principle and non-degeneracy condition

   On the limit we get ($\mu$ is a suitable variant of microlocal defect object):

$$
\begin{aligned}
(\forall\phi)\ \langle \mu, F\phi\rangle = 0 \ &\overset{F \text{ non-degenerate}}{\Longrightarrow} \quad \mu \equiv 0 \\
&\Longrightarrow \ \text{strong convergence of } \int_\alpha^\beta h_n(t,\mathbf{x},\lambda)\,d\lambda\,.
\end{aligned}
$$

   Non-degenerate condition:

$$
\operatorname*{ess\,sup}_{(t,\mathbf{x})\in\mathbb{R}^+\times\mathbb{R}^d} \ \sup_{|\boldsymbol{\xi}|=1} \operatorname{meas}\Bigl\{\lambda\in K \,:\, \tau + \langle \mathsf{f}'(t,\mathbf{x},\lambda) \,|\, \boldsymbol{\xi}\rangle = \langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0\Bigr\} = 0
$$

4. Adaptive H-measures

# Velocity averaging

1. Regularisation of the flux
2. Kinetic fomulation
3. Localisation principle and non-degeneracy condition
4. Adaptive H-measures

$$\mu(\varphi\psi) = \lim_{n\to\infty} \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,\mathbf{x}) h_n(t,\mathbf{x},\lambda) \overline{\mathcal{A}_{\bar\psi(\pi_P(\cdot,\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})} \, dt\, d\mathbf{x}\, d\lambda\,,$$

where

$$\pi_P(\tau,\boldsymbol{\xi},\lambda) := \frac{(\tau,\boldsymbol{\xi})}{|(\tau,\boldsymbol{\xi})| + \langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle}$$

# Existence of strong traces for $(\mathsf{DP}_1)$

$$(\mathsf{DP}_1) \qquad \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,.$$

$$\operatorname{ess\,lim}_{t \to 0^+} u(t,\cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad ???$$

Kwon (2009): scalar diffusion matrices $A(u) = a(u)I$ without non-degeneracy conditions

Aleksić, Mitrović (2013): traceable fluxes f and ultra-parabolic $A$ (i.e. $A = B \oplus 0$ where $B' > 0$) without non-degeneracy conditions

Frid, Li (2017): for $A = B \otimes 0$ where $cI \le B' \le \Lambda c I$ and $c \ge 0$.

"Fully degenerate" matrices $A'$ not covered, e.g.

$$A'(\lambda) = \left( \frac{1}{\sqrt{\lambda^2+1}} \begin{bmatrix} \lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 0 & \lambda^2+1 \end{bmatrix} \left( \frac{1}{\sqrt{\lambda^2+1}} \begin{bmatrix} \lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) = \begin{bmatrix} 1 & -\lambda \\ -\lambda & \lambda^2 \end{bmatrix}$$

# Existence of strong traces for $(\mathrm{DP}_1)$

$(\mathsf{DP_1})$
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## Theorem (E., Mitrović (2022))

*Let* $\mathsf{f} \in \mathrm{C}^1(\mathbb{R};\mathbb{R}^d)$ *and let* $A \in \mathrm{C}^{1,1}(\mathbb{R};\mathbb{R}^{d\times d})$ *be such that for any* $\lambda \in \mathbb{R}$ *we have* $A'(\lambda)$ *is symmetric and positive semi-definite.*
*Then any quasi-solution* $u \in \mathrm{L}^\infty_{loc}(\mathbb{R}^+;\mathrm{L}^p_{loc}(\mathbb{R}^d))$, *for some* $p > 1$, *to* $(DP_1)$ *admits the strong trace at* $t = 0$.

# Proof – an important point

$$(\mathsf{DP}_1) \qquad \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ .$$

Which scaling to choose with respect to $\mathbf{x}$ in

$$u_m(t, \mathbf{x}, \mathbf{y}) = u\Big(\frac{t}{m}, \frac{\mathbf{x}}{m} + \mathbf{y}\Big)\,?$$

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(DP$_1$) $$\partial_t u + \mathsf{div}_{\mathbf{x}}\mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ .$$

If $A'(\lambda) = \begin{bmatrix} \tilde{a}(\lambda) & 0 \\ 0 & 0 \end{bmatrix}$, for $\tilde{a}(\lambda) \in \mathbb{R}^{k\times k}$ $(k \in \{1, \dots, d\})$, and

$$\tilde{a}(\lambda) > 0$$

we use

$$u_m(t, \mathbf{x}, \mathbf{y}) = u\Big(\frac{t}{m}, \frac{\tilde{\mathbf{x}}}{\sqrt{m}} + \tilde{\mathbf{y}}, \frac{\bar{\mathbf{x}}}{m} + \bar{\mathbf{y}}\Big)$$

where $\mathbf{x} = (\tilde{\mathbf{x}}, \bar{\mathbf{x}}) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ and apply a compactness result from Holden et al (2009).

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$(*)$ $$(\forall \tilde{\boldsymbol{\xi}} \in \mathbb{R}^k \setminus \{0\})(\forall (\alpha', \beta') \subseteq \mathbb{R})$$
$$(\alpha', \beta') \ni \lambda \mapsto \langle \tilde{a}(\lambda) \tilde{\boldsymbol{\xi}} \,|\, \tilde{\boldsymbol{\xi}} \rangle \text{ is not indentically equal to zero.}$$

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where $\mathbf{x} = (\tilde{\mathbf{x}}, \bar{\mathbf{x}}) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ and apply a compactness result from Holden et al (2009).

If $(*)$ is not satisfied, we can reduce locally $a$ on some $(\alpha, \beta) \subseteq \mathbb{R}$ to that form, and then apply above for $s_{\alpha,\beta}(u) := \max\{\alpha, \min\{u, \beta\}\}$ instead of $u$.

# And...

# ...thank you for your attention :)

- Traces: E., Mitrović, SIAM J. Math. Anal, 54 (2022) 1775–1796.
- Existence via Velocity averaging: E., Mišur, Mitrović, to appear in J. Lond. Math. Soc., arXiv:2008.08310, 39 pp.