Entropy solutions of degenerate parabolic equations

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Degenerate parabolic equation - introduction

$$\begin{aligned}\partial_t u(t,\mathbf{x}) + \mathsf{div}_{\mathbf{x}}\mathsf{f}(t,\mathbf{x},u(t,\mathbf{x})) &= D^2_{\mathbf{x}} \cdot A(u(t,\mathbf{x})) \\ &= \mathsf{div}_{\mathbf{x}}\big(A'(u(t,\mathbf{x}))\nabla_{\mathbf{x}} u(t,\mathbf{x})\big)\,,\end{aligned}$$

where $\mathsf{f} : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $A : \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ are given and $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is unknown.

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- LHS: convection effects (f flux, $\mathsf{f} = \mathsf{f}(u)$ homogeneous case);
- RHS: diffusion effects (A' diffusion matrix – direction and intensity of the diffusion).

Motivation for the equation:

- flow in porous media (e.g. $\mathsf{f} = 0$ and $A(u) = u^m \mathrm{I}$ – porous media equation)
  - heterogeneous layers $\longrightarrow$ discontinuous flux and a lack of diffusion in some directions
- sedimentation-consolidation process

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Aim:

- Existence (and uniqueness) of solution to the Cauchy problem;
- Existence of traces of solutions, i.e. give meaning to $u(0,\cdot)$.

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Why traces:

- Formulation, well-posedness, optimal control, etc., for initial-boundary problems.
- Characterising the limit of hyperbolic relaxation towards a scalar conservation law.

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- degeneracy of A', i.e. $A' \geq 0$.

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We start with: $A = 0$

First order quasilinear equations

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,, \end{cases}$$

where $\mathsf{f} : \mathbb{R} \to \mathbb{R}^d$ (homogeneous) flux, $u : \mathbb{R}^{d+1}_+ \to \mathbb{R}$ unknown.

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Weak solutions: $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+)$ s.t. $\mathsf{f}(u) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^d)$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$

$$\int_{\mathbb{R}^{d+1}_+} u\varphi_t + \mathsf{f}(u) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} dt + \int_{\mathbb{R}^d} u_0 \varphi(0, \cdot) \, d\mathbf{x} = 0 \,.$$

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Even for smooth f's non-uniqueness:

$d = 1$, $f(\lambda) = \frac{\lambda^2}{2}$ (Burgers equation), $u_0(x) = \begin{cases} 0 \,, & x < 0 \\ 1 \,, & x \geq 0 \end{cases}$.

Both functions are a weak solution:

$$u_1(t,x) = \begin{cases} 0 \,, & x < t/2 \\ 1 \,, & x \geq t/2 \end{cases} \,, \qquad u_2(x) = \begin{cases} 0 \,, & x < 0 \\ x/t \,, & 0 \leq x < t \quad \text{(rarefaction wave)} \\ 1 \,, & x \geq t \end{cases}$$

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For the uniqueness we need to impose some conditions on discontinuities.

Vanishing Viscosity 1/2

Consider only those weak solutions that can be reached as a limit $\varepsilon \to 0^+$ of the sequence of solutions (u^ε):

$$\begin{cases} \partial_t u^\varepsilon + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u^\varepsilon) = \varepsilon \Delta u^\varepsilon \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u^\varepsilon|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

For $\eta \in \mathrm{C}^2(\mathbb{R})$ convex (i.e. $\eta'' \geq 0$) and $\varphi \in \mathrm{C}^2_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$, we multiply the equation by $-\eta'(u^\varepsilon)\varphi$ and integrate over $\mathbb{R}^{1+d}_+$:

$$-\int_{\mathbb{R}^{d+1}_+} \partial_t u^\varepsilon \eta'(u^\varepsilon)\varphi + \mathsf{f}'(u^\varepsilon) \cdot \nabla u^\varepsilon \eta'(u^\varepsilon)\varphi \, d\mathbf{x}dt = -\varepsilon \int_{\mathbb{R}^{d+1}_+} \Delta u^\varepsilon \eta'(u^\varepsilon)\varphi \, d\mathbf{x}dt$$

Vanishing Viscosity 2/2

Using

$$\partial_t u^\varepsilon \eta'(u^\varepsilon) = \partial_t\big(\eta(u^\varepsilon)\big) \quad \text{and} \quad \mathsf{f}'(u^\varepsilon)\cdot\nabla u^\varepsilon \eta'(u^\varepsilon) = \mathsf{div}\big(\mathsf{f}^\eta(u^\varepsilon)\big)\,,$$

where $\mathsf{f}^\eta(\lambda) = \int_0^\lambda \mathsf{f}'(s)\eta'(s)\,ds$, for the left hand side we have

$$-\int_{\mathbb{R}^{1+d}_+} \partial_t u^\varepsilon \eta'(u^\varepsilon)\varphi + \mathsf{f}'(u^\varepsilon)\cdot\nabla u^\varepsilon \eta'(u^\varepsilon)\varphi\, d\mathbf{x}dt$$

$$= \int_{\mathbb{R}^{1+d}_+} \eta(u^\varepsilon)\partial_t\varphi + \mathsf{f}^\eta(u^\varepsilon)\cdot\nabla\varphi\, d\mathbf{x}dt + \int_{\mathbb{R}^d} \eta(u_0)\varphi(0,\cdot)\, d\mathbf{x}\,.$$

The right hand side satisfies

$$-\varepsilon\int_{\mathbb{R}^{1+d}_+} \Delta u^\varepsilon \eta'(u^\varepsilon)\varphi\, d\mathbf{x}dt = \varepsilon\int_{\mathbb{R}^{1+d}_+} \eta'(u^\varepsilon)\nabla u^\varepsilon\cdot\nabla\varphi + \underbrace{|\nabla u^\varepsilon|^2\eta''(u^\varepsilon)\varphi}_{\geq 0}\, d\mathbf{x}dt$$

$$\geq -\varepsilon\int_{\mathbb{R}^{1+d}_+} \eta(u^\varepsilon)\Delta\varphi\, d\mathbf{x}dt$$

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Entropy solutions

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

Entropy solutions: u a weak solution and s.t. $\forall \eta \in \mathrm{C}(\mathbb{R})$ convex and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$,

$$\int_{\mathbb{R}^{d+1}_+} \eta(u)\varphi_t + \mathsf{f}^\eta(u) \cdot \nabla_{\mathbf{x}}\varphi \, d\mathbf{x}dt + \int_{\mathbb{R}^d} \eta(u_0)\varphi(0,\cdot)\, d\mathbf{x} \geq 0 \,,$$

here $\mathsf{f}^\eta(\lambda) = \int_0^\lambda \mathsf{f}'\eta' \, ds$ is an entropy-flux.

- η is called (mathematical) entropy ($-\eta$ corresponds to physical entropy)
- The above inequality is due to the fact that the physical entropy has a tendency to increase in time, i.e. the mathematical entropy decreases in time

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Entropy solutions: (Kružkov) $u \in \mathrm{L}^\infty(\mathbb{R}^{d+1}_+)$ s.t. $\forall \lambda \in \mathbb{R}$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$,

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$$\lambda = \|u\|_{L^\infty} \implies -\int_{\mathbb{R}^{d+1}_+} u\varphi_t + \mathsf{f}(u) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x}dt - \int_{\mathbb{R}^d} u_0 \varphi(0,\cdot) \, d\mathbf{x} \geq 0$$

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Kružkov (1970): existence and uniqueness of entropy solutions for smooth heterogeneous fluxes f.

- Existence: vanishing viscosity method; Uniqueness: method of doubling variables (used for developing numerical schemes as well)

Panov (2010): existence of entropy solutions for non-smooth heterogeneous fluxes under non-degeneracy assumptions

- u_n solution for the regularised flux f_n, and apply a compactness result

Strong traces ($A = 0$)

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$\Longleftrightarrow$

$$\text{(a.e. } \lambda \in \mathbb{R}) \qquad \partial_t |u-\lambda| + \mathsf{div}_{\mathbf{x}} \Big(\operatorname{sgn}(u-\lambda)(\mathsf{f}(u) - \mathsf{f}(\lambda)) \Big) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^{d+1}_+)\,,$$

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Vasseur (2001): existence of strong traces for entropy solutions for smooth fluxes f and with a non-degeneracy condition

Panov (2005, 2007): existence of strong traces for entropy solutions (without non-degeneracy conditions)

Neves, Panov, Silva (2018): existence of strong traces for entropy solutions for heterogeneous fluxes f and with a non-degeneracy condition

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The result does not hold for weak solutions!

Strong traces ($A = 0$) – comments

$$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ := \mathbb{R}^+ \times \mathbb{R}^d$$

Idea of the proof:

u admits the strong trace $\iff$

$$u_m(t, \mathbf{x}, \mathbf{y}) := u\Big(\frac{t}{m}, \frac{\mathbf{x}}{m} + \mathbf{y}\Big) \text{ is precompact in } \mathrm{L}^1_{loc}(\mathbb{R}^{d+1}_+ \times \mathbb{R}^d) .$$

Some applications:

- The strong boundary condition in the sense of Bardos, LeRoux, Nédélec for rough initial u_0 and boundary u_b data: $(\forall \lambda \in \mathbb{R})$

$$\big(\mathrm{sgn}(u-\lambda) + \mathrm{sgn}(\lambda - u_b)\big)\big(\mathsf{f}(u) - \mathsf{f}(\lambda)\big) \cdot \vec{\nu} \geq 0 \ \text{ on } \ \partial\Omega .$$

- Bürger, Frid, Karlsen (2007): The well-posedness of the initial-boundary problem with zero-flux boundary condition.
- Pfaff, Ulbrich (2015): The optimal control of initial-boundary value problems.

Degenerate parabolic equation ($A' \geq 0$)

$$
\text{(DP)} \qquad \begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,, & \Big(\mathsf{div}_{\mathbf{x}}(A'(u)\nabla_{\mathbf{x}} u)\Big) \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d)\,, \end{cases}
$$

where $\mathsf{f} : \mathbb{R} \to \mathbb{R}^d$, $A : \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, and $u : \mathbb{R}^{d+1}_+ \to \mathbb{R}$ unknown.

Definition of solutions (kinetic formulation)

(DP)
$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \, , \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \, . \end{cases}$$

Definition

$u \in \mathrm{L}^\infty(\mathbb{R}^{d+1}_+)$ is called a quasi-solution to (DP_1) if for a.e. $\lambda \in \mathbb{R}$

$$\partial_t |u - \lambda| + \mathsf{div}_{\mathbf{x}} \Big(\mathsf{sgn}(u-\lambda) \left(\mathsf{f}(u) - \mathsf{f}(\lambda) \right) \Big) - D^2_{\mathbf{x}} \cdot [\mathsf{sgn}(u-\lambda)(A(u) - A(\lambda))] = -\gamma(t, \mathbf{x}, \lambda) \, ,$$

holds in $\mathcal{D}'(\mathbb{R}^{d+1}_+)$, where $\gamma \in \mathrm{C}(\mathbb{R}_\lambda; \mathcal{M}(\mathbb{R}^{d+1}_+))$.

Vol'pert, Hudjaev (1969)

For $A = 0$ and $\gamma \geq 0$ coincides with the previous definition of entropy solutions.

Definition of solutions (kinetic formulation)

(DP) $$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

Theorem

If function u is a bounded quasi-solution to (DP$_1$), then the function

$$h(t, \mathbf{x}, \lambda) := \mathsf{sgn}(u(t, \mathbf{x}) - \lambda) = -\partial_\lambda |u(t, \mathbf{x}) - \lambda|$$

is a weak solution to the following linear equation:

$$\partial_t h + \mathsf{div}_{\mathbf{x}} \left(\mathsf{f}' \, h \right) - D^2_{\mathbf{x}} \cdot [A'(\lambda) h] = \partial_\lambda \gamma(t, \mathbf{x}, \lambda) \,.$$

Lions, Perthame, Tadmor (1994)

Existence of entropy solutions to (DP)

(DP) $$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ , \\ u\big|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) . \end{cases}$$

- smooth fluxes
  - Carrillo (1999): L^∞ solutions
  - Chen, Perthame (2003): L^1 solutions
  - Tadmor, Tao (2007): improved regularity under a non-degeneracy condition
  - Graf, Kunzinger, Mitrović (2017): on Riemannian manifolds
- non-smooth fluxes (under a non-degeneracy condition)
  - Sazhenkov (2006), Panov (2009): heterogeneous ultra-parabolic equations, i.e. $A(\lambda)$ satisfies an ellipticity assumption on a subspace of $\mathbb{R}^d$ uniformly in λ
  - Lazar, Mitrović (2012): the result for heterogeneous ultra-parabolic equations using a velocity averaging approach
  - Holden, Karlsen, Mitrović, Panov (2009): general but homogeneous A (in E., Mišur, Mitrović (submitted) a similar result via velocity averaging approach)

Existence of entropy solutions to (DP)

(DP) $$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,, \\ u\big|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

- smooth fluxes
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Velocity averaging

1. Regularisation of the flux
2. Kinetic fomulation
3. Localisation principle and non-degeneracy condition
4. Adaptive H-measures

Velocity averaging

1. Regularisation of the flux

   Replace f by f_n, which defines sequence of solutions (u_n).
   It is sufficient to get the strong convergence of (u_n).
2. Kinetic fomulation
3. Localisation principle and non-degeneracy condition
4. Adaptive H-measures

Velocity averaging

1. Regularisation of the flux
2. Kinetic fomulation

$h_n(t, \mathbf{x}, \lambda) := \operatorname{sgn}(u_n(t, \mathbf{x}) - \lambda).$

$$\partial_t h_n + \mathsf{div}_{\mathbf{x}} \left(\mathsf{f}_n' \, h_n \right) - D^2_{\mathbf{x}} \cdot [A'(\lambda) h_n] = \partial_\lambda \gamma_n(t, \mathbf{x}, \lambda) \ .$$

$$2u_n(t, \mathbf{x}) - \alpha - \beta = \int_\alpha^\beta h_n(t, \mathbf{x}, \lambda) \, d\lambda$$

3. Localisation principle and non-degeneracy condition
4. Adaptive H-measures

Velocity averaging

1. Regularisation of the flux

2. Kinetic fomulation

3. Localisation principle and non-degeneracy condition

   On the limit we get (μ is a suitable variant of microlocal defect object):

$$(\forall\phi)\ \langle\mu, F\phi\rangle = 0 \quad \overset{F \text{ non-degenerate}}{\Longrightarrow} \quad \mu \equiv 0$$

$$\Longrightarrow \text{ strong convergence of } \int_\alpha^\beta h_n(t,\mathbf{x},\lambda)\,d\lambda\,.$$

   Non-degenerate condition:

$$\operatorname*{ess\,sup}_{(t,\mathbf{x})\in\mathbb{R}^+\times\mathbb{R}^d} \sup_{|\boldsymbol{\xi}|=1} \operatorname{meas}\Big\{\lambda\in K : \tau + \langle \mathsf{f}'(t,\mathbf{x},\lambda) \,|\, \boldsymbol{\xi}\rangle = \langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0\Big\} = 0$$

4. Adaptive H-measures

Velocity averaging

1. Regularisation of the flux
2. Kinetic fomulation
3. Localisation principle and non-degeneracy condition
4. Adaptive H-measures

$$\mu(\varphi\psi) = \lim_{n\to\infty} \int_{\mathbb{R}^+\times\mathbb{R}^{d+1}} \varphi(t,\mathbf{x}) h_n(t,\mathbf{x},\lambda) \overline{\mathcal{A}_{\bar{\psi}(\pi_P(\cdot,\cdot,\lambda),\lambda)}(v_n)(\mathbf{x})} \, dt\, d\mathbf{x}\, d\lambda \,,$$

where

$$\pi_P(\tau,\boldsymbol{\xi},\lambda) := \frac{(\tau,\boldsymbol{\xi})}{|(\tau,\boldsymbol{\xi})| + \langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle}$$

Existence of strong traces for (DP_1)

$$(\mathrm{DP_1}) \qquad \partial_t u + \mathrm{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ .$$

$$\operatorname{ess\,lim}_{t\to 0^+} u(t,\cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad ???$$

Kwon (2009): scalar diffusion matrices $A(u) = a(u)I$ without non-degeneracy conditions

Aleksić, Mitrović (2013): traceable fluxes f and ultra-parabolic A (i.e. $A = B \oplus 0$ where $B > 0$) without non-degeneracy conditions

"Fully degenerate" matrices A' not covered, e.g.

$$A'(\lambda) = \left(\frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix}\lambda & 1\\ 1 & -\lambda\end{bmatrix}\right)\begin{bmatrix}0 & 0\\ 0 & \lambda^2+1\end{bmatrix}\left(\frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix}\lambda & 1\\ 1 & -\lambda\end{bmatrix}\right) = \begin{bmatrix}1 & -\lambda\\ -\lambda & \lambda^2\end{bmatrix}$$

Existence of strong traces for (DP$_1$)

(DP$_1$) $$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_{+} \,.$$

$$\operatorname{ess\,lim}_{t \to 0^+} u(t,\cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad ???$$

Theorem (E., Mitrović)

Let $\mathsf{f} \in \mathrm{C}^1(\mathbb{R};\mathbb{R}^d)$ *and let* $A \in \mathrm{C}^{1,1}(\mathbb{R};\mathbb{R}^{d\times d})$ *be such that for any* $\lambda \in \mathbb{R}$ *we have* $A'(\lambda)$ *is symmetric and positive semi-definite.*
Then any quasi-solution $u \in \mathrm{L}^\infty_{loc}(\mathbb{R}^+;\mathrm{L}^p_{loc}(\mathbb{R}^d))$*, for some* $p > 1$*, to (DP$_1$) admits the strong trace at* $t = 0$*.*

Proof – an important point

$$(\mathsf{DP}_1) \qquad \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_{+} \,.$$

Which scaling to choose with respect to $\mathbf{x}$ in

$$u_m(t, \mathbf{x}, \mathbf{y}) = u\Big(\frac{t}{m}, \frac{\mathbf{x}}{m} + \mathbf{y}\Big)\,?$$

Proof – an important point

(DP$_1$) $$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \, .$$

If $A'(\lambda) = \begin{bmatrix} \tilde{a}(\lambda) & 0 \\ 0 & 0 \end{bmatrix}$, for $\tilde{a}(\lambda) \in \mathbb{R}^{k\times k}$ $(k \in \{1, \dots, d\})$, and

$$\tilde{a}(\lambda) > 0$$

we use

$$u_m(t, \mathbf{x}, \mathbf{y}) = u\Big(\frac{t}{m}, \frac{\tilde{\mathbf{x}}}{\sqrt{m}} + \tilde{\mathbf{y}}, \frac{\bar{\mathbf{x}}}{m} + \bar{\mathbf{y}}\Big)$$

where $\mathbf{x} = (\tilde{\mathbf{x}}, \bar{\mathbf{x}}) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ and apply a compactness result from Holden et al (2009).

Proof – an important point

(DP$_1$) $$\partial_t u + \mathsf{div}_{\mathbf{x}}\mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_{+}\,.$$

If $A'(\lambda) = \begin{bmatrix} \tilde{a}(\lambda) & 0 \\ 0 & 0 \end{bmatrix}$, for $\tilde{a}(\lambda) \in \mathbb{R}^{k\times k}$ $(k \in \{1, \dots, d\})$, and

$(*)$ $$(\forall \tilde{\boldsymbol{\xi}} \in \mathbb{R}^k \setminus \{0\})(\forall (\alpha', \beta') \subseteq \mathbb{R})$$ $$(\alpha', \beta') \ni \lambda \mapsto \langle \tilde{a}(\lambda)\tilde{\boldsymbol{\xi}} \,|\, \tilde{\boldsymbol{\xi}} \rangle \text{ is not indentically equal to zero.}$$

we use

$$u_m(t, \mathbf{x}, \mathbf{y}) = u\Big(\frac{t}{m}, \frac{\tilde{\mathbf{x}}}{\sqrt{m}} + \tilde{\mathbf{y}}, \frac{\bar{\mathbf{x}}}{m} + \bar{\mathbf{y}}\Big)$$

where $\mathbf{x} = (\tilde{\mathbf{x}}, \bar{\mathbf{x}}) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ and apply a compactness result from Holden et al (2009).

Proof – an important point

$$(\mathsf{DP}_1) \qquad \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^{d+1}_+ \,.$$

If $A'(\lambda) = \begin{bmatrix} \tilde{a}(\lambda) & 0 \\ 0 & 0 \end{bmatrix}$, for $\tilde{a}(\lambda) \in \mathbb{R}^{k \times k}$ $(k \in \{1, \dots, d\})$, and

$$(*) \qquad \begin{aligned} &(\forall \tilde{\boldsymbol{\xi}} \in \mathbb{R}^k \setminus \{0\})(\forall (\alpha', \beta') \subseteq \mathbb{R}) \\ &\qquad (\alpha', \beta') \ni \lambda \mapsto \langle \tilde{a}(\lambda) \tilde{\boldsymbol{\xi}} \,|\, \tilde{\boldsymbol{\xi}} \rangle \text{ is not indentically equal to zero.} \end{aligned}$$

we use

$$u_m(t, \mathbf{x}, \mathbf{y}) = u\Big(\frac{t}{m}, \frac{\tilde{\mathbf{x}}}{\sqrt{m}} + \tilde{\mathbf{y}}, \frac{\bar{\mathbf{x}}}{m} + \bar{\mathbf{y}}\Big)$$

where $\mathbf{x} = (\tilde{\mathbf{x}}, \bar{\mathbf{x}}) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ and apply a compactness result from Holden et al (2009).

If $(*)$ is not satisfied, we can reduce locally a on some $(\alpha, \beta) \subseteq \mathbb{R}$ to that form, and then apply above for $s_{\alpha,\beta}(u) := \max\{\alpha, \min\{u, \beta\}\}$ instead of u.

...thank you for your attention :)

- Traces: E., Mitrović, accepted for publication in SIAM J. Math. Anal, 22 pp.
- Velocity averaging: E., Mišur, Mitrović, arXiv:2008.08310, submitted, 38 pp.

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